2022

# p-Adic analysis: A quick introduction

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ABSTRACT. These notes aim to provide a fast introduction to $p$-adic analysis assuming basic knowledge in algebra and analysis. The text corresponds to the lecture notes for a Mini-Course in the L. Santaló Research Summer School 2019. Palacio de la Magdalena, Santander, Spain, June 24-28, 2019.

CONTENTS



## 1. Introduction

These notes aim to provide a fast introduction to $p$-adic analysis assuming basic knowledge in algebra and analysis. We cannot provide detailed proofs. For an in-depth discussion, the reader may consult [**1**], [**23**], [**24**], see also [**12**], [**14**], [**21**], [**22**]. We focus on basic aspects of analysis involving complex-valued functions.

In the last thirty years $p$-adic analysis has received great attention due to its connections with physics, biology, cryptography, and several mathematical theories, see e.g. [**1**], [**2**], [**17**], [**18**], [**15**], [**16**], [**24**], [**25**], and the references therein. As a consequence of all this, nowadays, $p$-adic analysis is having a tremendous expansion. Let us mention a couple examples. First, the developing of the theory of $p$-adic pseudodifferential equations, which is a theory connected with several fields, see e.g. [**1**], [**18**], [**15**], [**16**], [**24**], [**25**] and the reference therein. Second, the deep

2020 *Mathematics Subject Classification.* Primary 43A70, 46S10, 11S40, 60G53.
The author was partially supported by UTRGV through the Debnath Endowed Professorship.

connection between local zeta functions and string amplitudes, see e.g. [**5**], [**6**], and the references therein.

The L. Santaló Research School 2019 aims to provide an introduction to the area of local zeta functions. In the Archimedean case, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the study of local zeta functions was initiated by Gel'fand and Shilov [**11**]. The meromorphic continuation of the local zeta functions was established, independently, by Atiyah [**3**] and Bernstein [**4**], see also [**10**, Theorem 5.5.1 and Corollary 5.5.1]. The main motivation was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. It is important to mention here, that in the $p$-adic framework, the existence of fundamental solutions for pseudodifferential operators is also a consequence of the fact that the Igusa local zeta functions admit a meromorphic continuation, see [**18**, Chapter 10] and [**25**, Chapter 5]. In the 70s, Igusa developed a uniform theory for local zeta functions over local fields of characteristic zero [**10**]. For an introduction to the basic aspects of local zeta functions the reader may consult [**20**].

## 2. $p$-Adic numbers: essential facts

**2.1. Basic facts.** In this section we summarize the basic aspects of the field of $p$-adic numbers, for an in-depth discussion the reader may consult [**1**, **12**–**14**, **21**–**23**] and [**24**].

Definition 2.1. Let $F$ be a field. A norm (or an absolute value) on $F$ is a real-valued function, $|\cdot|$, satisfying

(i) $|x| = 0 \Leftrightarrow x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x + y| \leq |x| + |y|$ (*triangle inequality*), for any $x, y \in F$.

Definition 2.2. A norm $|\cdot|$ is called non-Archimedean (or ultrametric), if it satisfies

$$|x + y| \leq \max\{|x|, |y|\}. \tag{2.1}$$

Notice that (2.1) implies the triangle inequality.

Example 2.3. The trivial norm is defined as

$$|x|_{\text{trivial}} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

From now on we will work only with non-trivial norms.

Definition 2.4. Let $p$ be a fixed prime number, and let $x$ be a nonzero rational number. Then, $x = p^k \frac{a}{b}$, with $p \nmid ab$, and $k \in \mathbb{Z}$. The $p$-adic absolute value (or $p$-adic norm) of $x$ is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Exercise 2.5. The function $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$. In addition, show that $|x + y|_p = \max\{|x|_p, |y|_p\}$ when $|x|_p \neq |y|_p$.

We set $\mathbb{R}_+ := \{x \in \mathbb{R};\ x \geq 0\}$. We denote by $\mathbb{N}$ the set of non-negative integers.

DEFINITION 2.6. Let $X$ be a non-empty set. A distance, or metric, on $X$ is a function $d : X \times X \to \mathbb{R}_+$ satisfying the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x$, $y$, $z$ in $X$.

The pair $(X, d)$ is called a metric space.

EXAMPLE 2.7. Let $F$ be a field endowed with a norm $|\cdot|$. The distance $d(x, y) := |x - y|$ , for $x$, $y$ in $F$, is called the induced distance by $|\cdot|$. The pair $(F, d)$ is a metric space.

DEFINITION 2.8. Let $(X, d)$ be a metric space. The metric d is called non-Archimedean if

$$d(x, y) \leq \max \{d(x, z), d(z, y)\} \text{ for any } x,\ y,\ z \in X.$$

EXAMPLE 2.9. Take $X = \mathbb{Q}$, and $d$ the distance induced by the $p$-adic norm $|\cdot|_p$, for a fixed prime $p$. Then $d$ is non-Archimedean.

DEFINITION 2.10. Let $(X, d)$ be a metric space. A sequence $\{a_i\}_{i\in\mathbb{N}}$ in $X$ is called a Cauchy sequence, if for any $\epsilon > 0$ there exists $N$ such that $d(a_m, a_n) < \epsilon$ whenever both $m > N$, $n > N$.

DEFINITION 2.11. Two metrics $d_1$ and $d_2$ on a set $X$ are called equivalent if a sequence is Cauchy with respect to $d_1$ if and only if it is Cauchy with respect to $d_2$. We say that two norms are equivalent if they induce equivalent metrics.

EXERCISE 2.12. Let $\alpha$ be a fixed positive real number. For $x \in \mathbb{Q}$, we define $\|x\| = |x|_\infty^\alpha$, where $|\cdot|_\infty$ denotes the standard absolute value. Show that $\|\cdot\|$ is a norm if and only if $\alpha \leq 1$, and that in that case it is equivalent to the norm $|\cdot|_\infty$ .

THEOREM 2.13 (Ostrowski, [**1**, Theorem 1.3.2]). *Any non trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ or to the standard absolute value $|\cdot|_\infty$.*

REMARK 2.14. (i) Let $F$ be a field endowed with a norm $|\cdot|$. We introduce a topology on $F$ by giving a basis of open sets consisting of the *open balls* $B_r(a)$ with center $a$ and radius $r > 0$:

$$B_r(a) = \{x \in F; |x - a| < r\}.$$

(ii) A sequence of points $\{x_i\}_{i\in\mathbb{N}} \subset F$ is called *Cauchy* if

$$|x_m - x_n| \to 0, \qquad m, n \to \infty.$$

(iii) A field $F$ with a non trivial absolute value $|\cdot|$ is said to be *complete* if any Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ has a limit point $x^* \in F$, i.e. if $|x_n - x^*| \to 0$, $n \to \infty$. This is equivalent to the fact that $(F, d)$, with $d(x, y) = |x - y|$, is *a complete metric space.*

(iv) Let $(X, d)$, $(Y, D)$ be two metric spaces. A bijection $\rho : X \to Y$ satisfying

$$D(\rho(x), \rho(x')) = d(x, x'),\ x,\ x' \in X,$$

is called an *isometry.*

The following fact is well-known, see e.g. [**19**].

THEOREM 2.15. *Let $(M, d)$ be a metric space. There exists a complete metric space $(\widetilde{M}, \widetilde{d})$, such that $M$ is isometric to a dense subset of $\widetilde{M}$. This space $\widetilde{M}$ is unique up to isometries, that is, if $\widetilde{M_0}$ is a complete metric space having $M$ as a dense subspace, then $\widetilde{M_0}$ is isometric to $\widetilde{M}$.*

EXERCISE 2.16. Let $(F, |\cdot|)$ be a valued field, where $|\cdot|$ is a non-Archimedean absolute value. Assume that $F$ is complete with respect to $|\cdot|$. Then, the series $\sum_{k\geq 0} a_k,\ a_k \in F$ converges if an only if $\lim_{k\to\infty} |a_k| = 0$.

**2.2. The field of $p$-adic numbers.** We set

$$\mathbb{Q}_p := \left\{ x = p^{\gamma} \sum_{i=0}^{\infty} x_i p^i;\ \gamma \in \mathbb{Z},\ x_i \in \{0, 1, \ldots, p-1\},\, x_0 \neq 0 \right\} \cup \{0\},$$

and $ord(x) := \gamma$, $|x|_p := p^{-\gamma}$ for $x \in \mathbb{Q}_p \smallsetminus \{0\}$. We set $|0|_p := 0$ and $ord(0) := +\infty$.

LEMMA 2.17. *With the above notation, the following assertions hold true:*

(i) *$\mathbb{Q}_p$ is a field of characteristic zero;*
(ii) *$|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_+$ is a norm;*
(iii) *$(\mathbb{Q}_p, |\cdot|_p)$ is a complete metric space;*
(iv) *$\mathbb{Q}$ is dense in $\mathbb{Q}_p$;*
(v) *the completion of $(\mathbb{Q}, |\cdot|_p)$ is $(\mathbb{Q}_p, |\cdot|_p)$.*

PROOF. (i) We set

$$\mathbb{Z}_p := \left\{ x \in \mathbb{Q}_p;\ x = p^{\gamma} \sum_{i=0}^{\infty} x_i p^i; \gamma \in \mathbb{N},\ x_0 \neq 0 \right\}.$$

We first show that $\mathbb{Z}_p$ is a ring. Take

$$x = p^{\alpha} \sum_{i=0}^{\infty} x_i p^i \text{ with } \alpha \in \mathbb{N},\ x_0 \neq 0, \quad y = p^{\beta} \sum_{i=0}^{\infty} y_i p^i \text{ with } \beta \in \mathbb{N},\ y_0 \neq 0.$$

And set $\gamma = \min\{\alpha, \beta\}$ and

$$z = p^{\gamma} \sum_{i=0}^{\infty} z_i p^i \text{ with } \gamma \in \mathbb{N},\ z_0 \neq 0.$$

We now define the digits $z_i$s by the following recursive formulae:

$$p^{\gamma} \sum_{i=0}^{L-1} z_i p^i \equiv p^{\alpha} \sum_{i=0}^{L-1} x_i p^i + p^{\beta} \sum_{i=0}^{L-1} y_i p^i \mod p^L \text{ for } L \geq \gamma + 1. \tag{2.2}$$

Here $A \equiv B \bmod p^L$ means $p^L$ divides $A - B$. Now, we define $x + y = z$. Notice that (2.2) determines uniquely all the digits $z_i$'s. For the product $xy$, we set

$$w = p^{\alpha+\beta} \sum_{i=0}^{\infty} w_i p^i \text{ with } w_0 \neq 0.$$

Then the digits $w_i$s are uniquely determined by the following recursive formulae:

$$p^{\alpha+\beta} \sum_{i=0}^{L-1} w_i p^i \equiv \left( p^{\alpha} \sum_{i=0}^{L-1} x_i p^i \right) \left( p^{\beta} \sum_{i=0}^{L-1} y_i p^i \right) \mod p^L \text{ for } L \geq \alpha + \beta + 1. \tag{2.3}$$

Now we define $xy = w$. It is not difficult to verify that $(\mathbb{Z}_p, +, \cdot)$ is a commutative ring with unity. Furthermore, by using (2.3) one verifies that $xy = 0$ implies that

$x = 0$ or $y = 0$. This means that $\mathbb{Z}_p$ is a domain. Finally, we notice that the field of fractions of $\mathbb{Z}_p$ is precisely $\mathbb{Q}_p$. In order to verify this assertion it is necessary to use Exercise 2.19.

(ii) By definition $|x|_p = 0$ if and only if $x = 0$. Now given $x = p^\alpha \widetilde{x}$, with $|\widetilde{x}|_p = 1$ and $y = p^\beta \widetilde{y}$, with $|\widetilde{y}|_p = 1$, see Exercise 2.19-(iii), we have $xy = p^{\alpha+\beta}\widetilde{z}$, with $|\widetilde{z}|_p = 1$, see Exercise 2.19-(ii). Then $|xy|_p = |x|_p |y|_p$. Finally, if $x,y \in \mathbb{Z}_p$, then $|x+y|_p \leq p^{-\min\{\alpha,\beta\}} = \max\left\{|x|_p\,,\ |y|_p\right\}$, by using the definition of the sum operation. In the general case, with $\alpha \leq \beta$,

$$\begin{aligned}|x+y|_p &= \left|p^\alpha\left(\widetilde{x}+p^{\beta-\alpha}\widetilde{y}\right)\right|_p = |p^\alpha|_p \left|\widetilde{x}+p^{\beta-\alpha}\widetilde{y}\right|_p \leq p^{-\alpha}\max\left\{|\widetilde{x}|_p\,, \left|p^{\beta-\alpha}\widetilde{y}\right|_p\right\}\\ &= p^{-\alpha}\max\left\{1, p^{-(\beta-\alpha)}\right\} = p^{-\alpha} = \max\left\{|x|_p\,,\ |y|_p\right\}.\end{aligned}$$

(iii) Let $\left\{x^{(m)}\right\}_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{Q}_p$, with

$$x^{(m)} = p^{\gamma_m}\sum_{i=0}^{\infty} x_i^{(m)} p^i.$$

Then, given $L$ there exists $M \in \mathbb{N}$ such that

$$\left|x^{(m)} - x^{(n)}\right|_p < p^{-L} \text{ for } m \geq n > M \Leftrightarrow ord(x^{(m)} - x^{(n)}) > L \text{ for } m \geq n > M,$$

which implies the existence of $\gamma \in \mathbb{Z}$, $x_i \in \{0,1,\ldots,p-1\}$ for $i = \gamma,\ldots,L$ such that

$$x^{(m)} = \sum_{i=\gamma}^{L} x_i p^i + p^{L+1}e^{(m)} = x_L + p^{L+1}e^{(m)}, \text{ for } m > M.$$

Then there exists $x = p^\gamma \sum_{i=0}^{\infty} x_i p^i = \lim_{L\to\infty} x_L \in \mathbb{Q}_p$ such that

$$\left|x - x^{(m)}\right|_p < p^{-L} \text{ for } m > M.$$

Which implies that $x^{(m)} \to x$.

(iv) Any $x \in \mathbb{Q}_p \smallsetminus \{0\}$ can be written as $x = p^\gamma \widetilde{x}$, with $\widetilde{x} \in \mathbb{Z}_p$ and $|\widetilde{x}|_p = 1$. Given $p^{-L}$, with $L \in \mathbb{N}$, there exists $\frac{a}{b} \in \mathbb{Q}$ such that $\left|x - \frac{a}{b}\right|_p < p^{-L}$. We take $b^{-1} = p^\gamma$ and $a \in \mathbb{Z}$ satisfying $|a - \widetilde{x}|_p < p^{-L+\gamma}$.

(v) It follows from (i)-(iii) by using Theorem 2.15. $\square$

As a consequence of Lemma 2.17, the field of $p$-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the distance induced by $|\cdot|_p$. Also, by Lemma 2.17, any $p$-adic number $x \neq 0$ has a unique representation of the form

$$x = p^\gamma \sum_{i=0}^{\infty} x_i p^i,$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_i \in \{0,1,\ldots,p-1\}$, $x_0 \neq 0$. The integer $\gamma$ is called the *$p$-adic order of $x$*, and it will be denoted as $ord(x)$. By definition $ord(0) = +\infty$.

REMARK 2.18. The unit ball

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p;\ |x|_p \leq 1\} = \{x \in \mathbb{Q}_p;\ x = \sum_{i=i_0}^{\infty} x_i p^i, i_0 \geq 0\},$$

is a domain of principal ideals. Any ideal of $\mathbb{Z}_p$ has the form

$$p^m\mathbb{Z}_p = \{x \in \mathbb{Z}_p; x = \sum_{i\geq m} x_i p^i\},\ m \in \mathbb{N}.$$

Indeed, let $I \subseteq \mathbb{Z}_p$ be an ideal. Set $m_0 = \min_{x\in I} ord(x) \in \mathbb{N}$, and let $x_0 \in I$ such that $ord(x_0) = m_0$. Then $I = x_0\mathbb{Z}_p$. From a geometric point of view, the ideals of the form $p^m\mathbb{Z}_p$, $m \in \mathbb{Z}$, constitute a fundamental system of neighborhoods around the origin in $\mathbb{Q}_p$.

The *residue field* of $\mathbb{Q}_p$ is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ (the finite field with $p$ elements). The *group of units* of $\mathbb{Z}_p$ is

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p; |x|_p = 1\}.$$

EXERCISE 2.19. (i) Show that $x = x_0 + x_1p + \cdots \in \mathbb{Z}_p$ is a unit if and only if $x_0 \neq 0$. (ii) $\mathbb{Z}_p^\times$ is an Abelian group under the multiplication operation defined in the proof of Lemma 2.17-(i). (iii) If $x \in \mathbb{Q}_p \setminus \{0\}$, then $x = p^m u$, $m \in \mathbb{Z}$, $u \in \mathbb{Z}_p^\times$. **Hint**. Suppose that $x_0 \neq 0$. One has to show the existence of an element $y = y_0 + y_1p + \cdots \in \mathbb{Z}_p^\times$ such that $xy = 1$. The digits $y_i$s are determined recursively by using the formulae $xy \equiv 1 \bmod p^L$, $L \geq 1$. Notice that $x_0y_0 \equiv 1 \bmod p$ has a solution $y_0 \in \{1, \ldots, p-1\}$.

EXAMPLE 2.20. The formula $\frac{-1}{(p-1)} = \sum_{i=0}^{\infty} p^i$ holds true in $\mathbb{Q}_p$, i.e.

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$

Indeed, set

$$\begin{aligned} z^{(n)} &:= (p-1) + (p-1)p + \cdots + (p-1)p^n \\ &= (p-1)\frac{p^{n+1}-1}{p-1} = p^{n+1} - 1. \end{aligned}$$

Then $\lim_{n\to\infty} z^{(n)} = \lim_{n\to\infty} p^{n+1} - 1 = 0 - 1 = -1$, since $|p^{n+1}|_p = p^{-n-1}$.

**2.3. Topology of $\mathbb{Q}_p$.** Define

$$B_r(a) = \{x \in \mathbb{Q}_p;\ |x-a|_p \leq p^r\},\ r \in \mathbb{Z},$$

as the *ball with center $a$ and radius $p^r$*, and

$$S_r(a) = \{x \in \mathbb{Q}_p;\ |x-a|_p = p^r\},\ r \in \mathbb{Z}$$

as the *sphere with center $a$ and radius $p^r$*.

The topology of $\mathbb{Q}_p$ is quite different from the usual topology of $\mathbb{R}$. First of all, since $|\cdot|_p : \mathbb{Q}_p \to \{p^m;\ m \in \mathbb{Z}\} \cup \{0\}$, the radii are always integer powers of $p$, for the sake of brevity we just use the power in the notation $B_r(a)$ and $S_r(a)$. On the other hand, since the powers of $p$ and zero form a discrete set in $\mathbb{R}$, in the definition of $B_r(a)$ and $S_r(a)$ we can always use '$\leq$'. Indeed,

$$\{x \in \mathbb{Q}_p;\ |x-a|_p < p^r\} = \{x \in \mathbb{Q}_p;\ |x-a|_p \leq p^{r-1}\} = B_{r-1}(a).$$

REMARK 2.21. Notice that $B_r(a) = a + p^{-r}\mathbb{Z}_p$ and $S_r(a) = a + p^{-r}\mathbb{Z}_p^\times$.

We declare that the $B_r(a)$, $r \in \mathbb{Z}$, $a \in \mathbb{Q}_p$, are open subsets. These sets form a basis for the topology of $\mathbb{Q}_p$.

PROPOSITION 2.22. *$S_r(a)$, $B_r(a)$ are open and closed sets in the topology of $\mathbb{Q}_p$.*

PROOF. We first show that $S_r(a)$ is open. Notice that

$$S_r(a) = \bigsqcup_{i\in\{1,\ldots,p-1\}} a + p^{-r}i + p^{-r+1}\mathbb{Z}_p = \bigsqcup_{i\in\{1,\ldots,p-1\}} B_{(r-1)}(a+p^{-r}i),$$

and consequently $S_r(a)$ is an open set.

In order to show that $S_r(a)$ is closed, we take a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of $S_r(a)$ converging to $\widetilde{x}_0 \in \mathbb{Q}_p$. We must show that $\widetilde{x}_0 \in S_r(a)$. Note that $x_n = a + p^{-r}u_n$, $u_n \in \mathbb{Z}_p^\times$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we have

$$|x_n - x_m|_p = p^r|u_n - u_m|_p \to 0, \qquad n,\, m \to \infty,$$

and consequently $\{u_n\}_{n\in\mathbb{N}}$ is also Cauchy. Since $\mathbb{Q}_p$ is complete $u_n \to \widetilde{u}_0$. Then $x_n \to a + p^{-r}\widetilde{u}_0$, so in order to conclude our proof we must verify that $\widetilde{u}_0 \in \mathbb{Z}_p^\times$. Because $u_m$ is arbitrarily close to $\widetilde{u}_0$, their $p$-adic expansions must agree up to a big power of $p$, hence $\widetilde{u}_0 \in \mathbb{Z}_p^\times$.

A similar argument shows that $B_r(a)$ is closed. $\square$

LEMMA 2.23. *If $b \in B_r(a)$ then $B_r(b) = B_r(a)$, i.e. any point of the ball $B_r(a)$ is its center.*

PROOF. Let $x \in B_r(b)$, then

$$|x-a|_p = |x-b+b-a|_p \le \max\{|x-b|_p, |b-a|_p\} \le p^r,$$

i.e. $B_r(b) \subseteq B_r(a)$. Since $a \in B_r(b)$ (i.e. $|b-a|_p = |a-b|_p \le p^r$), we can repeat the previous argument to show that $B_r(a) \subseteq B_r(b)$. $\square$

EXERCISE 2.24. Show that any two balls in $\mathbb{Q}_p$ are either disjoint or one is contained in another.

EXERCISE 2.25. Show that the boundary of any ball is the empty set.

EXERCISE 2.26. The set of balls of $\mathbb{Q}_p$ is countable. **Hint**. Given ball $B_r(a) = a + p^{-r}\mathbb{Z}_p$, with $a \in \mathbb{Q}_p$, $r \in \mathbb{Z}$, there is a rational number $\widetilde{a} \in \mathbb{Q}$ such that $B_r(a) = \widetilde{a} + p^{-r}\mathbb{Z}_p$. Then $B_r(a)$ is uniquely determined by the pair $(\widetilde{a}, r)$.

EXERCISE 2.27. Every open set in $\mathbb{Q}_p$ is a union at most of a countable set of disjoint balls.

A topological space $X$ is called *disconnected* if it can be represented as a union of two disjoint nonempty open subsets. Otherwise we say that $X$ is a *connected* topological space. A subset $A$ of $X$ is called *disconnected* if it can be represented as

$$A = (Y_1 \cap A) \bigsqcup (Y_2 \cap A)\,,$$

where $Y_1$, $Y_2$ are nonempty open subsets of $X$, $Y_1 \cap A \neq \emptyset$, $Y_2 \cap A \neq \emptyset$, and $\bigsqcup$ denotes disjoint union. A topological space $X$ is called *totally disconnected* if the only connected sets are the empty set and the points $\{a\}$, $a \in X$.

EXERCISE 2.28. The space $\mathbb{Q}_p$ is totally disconnected. **Hint**. Given a set $A \neq \{a\}$, with $a \in A$, show that there is a ball $B_r(a)$ such that $B_r(a) \cap A \neq A$. Now use that

$$A = (B_r(a) \cap A) \bigsqcup ((\mathbb{Q}_p \smallsetminus B_r(a)) \cap A)\,.$$

A subset $K \subset X$ is called *compact* if each of its open covers contains a finite subcover. A topological space $X$ is called *locally compact* if every point of $X$ has a compact neighborhood.

THEOREM 2.29. [**1**, Theorem 1.8.5] *A set* $K \subset \mathbb{Q}_p$ *is compact if and only if it is closed and bounded in* $\mathbb{Q}_p$.

REMARK 2.30. $\mathbb{Q}_p$ is a fractal. More precisely, $\mathbb{Q}_p$ is homeomorphic to a Cantor-like set of $\mathbb{R}$, see e.g. [**1**, Section 1.9] and the references therein.

**2.4. The $n-$dimensional $p$-adic space.** We extend the $p$-adic norm to $\mathbb{Q}_p^n$ by taking

$$\|x\|_p := \max_{1\leq i\leq n} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $\|x\|_p = p^{-ord(x)}$. The metric space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a separable complete ultrametric space (here, separable means that $\mathbb{Q}_p^n$ contains a countable dense subset, which is $\mathbb{Q}^n$ ).

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n \ ; \ \|x - a\|_p \leq p^r\}$ *the ball of radius* $p^r$ *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x_i \in \mathbb{Q}_p \ ; \ |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius $p^r$ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^n$ equals the product of $n$ copies of $B_0 = \mathbb{Z}_p$. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n \ ; \ \|x - a\|_p = p^r\}$ *the sphere of radius* $p^r$ *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^\times)^n \subsetneq S_0^n$.

As a topological space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is totally disconnected. Two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other. As in the one dimensional case, a subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$. Since the balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$, one has that $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a locally compact topological space. For further details, the reader may consult for instance [**1**], [**24**], [**21**].

## 3. Integration in $\mathbb{Q}_p^n$

**3.1. Measure theory: A basic dictionary.** The notion of measure of a set is a mathematical abstraction of the naive notions of the length of a segment, the area of a plane figure, and the volume of a body.

Let $X$ be a non-empty set. We want to introduce a notion of measure for a class of subsets of $X$. A suitable class is a $\sigma$-algebra of subsets of $X$. Denote by $\mathcal{P}(X)$ the power set of $X$, then a subset $\Sigma \subset \mathcal{P}(X)$ is called a $\sigma$-algebra, if it satisfies the following properties:

(i) $X \in \Sigma$;
(ii) $\Sigma$ is closed under complementation: if $A \in \Sigma$ then $A^c := X \smallsetminus A \in \Sigma$;
(iii) $\Sigma$ is closed under countable unions: if $A_i \in \Sigma$ for $i \in \mathbb{N}$, then $\cup_{i\in\mathbb{N}} A_i \in \Sigma$.

Notice that it follows from the above definition that $\varnothing \in \Sigma$, and that $\Sigma$ is closed under countable intersections. The elements of $\Sigma$ are called *measurable sets*, which means that we can assign a measure to these sets. The pair $(X, \Sigma)$ is called a *measurable space*. Assume that $(Y, \Lambda)$ is another measurable space and that $f : (X, \Sigma) \to (Y, \Lambda)$ is a function between measurable spaces. The function $f$ is called a *measurable function* if the preimage of every measurable set is measurable.

EXAMPLE 3.1. Let $X$ be a non-empty set. the following are some simple examples of $\sigma$-algebras.

(i) $\Sigma = \{X, \varnothing\}$, this is the trivial $\sigma$-algebra;
(ii) $\Sigma = \mathcal{P}(X)$, this is the discrete $\sigma$-algebra;

(iii) $\Sigma = \{X, \varnothing, A, A^c\}$ is the $\sigma$-algebra generated by the subset $A$.

Example 3.2. Let $F$ be a family of subsets of $X$. Then there exists a unique smallest $\sigma$-algebra $\sigma(F)$ which contains any set in $F$. The $\sigma$-algebra $\sigma(F)$ is called the $\sigma$-algebra generated by $F$, it agrees with intersection of all the $\sigma$-algebras containing $F$. If $(X, d)$ is a metric space, the $\sigma$-algebra generated by the open balls is called the Borel $\sigma$-algebra of $X$.

Definition 3.3. Let $(X, \Sigma)$ be a *measurable space. A function* $\mu : \Sigma \to [0, +\infty]$ is called a *measure* if it satisfies the following properties:
(i) $\mu(\varnothing) = 0$;
(ii) for any countable collection $A_i$, $i \in \mathbb{N}$, of pairwise disjoint sets in $\Sigma$,

$$\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

Let $\mu$ be a measure on $(X, \Sigma)$. The following are some basic properties of $\mu$:
(i) monotonicity: if $A_1$ and $A_2$ are measurable sets with $A_1 \subset A_2$, then $\mu(A_1) \leq \mu(A_2)$;
(ii) subadditivity: for any countable collection $A_i$, $i \in \mathbb{N}$, of measurable sets in $\Sigma$,

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mu(A_i);$$

(iii) continuity from below: if $A_i$, $i \in \mathbb{N}$, are measurable sets in $\Sigma$ such that $A_i \subset A_{i+1}$ for all $i$, then the union of the sets $A_i$ is measurable, and

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i);$$

(iv) continuity from above: if $A_i$, $i \in \mathbb{N}$, are measurable sets in $\Sigma$ such that $A_{i+1} \subset A_i$ for all $i$, then the intersection of the sets $A_i$ is measurable. In addition, if $A_0$ has finite measure, then

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

### 3.2. The Haar measure.

Theorem 3.4. [**9**, Theorem B. Section 58] *Let $(G, \cdot)$ be a locally compact topological Abelian group. There exists a regular Borel measure $\mu_{Haar}$ (called a Haar measure of $G$), unique up to multiplication by a positive constant, such that $\mu_{Haar}(U) > 0$ for every non empty open set $U$, and $\mu_{Haar}(x \cdot E) = \mu_{Haar}(E)$, for every Borel set $E$.*

Notation 1. We will denote the Haar measure by $dx$, then $\mu_{\mathrm{Haar}}(U) = \int_U dx$.

Exercise 3.5. Prove that $(\mathbb{Q}_p, +)$, respectively $(\mathbb{Q}_p^\times, \cdot)$, are locally compact topological groups. Since $\mathbb{Q}_p$, respectively $\mathbb{Q}_p^\times = \mathbb{Q}_p \backslash \{0\}$, are metric spaces, the continuity of the sum, respectively of the product, means that if $x_n \to x$, and $y_n \to y$, then $x_n + y_n \to x + y$, respectively $x_n y_n \to xy$.

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, by Theorem 3.4 there exists a measure $dx$, which is invariant under translations, i.e. $\int_B dx = \int_{a+B} dx$. We denote this fact symbolically as $d(x+a) = dx$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then $dx$ is unique.

In the $n$-dimensional case, $(\mathbb{Q}_p^n, +)$ is also locally compact topological group. We denote by $d^n x$ the Haar measure normalized by the condition $\int_{\mathbb{Z}_p^n} d^n x = 1$. This measure agrees with the product measure $dx_1 \cdots dx_n$, and it also satisfies that $d^n(x+a) = d^n x$, for $a \in \mathbb{Q}_p^n$.

The balls of $\mathbb{Q}_p^n$ generate the Borel $\sigma$-algebra of $\mathbb{Q}_p^n$. The measure $d^n x$ assigns to each open compact subset $U$ a nonnegative real number $\int_U d^n x$, which satisfies

$$\int_{\cup_{k=1}^{\infty} U_k} d^n x = \sum_{k=1}^{\infty} \int_{U_k} d^n x, \tag{3.1}$$

for all compact open subsets $U_k$ in $\mathbb{Q}_p^n$, which are pairwise disjoint, and sucht that $\cup_{k=1}^{\infty} U_k$ is still compact.

Remark 3.6. Let $\mathcal{B}(\mathbb{Q}_p^n)$ be the Borel $\sigma$-algebra on $\mathbb{Q}_p^n$. Let $d^n x$ be the normalized Haar measure on $\left(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n)\right)$. The Haar measure of a Borel set $A$ is denoted as $\mu_{\text{Haar}}^{(n)}(A)$. The fact that $d^n x$ is a regular measure means that for any measurable subset $A$ of $\mathbb{Q}_p^n$ is holds that

$$\begin{aligned}\mu_{\text{Haar}}^{(n)}(A) &= \sup\left\{\mu_{\text{Haar}}^{(n)}(F)\,; F \subset A,\, F \text{ compact and measurable}\right\} \\ &= \inf\left\{\mu_{\text{Haar}}^{(n)}(G); G \supset A,\, G \text{ open and measurable}\right\}.\end{aligned}$$

**3.3. Integration of locally constant functions.** A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is said to be *locally constant* if for every $x \in \mathbb{Q}_p^n$ there exists an open compact subset $U$, containing $x$, and such that $f(x) = f(u)$ for all $u \in U$.

Exercise 3.7. Every locally constant function is continuous.

Any locally constant function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form

$$\varphi(x) = \sum_{k=1}^{\infty} c_k 1_{U_k}(x), \tag{3.2}$$

where $c_k \in \mathbb{C}$,

$$1_{U_k}(x) = \begin{cases} 1 & \text{if} \quad x \in U_k \\ \\ 0 & \text{if} \quad x \notin U_k, \end{cases}$$

and $U_k \subseteq \mathbb{Q}_p^n$ is an open compact for every $k$. Indeed, there exists a covering $\{U_i\}_{i\in\mathcal{N}}$ of $\mathbb{Q}_p^n$ such that each $U_i$ is open compact and $\varphi\mid_{U_i}$ is a constant function. Since $\left(\mathbb{Q}_p^n, \|\cdot\|_p\right)$ is a separable metric space any open cover has a countable subcover, consequently we may take $\mathcal{N} = \mathbb{N}$.

Let $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ be a locally constant function as in (3.2). Assume that $A = \bigsqcup_{i=1}^{k} U_i$, the symbol $\bigsqcup$ means disjoint union, i.e. the sets $U_i$ are pairwise disjoint, with $U_i$ open compact. Then, we define

$$\int_A \varphi(x)\, d^n x = c_1 \int_{U_1} d^n x + \cdots + c_k \int_{U_k} d^n x. \tag{3.3}$$

We recall that given a function $\varphi:\mathbb{Q}_p^n\to\mathbb{C}$ the *support of* $\varphi$ is the set

$$Supp(\varphi)=\overline{\{x\in\mathbb{Q}_p^n\ :\ \varphi(x)\neq 0\}}.$$

A locally constant function with compact support is called a *test function* or a *Bruhat-Schwartz function.* These functions form a $\mathbb{C}$-vector space denoted as $\mathcal{D}:=\mathcal{D}\left(\mathbb{Q}_p^n\right)$. From (3.3) one has that the mapping

$$\begin{array}{ccc}\mathcal{D} & \longrightarrow & \mathbb{C}\\ \varphi & \longmapsto & \int_{\mathbb{Q}_p^n}\varphi\ d^nx,\end{array}\tag{3.4}$$

is a well-defined linear functional.

**3.4. Integration of continuous functions with compact support.** We now extend the integration to a larger class of functions. Let $U$ be an open compact subset of $\mathbb{Q}_p^n$. We denote by $C(U,\mathbb{C})$ the space of all the complex-valued continuous functions supported on $U$, endowed with the supremum norm, i.e. for $\varphi\in C(U,\mathbb{C})$, we set

$$\|\varphi\|:=\sup_{x\in U}|\varphi(x)|.$$

We denote by $C_0(\mathbb{Q}_p^n,\mathbb{C})$ the space of all the complex-valued continuous functions vanishing at infinity, endowed also with the supremum norm. The function $\varphi$ vanishes at infinity, if given $\varepsilon>0$, there exists a compact subset $K$ such that $|\varphi(x)|<\varepsilon$, if $x\notin K$.

It is known that $\mathcal{D}$ is dense in $C_0(\mathbb{Q}_p^n,\mathbb{C})$, see e.g. [**23**, Proposition 1.3]. We identify $C(U,\mathbb{C})$ with a subspace of $C_0(\mathbb{Q}_p^n,\mathbb{C})$, therefore $\mathcal{D}$ is dense in $C(U,\mathbb{C})$.

Now, the functional (3.4) satisfies

$$\left|\int_U\varphi\ d^nx\right|\leq\sup_{x\in U}|\varphi(x)|\int_U d^nx,\text{ for }\varphi\in\mathcal{D},$$

and then it has a unique extension to $C(U,\mathbb{C})$. Indeed, given $f\in C(U,\mathbb{C})$ by the density of $\mathcal{D}$ there is sequence $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{D}$ approaching $f$ in the supremum norm. Now we take

$$\int_{\mathbb{Q}_p^n}f(x)\ d^nx:=\lim_{m\to\infty}\int_{\mathbb{Q}_p^n}f_m(x)\ d^nx.\tag{3.5}$$

Notice that the convergence of the sequence in the right-hand side of (3.5) follows from

$$\left|\int_{\mathbb{Q}_p^n}(f(x)-f_m(x))\ d^nx\right|\leq\int_{\mathbb{Q}_p^n}|f(x)-f_m(x)|\ d^nx\leq C(U)\|f-f_m\|\to 0$$
$$\text{as } m\to\infty.$$

More generally, if $f_m\ \underset{\longrightarrow}{\|\cdot\|}\ f$, with $f$, $f_m\in C(U,\mathbb{C})$ for $m\in\mathbb{N}$, then (3.5) holds true.

3.4.1. *Some remarks on uniform convergence.* We recall the notion of uniform convergence. Let $E$ be a non-empty set. Let $f_n:E\to\mathbb{C}$, $n\in\mathbb{N}$ be a sequence of complex-valued functions. We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent on $E$ with limit $f:E\to\mathbb{C}$, if for every $\epsilon>0$, there exists a natural number $N$ such that for all $n>N$ and any $x\in E$, $|f_n(x)-f(x)|<\epsilon$, which is equivalent to say for every $\epsilon>0$, there exists a natural number $N$ such that for all $n>N$, $\sup_{x\in E}|f_n(x)-f(x)|<\epsilon$.

The Weierstrass M-test is a very useful criterion for determining the uniform convergence of sequences. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n : E \to \mathbb{C}$ and let $M_n$ be a sequence of positive real numbers such that $|f_n(x)| < M_n$ for all $x \in E$ and $n \in \mathbb{N}$. If $\sum_n M_n$ converges, then $\sum_n f_n$ converges uniformly on $E$.

3.4.2. *Some remarks on convergent power series.* Let $(F, |\cdot|)$ be a complete field. Let us denote by $F[[z_1, \dots, z_n]]$, the ring of formal power series with coefficients in $F$. An element of this ring has the form

$$\sum_i c_i z^i = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} c_{i_1,\dots,i_n} z_1^{i_1} \dots x z_n^{i_n},$$

where $i = (i_1, \dots, i_n) \in \mathbb{N}^n$, and the $c_{i_1,\dots,i_n}$'s are in $F$.

A formal series $\sum_i c_i z^i$ is said to be convergent if there exists a positive real number $r$ such that $\sum_i c_i a^i$ converges for any $a = (a_1, \dots, a_n) \in F^n$ satisfying $\max_i |a_i| < r$. The convergent series form a subring of $F[[z_1, \dots, z_n]]$, which will be denoted as $F\langle\langle z_1, \dots, z_n\rangle\rangle$.

If for $\sum_i c_i z^i$ there exists $\sum_i c_i^{(0)} z^i \in \mathbb{R}\langle\langle z_1, \dots, z_n\rangle\rangle$ (a real convergent series) such that $|c_i| \le c_i^{(0)}$ for all $i \in \mathbb{N}^n$, we say that $\sum_i c_i^{(0)} z^i$ is a *dominant series* for $\sum_i c_i z^i$ and write

$$\sum c_i z^i << \sum c_i^{(0)} x^i.$$

Exercise 3.8. A formal power series is convergent if and only if it has a dominant series.

Example 3.9. We set $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $|k| := k_1 + \dots + k_n$. Let $f(z) = \sum_k c_k z^k$ be a complex convergent power series on $\max_i |x_i| < r$. This series has a dominant series, and by the Weierstrass M-test, the sequence $\sum_{|k|\le m} c_k z^k$ converges uniformly to $f(z)$ on $\max_i |z_i| < r$.

We construct a 'radial-type function' on $|x_i|_p \le p^L < r$ for $i = 1, \dots, n$, i.e. on the ball $B_L^n$, by taking

$$f(|x_1|_p, \dots, |x_n|_p) = \sum_k c_k |x|_p^k := \sum_{(k_1,\dots,k_n)} c_{(k_1,\dots,k_n)} \prod_{i=1}^n |x_i|_p^{k_i},$$

for $x = (x_1, \dots, x_n) \in B_L^n$. Notice that

$$\begin{aligned}
&\sup_{x\in B_L^n} \left| f(|x_1|_p, \dots, |x_n|_p) - \sum_{|k|\le m} c_{(k_1,\dots,k_n)} \prod_{i=1}^n |x_i|_p^{k_i} \right| \\
&\le \sup_{\max_i |z_i| < r} \left| f(z_1, \dots, z_n) - \sum_{|k|\le m} c_{(k_1,\dots,k_n)} \prod_{i=1}^n z_i^{k_i} \right|,
\end{aligned}$$

and consequently $\sum_{|k|\le m} c_{(k_1,\dots,k_n)} \prod_{i=1}^n |x_i|_p^{k_i}$ converges uniformly to $f(|x_1|_p, \dots, |x_n|_p)$ on $B_L^n$. Then

$$\begin{aligned}
\int_{B_L^n} f(|x_1|_p, \dots, |x_n|_p) d^n x &= \lim_{m\to\infty} \sum_{|k|\le m} c_{(k_1,\dots,k_n)} \int_{B_L^n} \prod_{i=1}^n |x_i|_p^{k_i} \, d^n x \\
&= \lim_{m\to\infty} \sum_{|k|\le m} c_{(k_1,\dots,k_n)} \prod_{i=1}^n \int_{B_L} |x_i|_p^{k_i} \, dx_i. \qquad (3.6)
\end{aligned}$$

**3.5. The change of variables formula in dimension one.** Let us start by establishing the formula:

$$d\,(ax) = |a|_p\,dx,\; a \in \mathbb{Q}_p^{\times}, \tag{3.7}$$

which means

$$\int_{aU} dx = |a|_p \int_U dx,$$

for every Borel set $U \subseteq \mathbb{Q}_p$, for instance an open compact subset. Indeed, consider

$$\begin{array}{cccc} T_a : & \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p \\ & x & \longmapsto & ax, \end{array}$$

with $a \in \mathbb{Q}_p^{\times}$. $T_a$ is a topological and algebraic isomorphism. Then $U \mapsto \int_{aU} dx$ is a Haar measure for $(\mathbb{Q}_p, +)$, and by the uniqueness of such measure, there exists a positive constant $C(a)$ such that $\int_{aU} dx = C(a) \int_U dx$. To compute $C(a)$ we can pick any open compact set, for instance $U = \mathbb{Z}_p$. Now we show that

$$\int_{a\mathbb{Z}_p} dx = C(a) = |a|_p\,.$$

Let us consider first the case $a \in \mathbb{Z}_p$, i.e. $a = p^l u$, $l \in \mathbb{N}$, $u \in \mathbb{Z}_p^{\times}$. Fix a system of representatives $\{b\}$ of $\mathbb{Z}_p/p^l\mathbb{Z}_p$ in $\mathbb{Z}_p$, then $\mathbb{Z}_p = \bigsqcup_{b\in\mathbb{Z}_p/p^l\mathbb{Z}_p} \left(b + p^l\mathbb{Z}_p\right)$, and

$$1 = \int_{\mathbb{Z}_p} dx = \sum_{b\in\mathbb{Z}_p/p^l\mathbb{Z}_p} \int_{b+p^l\mathbb{Z}_p} dx = \sum_{b\in\mathbb{Z}_p/p^l\mathbb{Z}_p} \int_{p^l\mathbb{Z}_p} dx = \#\left(\mathbb{Z}_p/p^l\mathbb{Z}_p\right) \int_{p^l\mathbb{Z}_p} dx,$$

i.e. $|a|_p = p^{-l} = \int_{a\mathbb{Z}_p} dx$. The case $a \notin \mathbb{Z}_p$ is treated in a similar way.

Now, for $f : U \to \mathbb{C}$, where $U$ is a Borel set, we have

$$\int_U f(x)\,dx = |a|_p \int_{a^{-1}U - a^{-1}b} f\,(ay+b)\;dy, \text{ for any } a \in \mathbb{Q}_p^{\times}, b \in \mathbb{Q}_p.$$

The formula follows by changing variables as $x = ay + b$, and by using $dx = d\,(ay + b) = d(ay) = |a|_p\,dy$, since the Haar measure is invariant under translations.

Example 3.10. For any $r \in \mathbb{Z}$,

$$\int_{B_r} dx = \int_{p^{-r}\mathbb{Z}_p} dx = p^r \int_{\mathbb{Z}_p} dy = p^r.$$

Example 3.11. For any $r \in \mathbb{Z}$,

$$\int_{S_r} dx = \int_{B_r} dx - \int_{B_{r-1}} dx = p^r - p^{r-1} = p^r\left(1 - p^{-1}\right).$$

Example 3.12. Take $U = \mathbb{Z}_p \smallsetminus \{0\}$. Then

$$\int_U dx = \int_{\mathbb{Z}_p} dx = 1.$$

Notice that $U$ is not compact, since the sequence $\{p^n\}_{n\in\mathbb{N}} \subseteq U$ converges to $0 \notin U$. Now, by using

$$\mathbb{Z}_p \smallsetminus \{0\} = \bigsqcup_{j=0}^{\infty} \left\{x \in \mathbb{Z}_p;\ |x|_p = p^{-j}\right\},$$

and by changing variables as $x = p^j y$, $dx = p^{-j}dy$, we have

$$\int_{\mathbb{Z}_p \smallsetminus \{0\}} dx = \sum_{j=0}^{\infty} \int_{p^j \mathbb{Z}_p^\times} dx = \left( \sum_{j=0}^{\infty} p^{-j} \right) \int_{\mathbb{Z}_p^\times} dy = \frac{1 - p^{-1}}{1 - p^{-1}} = 1.$$

This calculation shows that $\mathbb{Z}_p \smallsetminus \{0\}$ has Haar measure 1. In particular, $\{0\}$ has Haar measure 0.

NOTATION 2. For $s \in \mathbb{C}$, $a > 0$, we set $a^s := e^{s \ln a}$.

EXAMPLE 3.13. Set

$$Z(s) := \int_{\mathbb{Z}_p \smallsetminus \{0\}} |x|_p^s \ dx, \ s \in \mathbb{C} \text{ for } \operatorname{Re}(s) > -1.$$

We prove that $Z(s)$ has a meromorphic continuation to the whole complex plane as a rational function of $p^{-s}$.

Indeed,

$$\begin{aligned} Z(s) &= \int_{\mathbb{Z}_p \smallsetminus \{0\}} |x|_p^s \ dx = \sum_{j=0}^{\infty} \int_{|x|_p = p^{-j}} |x|_p^s \ dx = \sum_{j=0}^{\infty} p^{-js} \int_{|x|_p = p^{-j}} dx \\ &= \left(1 - p^{-1}\right) \sum_{j=0}^{\infty} p^{-j(s+1)} \text{ (here we need the hypothesis } \operatorname{Re}(s) > -1) \\ &= \frac{\left(1 - p^{-1}\right)}{1 - p^{-1-s}}, \qquad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

We now note that the right hand-side is defined for any complex number $\operatorname{Re}(s) \neq -1$, therefore, it gives a meromorphic continuation of $Z(s)$ to the half-plane $\operatorname{Re}(s) < -1$. Thus, we have shown that integral $Z(s)$ has a meromorphic continuation to the whole $\mathbb{C}$ with simple poles satisfying $\operatorname{Re}(s) = -1$.

EXAMPLE 3.14. Take $p \neq 2$. We compute

$$Z(s, x^2 - 1) = \int_{\mathbb{Z}_p} \left| x^2 - 1 \right|_p^s \ dx, \quad \text{for} \quad \operatorname{Re}(s) > -1.$$

Let us take $\{0, 1, \ldots, p-1\} \subset \mathbb{Z} \subset \mathbb{Z}_p$ as a system of representatives of $\mathbb{F}_p \simeq \mathbb{Z}_p / p\mathbb{Z}_p$. Then $\mathbb{Z}_p = \bigsqcup_{j=0}^{p-1} (j + p\mathbb{Z}_p)$, and

$$\begin{aligned} Z(s, x^2 - 1) &= \sum_{j=0}^{p-1} \int_{j + p\mathbb{Z}_p} |(x-1)(x+1)|_p^s \, dx \\ &= p^{-1} \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} |(j - 1 + py)(j + 1 + py)|_p^s \, dy, \quad (x = j + py). \end{aligned}$$

Let us consider first the integrals in which $j \mp 1 + py \in \mathbb{Z}_p^\times$, i.e. the reduction mod $p$ of $j \mp 1$ is a nonzero element of $\mathbb{F}_p$. In this case,

$$\int_{\mathbb{Z}_p} |(j - 1 + py)(j + 1 + py)|_p^s \, dy = 1,$$

and since $p \neq 2$ there are exactly $p-2$ of those $j$s, then

$$\begin{aligned} Z(s, x^2-1) &= (p-2)\, p^{-1} + p^{-1} \int_{\mathbb{Z}_p} |py\,(2+py)|_p^s \, dy + p^{-1} \int_{\mathbb{Z}_p} |(-2+py)\, py|_p^s \, dy \\ &= (p-2)\, p^{-1} + 2p^{-1-s} \int_{\mathbb{Z}_p} |y|_p^s \, dy = (p-2)\, p^{-1} + 2p^{-1-s} \frac{1-p^{-1}}{1-p^{-1-s}}. \end{aligned}$$

Exercise 3.15. Take $q(x) = \prod_{i=1}^{r} (x-\alpha_i)^{e_i} \in \mathbb{Z}_p[x]$, $\alpha_i \in \mathbb{Z}_p$, $e_i \in \mathbb{N} \smallsetminus \{0\}$. Assume that $\alpha_i \not\equiv \alpha_j \mod p$. By using the methods presented in Examples 3.13 and 3.14 compute the integral

$$Z(s, q(x)) = \int_{\mathbb{Z}_p \smallsetminus \{0\}} |q(x)|_p^s \; dx.$$

**3.6. Improper integrals.** Our next task is the integration of functions that do not have compact support. A function $f : \mathbb{Q}_p^n \to \mathbb{C}$ is said to be *locally integrable*, $f \in L^1_{loc}$, if

$$\int_K f(x)\, d^n x < \infty \text{ for every compact subset } K \subset \mathbb{Q}_p^n.$$

Definition 3.16 (Improper Integral). A function $f \in L^1_{loc}$ is said to be integrable in $\mathbb{Q}_p^n$, if

$$\lim_{l \to +\infty} \int_{B_l^n} f(x)\, d^n x = \lim_{l \to +\infty} \sum_{j=-\infty}^{l} \int_{S_j^n} f(x)\, d^n x$$

exists. If the limit exists, it is denoted as $\int_{\mathbb{Q}_p^n} f(x)\, d^n x$, and we say that the *improper integral exists.* Note that in this case,

$$\int_{\mathbb{Q}_p^n} f(x)\, d^n x = \sum_{j=-\infty}^{+\infty} \int_{S_j^n} f(x)\, d^n x.$$

Example 3.17. The function $|x|_p$ is locally integrable but not integrable.

Example 3.18. Let $f : \mathbb{Q}_p \to \mathbb{C}$ be a *radial function* i.e. $f(x) = f(|x|_p)$. If $\sum_{j=-\infty}^{+\infty} f(p^j)\, p^j < \infty$. Then

$$\int_{\mathbb{Q}_p} f\left(|x|_p\right) dx = \sum_{j=-\infty}^{+\infty} \int_{|x|_p = p^j} f\left(|x|_p\right) dx = \left(1-p^{-1}\right) \sum_{j=-\infty}^{+\infty} f\left(p^j\right) p^j. \tag{3.8}$$

Exercise 3.19. Generalize formula (3.8) to the $n$-dimensional case.

Exercise 3.20. By using $\sum_{r=0}^{+\infty} r p^{-r} = \frac{p}{(p-1)^2}$, show that

$$\int_{\mathbb{Z}_p} \ln\left(|x|_p\right) \; dx = -\frac{\ln p}{p-1}.$$

**3.7. Further remarks on integrals of continuous functions with compact support.**

Example 3.21 (Continuation of Example 3.9). With the notation given in Example 3.9 and using formula (3.6) and Example 3.13, we have

$$\begin{aligned}\int_{B_L^n} f(|x_1|_p, \ldots, |x_n|_p) d^n x &= \lim_{m\to\infty} \sum_{|k|\leq m} c_{(k_1,\ldots,k_n)} \int_{B_L^n} \prod_{i=1}^n |x_i|_p^{k_i}\, d^n x \\ &= \lim_{m\to\infty} \sum_{|k|\leq m} c_{(k_1,\ldots,k_n)} \prod_{i=1}^n \int_{B_L} |x_i|_p^{k_i}\, dx_i \\ &= \lim_{m\to\infty} \sum_{|k|\leq m} c_{(k_1,\ldots,k_n)} \prod_{i=1}^n p^{L+Lk_i} \int_{\mathbb{Z}_p} |y_i|_p^{k_i}\, dx_i \\ &= \lim_{m\to\infty} \sum_{|k|\leq m} c_{(k_1,\ldots,k_n)} \prod_{i=1}^n p^{L+Lk_i} \frac{(1-p^{-1})}{1-p^{-1-k_i}} \\ &= \left(1-p^{-1}\right)^n \sum_{k\in\mathbb{N}^n} \frac{p^{L\left(n+\sum_{i=1}^n k_i\right)} c_{(k_1,\ldots,k_n)}}{\prod_{i=1}^n (1-p^{-1-k_i})}.\end{aligned}$$

Example 3.22. Let $f(x) = e^{-|x|_p}\Omega\left(|x|_p\right)$, where $\Omega\left(|x|_p\right)$ denotes the characteristic function of $\mathbb{Z}_p$. We compute first $\int f(x)dx$ by using Example 3.18:

$$\begin{aligned}\int_{\mathbb{Q}_p} e^{-|x|_p}\Omega\left(|x|_p\right) dx &= \int_{\mathbb{Z}_p} e^{-|x|_p} dx = \sum_{j=0}^{\infty} \int_{p^j\mathbb{Z}_p^\times} e^{-|x|_p} dx \\ &= \sum_{j=0}^{\infty} e^{-p^{-j}} \int_{p^j\mathbb{Z}_p^\times} dx = \sum_{j=0}^{+\infty} p^{-j}\left(1-p^{-1}\right) e^{-p^{-j}}.\end{aligned}$$

We now compute the integral by using Example 3.21:

$$\int_{\mathbb{Z}_p} e^{-|x|_p} dx = \lim_{M\to\infty} \sum_{k=0}^{M} \frac{(-1)^k}{k!} \int_{\mathbb{Z}_p} |x|_p^k\, dx = \left(1-p^{-1}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(1-p^{-1-k})}.$$

Consequently,

$$\sum_{j=0}^{\infty} p^{-j}\left(1-p^{-1}\right) e^{-p^{-j}} = \left(1-p^{-1}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(1-p^{-1-k})}.$$

We invite the reader to verify this identity directly .

Exercise 3.23. Consider the integral

$$I(s) = \int_{\mathbb{Z}_p\smallsetminus\{0\}} e^{-|x|_p^s} dx \text{ for } \operatorname{Re}(s) > 0.$$

Show that $I(s)$ has a meromorphic continuation to the whole complex plane with poles belonging to the set $\cup_{j=1}^{\infty}\left\{\operatorname{Re}(s) = \frac{-1}{j}\right\}$. Is the meromorphic continuation a rational function of $p^{-s}$? **Hint**. Recall that the uniform limit of holomorphic functions is holomorphic.

## 4. Change of variables formula

Let $U \subseteq \mathbb{Q}_p^n$ be *an open subset.* A function $h : U \to \mathbb{Q}_p$ is said to be *analytic in* $U$, if for every $b = (b_1, \ldots, b_n) \in U$ there exists an open subset $\widetilde{U} \subset U$, with $b \in \widetilde{U}$, and a convergent power series such that $h(x) = \sum_{i \in \mathbb{N}^n} a_i (x-b)^i$ for $x \in \widetilde{U}$, with $i = (i_1, \ldots, i_n)$ and $(x-b)^i = \prod_{j=1}^n (x_j - b_j)^{i_j}$. In this case, $\frac{\partial}{\partial x_l} h(x) = \sum_{i \in \mathbb{N}^n} a_i \frac{\partial}{\partial x_l} (x-b)^i$ is a convergent power series, see e.g. [**1**, Proposition 2.2.6]. Here $\frac{\partial}{\partial x_l}$ denotes the standard the derivative of a series.

Let $U$, $V$ open subsets in $\mathbb{Q}_p^n$. A mapping $H : U \to V$, $H = (H_1, \ldots, H_n)$ is called *analytic* if each $H_i$ is analytic. The mapping $H$ is said to be *bi-analytic* if $H$ and $H^{-1}$ are analytic.

Theorem 4.1. *Let $K_0, K_1 \subset \mathbb{Q}_p^n$ be open compact subsets, and consider a bi-analytic map $H = (H_1, \ldots, H_n) : K_1 \to K_0$ such that*

$$\det \left[ \frac{\partial H_i}{\partial y_j}(z) \right] \neq 0, \textit{ for any } z \in K_1.$$

*If $f$ is a continuous function on $K_0$, then*

$$\int_{K_0} f(x)\ d^n x = \int_{K_1} f(H(y)) \left| \det \left[ \frac{\partial H_i}{\partial y_j}(y) \right] \right|_p d^n y, \qquad (x = H(y)).$$

For the proof of this theorem the reader may consult [**10**, Prop. 7.4.1] or [**8**, Section 10.1.2].

Example 4.2. Set $U := \mathbb{Q}_p \smallsetminus \left\{ \frac{-d}{c} \right\}$ and $V := \mathbb{Q}_p \smallsetminus \left\{ \frac{a}{c} \right\}$, where $a$, $b$, $c$, $d \in \mathbb{Q}_p$, with $c \neq 0$. Consider the function

$$\begin{array}{ccc} U & \to & V \\ & & \\ x & \to & y = \frac{ax+b}{cx+d}, \end{array}$$

this is an analytic function in $U$, with inverse $x = \frac{dy-b}{-cy+a}$, which is analytic in $V$. Assume that $ad - bc \neq 0$, and take $\varphi : \mathbb{Q}_p \to \mathbb{C}$ a Bruhat-Schwartz function with support contained in $V$, then

$$\int_V \varphi(y)\, dy = \int_U \varphi\left( \frac{ax+b}{cx+d} \right) \frac{|ad-bc|_p}{|cx+d|_p^2} dx.$$

Example 4.3. Compute $\int_{B_r^n} d^n x$, where $B_r^n = \left\{ x \in \mathbb{Q}_p^n ; \|x\|_p \leq p^r \right\}$. We first recall that

$$B_r^n = p^{-r}\mathbb{Z}_p^n = \underbrace{\begin{array}{c} p^{-r}\mathbb{Z}_p \times \ldots \times p^{-r}\mathbb{Z}_p \\ n - copies \end{array}}.$$

By changing variables as $x_i = p^{-r} y_i$ for $i = 1, \ldots, n$, we have $d^n x = p^{nr} d^n y$, and

$$\int_{B_r^n} d^n x = p^{nr} \int_{\mathbb{Z}_p^n} d^n x = p^{nr}.$$

Exercise 4.4. Show that

$$\int_{S_r^n} d^n x = p^{nr} \left( 1 - p^{-n} \right).$$

EXERCISE 4.5. Set

$$Z(x,t) := (1-p^{-n})\|x\|_p^{-n}\sum_{k=0}^{\infty} q^{-kn}e^{-at(q^{-k}\|x\|_p^{-1})^{\alpha}} - \|x\|_p^{-n}e^{-at(p\|x\|_p^{-1})^{\alpha}},$$

for  and $x\in\mathbb{Q}_p^n\smallsetminus\{0\}$, $t>0$, $a>0$, $\alpha>0$. Show the formula

$$Z(x,t)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,\frac{1-p^{\alpha m}}{1-p^{-\alpha m-n}}\,(at)^m\|x\|_p^{-\alpha m-n}.$$

**Hint.** Use the technique presented in Examples 3.9, 3.21. First show that

$$(1-p^{-n})\sum_{k=0}^{\infty} q^{-kn}e^{-at(q^{-k}\|x\|_p^{-1})^{\alpha}} = \int_{\mathbb{Z}_p^n} e^{-at\|y\|_p^{\alpha}\|x\|_p^{-\alpha}}d^ny.$$

EXERCISE 4.6. Generalize Example 3.13 to $\mathbb{Q}_p^n$, i.e. to show that the integral

$$Z(s)=\int_{\mathbb{Z}_p^n\smallsetminus\{0\}}\|x\|_p^s\,d^nx,\text{ for }\operatorname{Re}(s)>-n,$$

admits an analytic continuation to the whole complex plane as a rational function of $p^{-s}$.

EXERCISE 4.7. Let $\alpha$ be a real number. Show that

$$I(\alpha)=\int_{\mathbb{Z}_p^n\smallsetminus\{0\}}\frac{1}{\|x\|_p^{\alpha}}d^nx<\infty\text{ if }\alpha<n.$$

EXERCISE 4.8. Let $\beta$ be a real number. Show that

$$J(\beta)=\int_{\mathbb{Q}_p^n\smallsetminus\mathbb{Z}_p^n}\frac{1}{\|x\|_p^{\beta}}d^nx<\infty\text{ if }\beta>n.$$

EXAMPLE 4.9. Take $N\geq 4$ and complex variables $s_{1j}$ and $s_{(N-1)j}$ for $2\leq j\leq N-2$ and $s_{ij}$ for $2\leq i<j\leq N-2$. Put $\boldsymbol{s}:=(s_{ij})\in\mathbb{C}^d$, where $d=\frac{N(N-3)}{2}$ denotes the total number of indices $ij$. The Koba-Nielsen local zeta functions is defined as follows:

$$Z^{(N)}(\boldsymbol{s})=\int_{\mathbb{Q}_p^{N-3}}\prod_{i=2}^{N-2}|x_j|_p^{s_{1j}}\,|1-x_j|_p^{s_{(N-1)j}}\prod_{2\leq i<j\leq N-2}|x_i-x_j|_p^{s_{ij}}\prod_{i=2}^{N-2}dx_i,$$

where $\prod_{i=2}^{N-2}dx_i$ is the normalized Haar measure on $\mathbb{Q}_p^{N-3}$. If $N=4$, we have

$$Z^{(4)}(\boldsymbol{s})=\int_{\mathbb{Q}_p}|x|_p^{s_{12}}\,|1-x|_p^{s_{32}}\;dx. \tag{4.1}$$

Notice that in integral (4.1) does not contain a test function, and consequently its convergence is not direct. In order to regularize it, we proceed as follows:

$$\begin{aligned}Z^{(4)}(s_{12},s_{32})&=\int_{\mathbb{Z}_p}|x|_p^{s_{12}}\,|1-x|_p^{s_{32}}\;dx+\int_{\mathbb{Q}_p\smallsetminus\mathbb{Z}_p}|x|_p^{s_{12}}\,|1-x|_p^{s_{32}}\;dx\\&=:Z_0^{(4)}(s_{12},s_{32})+Z_1^{(4)}(s_{12},s_{32}).\end{aligned}$$

To study integral $Z_0^{(4)}(s_{12},s_{32})$, we use that $\mathbb{Z}_p=\bigsqcup_{j=0}^{p-1}j+p\mathbb{Z}_p$, to get

$$Z_0^{(4)}(s_{12},s_{32})=\sum_{j=0}^{p-1}\int_{j+p\mathbb{Z}_p}|x|_p^{s_{12}}\,|1-x|_p^{s_{32}}\;dx=:\sum_{j=0}^{p-1}Z_{0,j}^{(4)}(s_{12},s_{32}).$$

Now, for $j \neq 0,1$,

$$Z_{0,j}^{(4)}\left(s_{12}, s_{32}\right)=p^{-1}.$$

In the case $j=0$,

$$Z_{0,0}^{(4)}\left(s_{12}, s_{32}\right)=\int_{p\mathbb{Z}_p}|x|_p^{s_{12}}\; dx=p^{-1-s_{12}}\left(\frac{1-p^{-1}}{1-p^{-1-s_{12}}}\right).$$

In the case $j=1$,

$$\begin{aligned} Z_{0,1}^{(4)}\left(s_{12}, s_{32}\right) &=\int_{1+p\mathbb{Z}_p}|1-x|_p^{s_{32}}\, dx=p^{-1-s_{32}}\int_{\mathbb{Z}_p}|y|_p^{s_{32}}\, dy \\ &=p^{-1-s_{32}}\left(\frac{1-p^{-1}}{1-p^{-1-s_{32}}}\right). \end{aligned}$$

Consequently,

$$Z_{0}^{(4)}\left(s_{12}, s_{32}\right)=(p-2)\, p^{-1}+p^{-1-s_{12}}\left(\frac{1-p^{-1}}{1-p^{-1-s_{12}}}\right)+p^{-1-s_{32}}\left(\frac{1-p^{-1}}{1-p^{-1-s_{32}}}\right).$$

Notice that $Z_{0}^{(4)}\left(s_{12}, s_{32}\right)$ is holomorphic, and consequently the underlying integrals converge, in

$$\operatorname{Re}\left(s_{12}\right)>-1 \text{ and } \operatorname{Re}(s_{32})>-1. \tag{4.2}$$

To study $Z_{1}^{(4)}\left(s_{12}, s_{32}\right)$, we first notice that

$$Z_{1}^{(4)}\left(s_{12}, s_{32}\right)=\int_{\mathbb{Q}_p\smallsetminus\mathbb{Z}_p}|x|_p^{s_{12}}\,|1-x|_p^{s_{32}}\, dx=\int_{\mathbb{Q}_p\smallsetminus\mathbb{Z}_p}|x|_p^{s_{12}+s_{32}}\, dx.$$

If $\operatorname{Re}(s_{12})+\operatorname{Re}\left(s_{32}\right)+1<0$, by Exercise 4.8, integral $Z_{1}^{(4)}\left(s_{12}, s_{32}\right)$ exists and it can be computed as

$$\begin{aligned} Z_{1}^{(4)}\left(s_{12}, s_{32}\right) &=\sum_{j=1}^{\infty}\int_{|x|_p=p^j}|x|_p^{s_{12}+s_{32}}\, dx=\left(1-p^{-1}\right)\sum_{j=1}^{\infty}p^{j(s_{12}+s_{32}+1)} \\ &=\left(1-p^{-1}\right)\frac{p^{s_{12}+s_{32}+1}}{1-p^{s_{12}+s_{32}+1}}. \end{aligned}$$

Thus $Z_{1}^{(4)}\left(s_{12}, s_{32}\right)$ is holomorphic in

$$\operatorname{Re}(s_{12})+\operatorname{Re}\left(s_{32}\right)+1<0. \tag{4.3}$$

Finally, it is not difficult to see that conditions (4.2)–(4.3) define an open set in $\mathbb{C}$.

Exercise 4.10. Consider the integral

$$J(s)=\int_{\mathbb{Q}_p^n}e^{-\|x\|_p^2}\,\|x\|_p^s\, d^n x \text{ for } \operatorname{Re}(s)>0.$$

Show that $J(s)$ has a meromorphic continuation to the whole complex plane. Is the meromorphic continuation a rational function of $p^{-s}$? **Hint**. Express $J(s)=J_1(s)+J_2(s)$, where

$$J_1(s):=\int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}e^{-\|x\|_p^2}\,\|x\|_p^s\, d^n x,\quad J_2(s):=\int_{\mathbb{Z}_p^n}e^{-\|x\|_p^2}\,\|x\|_p^s\, d^n x.$$

Show that $J_1(s)$ is holomorphic function in $\mathbb{C}$, see Lemma below. For $\mathrm{Re}(s) > 0$ justify that

$$J_2(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{\mathbb{Z}_p^n} \|x\|_p^{s+2j} \, d^n x.$$

LEMMA 4.11 ([**10**, Lemma 5.3.1]). *Let $U$ be an open subset of $\mathbb{C}$ and let $f : \mathbb{Q}_p^n \times U \to \mathbb{C}$ be a continuous function on $\mathbb{Q}_p^n \times U$ having the following properties: (i) if $K$ is a compact subset of $U$, there exists an integrable function $\varphi_K \geq 0$ on $\mathbb{Q}_p^n$ satisfying $|f(x,s)| \leq \varphi_K(x)$ for all $(x,s) \in \mathbb{Q}_p^n \times K$; $f(x,\cdot)$ is a holomorphic on $U$ for every $x \in \mathbb{Q}_p^n$. Then $\int_{\mathbb{Q}_p^n} f(x,s)\, d^n x$ defines a holomorphic function on $U$.*

## 5. Additive characters

Given a nonzero $p$-adic number

$x = x_{-m}p^{-m} + x_{-m+1}p^{-m+1} + \ldots + x_{-1}p^{-1} + x_0 + xp + \ldots$ with $x_{-m} \neq 0$ and $m > 0$,

we define its *fractional part as*

$$\{x\}_p = x_{-m}p^{-m} + x_{-m+1}p^{-m+1} + \ldots + x_{-1}p^{-1} \in \mathbb{Q}.$$

If $x \in \mathbb{Z}_p$, we set $\{x\}_p := 0$. Now the function

$$\chi_p(x) := \exp 2\pi i \{x\}_p$$

is called the *standard additive character* of $\mathbb{Q}_p$ (more precisely of $(\mathbb{Q}_p, +)$). Notice that

$$\chi_p : (\mathbb{Q}_p, +) \to (S, \cdot)$$

is a continuous homomorphism from $(\mathbb{Q}_p, +)$ into the unit complex circle considered as a multiplicative group, i.e. $\chi_p$ satisfies the following:

(i) $|\chi_p(x)| = 1$ for $x \in \mathbb{Q}_p$;
(ii) $\chi_p(x+y) = \chi_p(x)\chi_p(y)$ for $x, y \in \mathbb{Q}_p$;
(iii) $\overline{\chi_p(x)} = \frac{1}{\chi_p(x)} = \chi_p(-x)$ for $x \in \mathbb{Q}_p$, where the bar means complex conjugate;
(iv) $\chi_p(x) \not\equiv 1$ for $x \in \mathbb{Q}_p \smallsetminus \mathbb{Z}_p$.

EXAMPLE 5.1. Let $r$ be an integer. To show that

$$\int_{B_r} \chi_p(\xi x)\, dx = \begin{cases} p^r & \text{if} \quad |\xi|_p \leq p^{-r} \\ \\ 0 & \text{if} \quad |\xi|_p \geq p^{-r+1}. \end{cases}$$

If $|\xi|_p \leq p^{-r}$, then $|\xi x|_p \leq 1$ which means that $\xi x \in \mathbb{Z}_p$ and thus $\chi_p(\xi x) \equiv 1$,

$$\int_{B_r} \chi_p(\xi x)\, dx = \int_{B_r} dx = p^r.$$

If $|\xi|_p \geq p^{-r+1}$, there exists $x_0 \in S_r$, i.e. $|x_0|_p = p^r$, such that $|x_0\xi|_p \geq p$, i.e. $x_0\xi \in \mathbb{Q}_p \smallsetminus \mathbb{Z}_p$ and thus we may assume that $\chi_p(x_0\xi) \neq 1$. We now change variables as $x = y + x_0$, notice that $y$ runs through $p^{-r}\mathbb{Z}_p$, to get

$$\int_{|x|_p \leq p^r} \chi_p(\xi x)\, dx = \chi_p(\xi x_0) \int_{|y|_p \leq p^r} \chi_p(\xi y)\, dy,$$

which implies the announced formula.

EXERCISE 5.2. Let $r$ be an integer. For $x=(x_1,\dots,x_n)$, $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{Q}_p^n$, we set $x\cdot\xi:=\sum_i x_i\xi_i$. To show that

$$\int_{B_r^n}\chi_p(x\cdot\xi)\,d^nx=\begin{cases}p^{rn} & \text{if} \quad \|\xi\|_p\le p^{-r}\\ 0 & \text{if} \quad \|\xi\|_p\ge p^{-r+1}.\end{cases}$$

Hint: remember that $x=p^{ord(x)}\widetilde{x}$, with $\|\widetilde{x}\|_p=1$ and $\xi=p^{ord(\xi)}\widetilde{\xi}$, with $\left\|\widetilde{\xi}\right\|_p=1$. Thus $x\cdot\xi=p^{ord(x)+ord(\xi)}\widetilde{x}\cdot\widetilde{\xi}$.

EXERCISE 5.3. Let $r$ be an integer. To show that

$$\int_{S_r^n}\chi_p(x\cdot\xi)\,dx=\begin{cases}p^{nr}\left(1-p^{-n}\right) & \text{if} \quad \|\xi\|_p\le p^{-r}\\ -p^{rn-n} & \text{if} \quad \|\xi\|_p=p^{-r+1}\\ 0 & \text{if} \quad \|\xi\|_p\ge p^{-r+2}.\end{cases}\tag{5.1}$$

Hint: use Example 5.1 and

$$\int_{S_r^n}\chi_p(\xi x)\,d^nx=\int_{B_r^n}\chi_p(\xi x)\,d^nx-\int_{B_{r-1}^n}\chi_p(\xi x)\,d^nx.$$

EXERCISE 5.4. Extend the formula given in Example 3.18 to $n$-dimensional case, i.e. for radial functions $f:\mathbb{Q}_p^n\to\mathbb{C}$ i.e. $f(x)=f(\|x\|_p)$.

LEMMA 5.5. *(i) Let $f:\mathbb{R}\to\mathbb{C}$ be a function such that $\sum_{r=0}^{\infty}|f(p^{-r})|\,p^{-nr}<\infty$. Then*

$$\int_{\mathbb{Q}_p^n}f\left(\|x\|_p\right)\chi_p(\xi\cdot x)\,d^nx=\frac{(1-p^{-n})}{\|\xi\|_p^n}\sum_{r=0}^{\infty}f\left(\frac{p^{-r}}{\|\xi\|_p}\right)p^{-nr}-\frac{1}{\|\xi\|_p^n}f\left(\frac{p}{\|\xi\|_p}\right)\tag{5.2}$$

*for $\xi\neq0$, in the sense of improper integrals. (ii) If $f\left(\|x\|_p\right)$ is integrable, then $f\left(\|x\|_p\right)\chi_p(\xi\cdot x)$ is integrable with respecto $x$ for any fixed $\xi$ and the formula* (5.2) *holds true.*

PROOF. (ii) Suppose that $\|\xi\|_p=p^N$, by using (5.1),

$$\begin{aligned}\int_{\mathbb{Q}_p^n}f\left(\|x\|_p\right)\chi_p(\xi\cdot x)\,d^nx&=\sum_{r=-\infty}^{\infty}f(p^r)\int_{S_r^n}\chi_p(\xi\cdot x)\,d^nx\\&=\sum_{r=-\infty}^{-N}f(p^r)p^{nr}\left(1-p^{-n}\right)-p^{-nN}f(p^{-N+1})\\&=\frac{(1-p^{-n})}{\|\xi\|_p^n}\sum_{r=0}^{\infty}f\left(\frac{p^{-r}}{\|\xi\|_p}\right)p^{-nr}-\frac{1}{\|\xi\|_p^n}f\left(\frac{p}{\|\xi\|_p}\right).\end{aligned}$$

(ii) Notice that

$$\left\|f\left(\|-\|_p\right)\chi_p(\xi\cdot-)\right\|_1=\|f\|_1=\int_{\mathbb{Q}_p^n}\left|f\left(\|x\|_p\right)\right|d^nx<\infty$$

implies that

$$\int_{\mathbb{Q}_p^n \smallsetminus p\mathbb{Z}_p^n} \left| f\left( \|x\|_p \right) \right| d^n x = \sum_{r=0}^{\infty} \int_{S_r^n} \left| f\left( \|x\|_p \right) \right| d^n x = \sum_{r=0}^{\infty} \left| f\left( p^{-r} \right) \right| p^{-nr} \leq \|f\|_1 .$$

□

By using this formula with $f \equiv 1$, we get that

$$\int_{\mathbb{Q}_p^n} \chi_p\left( \xi \cdot x \right) dx = \begin{cases} \infty & \text{if} \quad \xi = 0 \\ \\ 0 & \text{if} \quad \xi \neq 0 \end{cases} =: \delta\left( \xi \right), \text{ the Dirac distribution.}$$

The Dirac distribution (or Dirac delta function) is the Fourier transform of the constant function 1.

Exercise 5.6. Set

$$Z(x,t) = \int_{\mathbb{Q}_p^n} \chi_p(x \cdot \xi) e^{-at\|\xi\|_p^\alpha} \, d^n \xi, \text{ for } t > 0 \text{ and } x \in \mathbb{Q}_p^n.$$

Show the formula

$$Z(x,t) = (1 - p^{-n}) \|x\|_p^{-n} \sum_{k=0}^{\infty} q^{-kn} e^{-at(q^{-k}\|x\|_p^{-1})^\alpha} - \|x\|_p^{-n} e^{-at(p\|x\|_p^{-1})^\alpha},$$

for $t > 0$ and $x \in \mathbb{Q}_p^n \smallsetminus \{0\}$.

## 6. Fourier Analysis on $\mathbb{Q}_p^n$

**6.1. Some function spaces.** For $1 \leq \rho < \infty$, we denote by $L^\rho := L^\rho\left( \mathbb{Q}_p^n, d^n x \right)$, the $\mathbb{C}$-vector space of all the complex-valued and Borel measurable functions $f$ satisfying

$$\|f\|_\rho := \left\{ \int_{\mathbb{Q}_p^n} |f(x)|^\rho \, d^n x \right\}^{\frac{1}{\rho}} < \infty.$$

For $\rho = \infty$, $f \in L^\infty := L^\infty\left( \mathbb{Q}_p^n, d^n x \right)$, if

$$\|f\|_\infty := \text{ess sup}_{x \in \mathbb{Q}_p^n} \; |f(x)| < \infty. \tag{6.1}$$

The condition appearing on the right-hand side in (6.1) means that function $f$ is bounded almost everywhere, i.e. this condition may be false in a set of measure zero. $L^\rho$ is a Banach space if we identify functions $f$ and $g$ satisfying $f(x) = g(x)$ almost everywhere. This identification implies that $\|\cdot\|_\rho$ is norm, and that $L^\rho$ is a complete metric space for the distance induced by $\|\cdot\|_\rho$.

We denote by $C_0 := C_0(\mathbb{Q}_p^n, \mathbb{C})$ the $\mathbb{C}$-vector space of continuous functions on $\mathbb{Q}_p^n$ that vanish at infinity endowed with the $L^\infty$-norm. The condition 'vanishing at infinity' means that for any $\epsilon > 0$ there exists a compact subset $K \subset \mathbb{Q}_p^n$ such that

$$|f(x)| < \epsilon \text{ for } x \in \mathbb{Q}_p^n \smallsetminus K.$$

Remark 6.1. Lebesgue's dominated convergence theorem. Let $f_m : \mathbb{Q}_p^n \to \mathbb{C}$, $m \in \mathbb{N}$, be a sequence of complex-valued Borel measurable functions. Suppose that the sequence converges pointwise to a function $f$ and that there exists an integrable function $g$ such that $|f_m(x)| \leq g(x)$ for any $x \in \mathbb{Q}_p^n$ and all $m$. Then $f$ is integrable and

$$\lim_{m\to\infty} \int_{\mathbb{Q}_p^n} f_m(x)\, d^n x = \int_{\mathbb{Q}_p^n} \lim_{m\to\infty} f_m(x)\, d^n x = \int_{\mathbb{Q}_p^n} f(x) d^n x.$$

EXERCISE 6.2. Let $I = (a, b) \subset \mathbb{R}$ and let $f(x, t)$ be a continuous function on $\mathbb{Q}_p^n \times I$ satisfying the following conditions: (i) $f(\cdot, t) \in L^1$ for any $t \in I$; (ii) $\frac{\partial f(x,t)}{\partial t}$ is continuous in $t$ for any $x \in \mathbb{Q}_p^n$; (iii) $\left|\frac{\partial f(x,t)}{\partial t}\right| \leq h(x) \in L^1$ for every $t \in I$. Then

$$\frac{\partial}{\partial t} \int_{\mathbb{Q}_p^n} f(x, t) d^n x = \int_{\mathbb{Q}_p^n} \frac{\partial}{\partial t} f(x, t) d^n x.$$

**Hint**. For every $t$, $t' \in I$, with $t' > t$, $f(x, t') = f(x, t) + (t' - t)\frac{\partial}{\partial t} f(x, t)$.

**6.2. The Fourier transform.** Given both $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, we define

$$x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n.$$

If $f \in L^1$ its Fourier transform is the function $\widehat{f}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p^n} f(x)\, \chi_p(x \cdot \xi)\, d^n x.$$

We also use the notation $\mathcal{F}_{x\to\xi}(f)$, $\mathcal{F}(f)$ to denote the Fourier transform of $f$.

LEMMA 6.3. *(i) The mapping $f \to \widehat{f}$ is a bounded linear mapping from $L^1$ to $L^\infty$ satisfying $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$.*

*(ii) If $f \in L^1$, then $\widehat{f}$ is uniformly continuous.*

PROOF. (i)

$$\left|\widehat{f}(\xi)\right| = \left|\int_{\mathbb{Q}_p^n} f(x)\, \chi_p(x \cdot \xi)\, d^n x\right| \leq \int_{\mathbb{Q}_p^n} |f(x)|\, d^n x = \|f\|_1.$$

(ii) Notice that

$$\widehat{f}(\xi + h) - \widehat{f}(\xi) = \int_{\mathbb{Q}_p^n} f(x)\, \chi_p(x \cdot \xi) \{\chi_p(x \cdot h) - 1\}\, d^n x,$$

and since $|f(x)\, \chi_p(x \cdot \xi) \{\chi_p(x \cdot h) - 1\}| \leq 2|f(x)| \in L^1$, by using the dominated convergence theorem,

$$\lim_{h\to 0} \left|\widehat{f}(\xi + h) - \widehat{f}(\xi)\right| \leq \int_{\mathbb{Q}_p^n} |f(x)| \lim_{h\to 0} |\chi_p(x \cdot h) - 1|\, d^n x = 0,$$

i.e. for any $\epsilon > 0$, there is $\delta > 0$, such that for any $\xi'$, $\xi \in \mathbb{Q}_p^n$, with $\|\xi' - \xi\|_p = \|h\|_p < \delta$, it holds that $\left|\widehat{f}(\xi') - \widehat{f}(\xi)\right| < \epsilon$. $\square$

EXERCISE 6.4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function such that $f(\|\xi\|_p)$ satisfies the following properties: (i) $f(\|\xi\|_p) = \lambda$ for $\|\xi\|_p \leq p^r$, where $\lambda$, $r \in \mathbb{Z}$ are fixed real numbers; (ii) $f(\|\xi\|_p)$ is a non-decreasing function in $\|\xi\|_p$; (iii) there exist positive constants $C_0$, $C_1$, $\gamma_0$, $\gamma_1$ and $l \in \mathbb{N}$ such that

$$C_0 \|\xi\|_p^{\gamma_0} \leq f(\|\xi\|_p) \leq C_1 \|\xi\|_p^{\gamma_1} \text{ for } \|\xi\|_p \geq p^l > p^r.$$

We now set

$$Z(x,t;f,\lambda,r):=Z(x,t)=\int_{\mathbb{Q}_p^n}\chi_p\left(-x\cdot\xi\right)e^{-t\left(f(\|\xi\|_p)-\lambda\right)}d^n\xi$$

$$=\int_{p^{-r}\mathbb{Z}_p^n}\chi_p\left(-x\cdot\xi\right)d^n\xi+e^{t\lambda}\int_{\mathbb{Q}_p^n\smallsetminus p^{-r}\mathbb{Z}_p^n}\chi_p\left(-x\cdot\xi\right)e^{-tf(\|\xi\|_p)}d^n\xi,$$

for $t>0$, and $x\in\mathbb{Q}_p^n$. Show the following assertions: first, for each $t>0$ fixed, $Z(x,t)\in L^1\left(\mathbb{Q}_p^n\right)\cap C\left(\mathbb{Q}_p^n\right)$; $Z(x,t)\geq 0$ for any $x$, $t\geq 0$. **Hint.** Use that $\mathbb{Q}_p^n=\bigsqcup_{k=0}^{\infty}S_k^n$. Second, show the formulae:

$$Z(x,t)=\Omega\left(p^r\|x\|_p\right)\left\{p^{rn}-\frac{e^{-tf(p\|x\|_p^{-1})}}{\|x\|_p^n}+\sum_{j=-r+1}^{ord(x)}p^{jn}e^{-tf(p^j)}\right\},$$

for $x\neq 0$ and $t>0$, and

$$Z(0,t)=p^{rn}+\sum_{j=-r+1}^{\infty}p^{jn}e^{-tf(p^j)},\text{ for }t>0.$$

Remark 6.5. The translation operator $\mathcal{T}_h$, $h\in\mathbb{Q}_p^n$, is defined by $\left(\mathcal{T}_hf\right)(x)=f\left(x-h\right)$. If $f\in L^1$, then $\widehat{\left(\mathcal{T}_hf\right)}\left(\xi\right)=\chi_p\left(\xi\cdot h\right)\widehat{f}\left(\xi\right)$, and $\mathcal{F}_{x\to\xi}(\chi_p\left(x\cdot h\right)f\left(x\right))=\left(\mathcal{T}_{-h}\widehat{f}\right)\left(\xi\right)=\widehat{f}\left(\xi+h\right)$.

We denote by $\Delta_k\left(x\right):=\Omega\left(p^{-k}\|x\|_p\right)$ the characteristic function of the ball $B_k^n=p^{-k}\mathbb{Z}_p^n$ for $k\in\mathbb{Z}$. A locally constant function with compact support (a test function) $\varphi:\mathbb{Q}_p^n\to\mathbb{C}$ is a linear combination of characteristic functions of balls, then

$$\varphi\left(x\right)=\sum_{i=1}^{m}c_i\mathcal{T}_{h_i}\Delta_{k_i}\left(x\right).$$

Notice that $\mathcal{T}_{h_i}\Delta_{k_i}\left(x\right)$ is the characteristic function of the ball $h_i+p^{-k_i}\mathbb{Z}_p^n$. We denote by $\mathcal{D}(\mathbb{Q}_p^n)$ the $\mathbb{C}$–vector space of test functions (the Bruhat-Schwartz).

We set $\delta_k\left(x\right)=p^{kn}\Omega\left(p^k\|x\|_p\right)$ for $k\in\mathbb{Z}$. Notice that $\delta_k$ satisfies

$$\int_{\mathbb{Q}_p^n}\delta_k\left(x\right)d^nx=1\text{ for any }k\in\mathbb{Z}.$$

Exercise 6.6. To show that $\widehat{\Delta}_k\left(\xi\right)=\delta_k\left(\xi\right)$. Hence if $\varphi\left(x\right)=\sum_{i=1}^{m}c_i\mathcal{T}_{h_i}\Delta_{k_i}\left(x\right)$ is a test function, then $\widehat{\varphi}\left(\xi\right)=\sum_{i=1}^{m}c_i\chi_p\left(\xi\cdot h_i\right)\delta_k\left(\xi\right)$. Consequently, $\widehat{\varphi}\left(\xi\right)$ is also a test function.

Remark 6.7. Notice that $\lim_{k\to\infty}\Delta_k\left(x\right)=1$, and that

$$\lim_{k\to\infty}\delta_k\left(x\right)=\begin{cases}\infty & \text{if}\quad x=0\\ \\ 0 & \text{if}\quad x\neq 0.\end{cases}$$

Exercise 6.8. Set $h_t\left(x\right):=\int_{\mathbb{Q}_p}\chi_p\left(-\xi x\right)e^{-t|\xi|_p^{\alpha}}d\xi$, for $t>0$, $\alpha>0$. Show that $h_t\left(\xi\right)$ is a continuous function in $\xi$ for $t>0$ fixed.

Proposition 6.9 ([**23**, Chap. I, Proposition 1.3]). *$\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $C_0$ as well as in $L^{\rho}$, $1\leq\rho<\infty$.*

Proposition 6.10 (Riemann-Lebesgue Theorem). *If $f \in L^1$, then $\widehat{f}(\xi) \to 0$ as $\|\xi\|_p \to \infty$.*

Proof. Given $\epsilon > 0$ and $f \in L^1(\mathbb{Q}_p^n)$, there exists $g_\epsilon \in \mathcal{D}(\mathbb{Q}_p^n)$ such that $\|f - g_\epsilon\|_1 < \epsilon$, cf. Proposition 6.9 . We now use that $\widehat{g_\epsilon}$ has compact support. For $\xi \notin$supp $\widehat{g_\epsilon}$ we have

$$\left|\widehat{f}(\xi)\right| = \left|\widehat{f}(\xi) - \widehat{g}_\epsilon(\xi)\right| \leq \left\|\widehat{(f - g_\epsilon)}\right\|_\infty \leq \|f - g_\epsilon\|_1 < \epsilon.$$

□

Definition 6.11. Given $f$, $g : \mathbb{Q}_p^n \to \mathbb{C}$ their convolution is the function

$$\begin{aligned} h(x) = f(x) * g(x) &= \int_{\mathbb{Q}_p^n} f(x-z)g(z)d^n z \\ &= \int_{\mathbb{Q}_p^n} f(z)g(x-z)d^n z, \end{aligned}$$

when the defining integral exists.

Remark 6.12. Young's inequality. Assume that $f \in L^\rho$, $g \in L^\sigma$ and $\frac{1}{\rho} + \frac{1}{\sigma} = \frac{1}{\gamma} + 1$ with $1 \leq \rho, \sigma, \gamma \leq \infty$. Then $\|f * g\|_\gamma \leq \|f\|_\rho \|g\|_\sigma$.

The following proposition is left as an exercise to the reader.

Proposition 6.13. *If $f \in L^\rho$, $1 \leq \rho \leq \infty$, and $g \in L^1$, then $f * g \in L^\rho$ and $\|f * g\|_\rho \leq \|f\|_\rho \|g\|_1$.*

Remark 6.14. Fubini's theorem. Let $f : \mathbb{Q}_p^{n+m} \to \mathbb{C}$ be a function such that the repeated integral

$$\int_{\mathbb{Q}_p^n} \left( \int_{\mathbb{Q}_p^m} f(x,y) d^m y \right) d^n x$$

exists, then $f \in L^1\left(\mathbb{Q}_p^{n+m}, d^{n+m}x\right)$, and the following formulae hold:

$$\begin{aligned} \int_{\mathbb{Q}_p^n} \left( \int_{\mathbb{Q}_p^m} f(x,y) d^m y \right) d^n x &= \int_{\mathbb{Q}_p^{n+m}} f(x,y) d^{n+m}(x,y) \\ &= \int_{\mathbb{Q}_p^m} \left( \int_{\mathbb{Q}_p^n} f(x,y) d^n x \right) d^m y. \end{aligned}$$

Lemma 6.15. *If $f$, $g \in L^1$, then $\widehat{f * g} = \widehat{f}\, \widehat{g}$.*

Proof. By Proposition 6.13, $f * g \in L^1$. The formula for the Fourier transform follows from Fubini's theorem. We invite the reader to verify this calculation. □

Lemma 6.16. *If $f$, $g \in L^1$, then*

$$\int_{\mathbb{Q}_p^n} \widehat{f}(y) g(y)\, d^n y = \int_{\mathbb{Q}_p^n} f(y) \widehat{g}(y)\, d^n y.$$

PROOF. By using that $\left\Vert \widehat{f}\right\Vert_{\infty} \leq \Vert f \Vert_1 < \infty$, Fubini's theorem and the definition of Fourier transform, we have

$$\int_{\mathbb{Q}_p^n} \widehat{f}(y) g(y)\, d^n y = \int_{\mathbb{Q}_p^n} \left\{ \int_{\mathbb{Q}_p^n} \chi_p(x \cdot y) f(x)\, d^n x \right\} g(y)\, d^n y$$
$$= \int_{\mathbb{Q}_p^n} \left\{ \int_{\mathbb{Q}_p^n} \chi_p(x \cdot y) g(y)\, d^n y \right\} f(x)\, d^n x = \int_{\mathbb{Q}_p^n} f(x) \widehat{g}(x)\, d^n x.$$

□

**6.3. The Fourier transform on the space of test functions.** Let $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ be a locally constant function, this means that for each $x \in \mathbb{Q}_p^n$, there exists an integer $l = l(x)$ such that

$$\varphi(x + x') = \varphi(x) \text{ for any } x' \in B_l^n. \tag{6.2}$$

Since $B_l^n = \left\{ x \in \mathbb{Q}_p^n ; \Vert x \Vert_p \leq p^l \right\} = p^{-l}\mathbb{Z}_p^n$, condition (6.2) is equivalent to

$$\varphi \mid_{x + p^{-l}\mathbb{Z}_p^n} \equiv \varphi(x).$$

If $\varphi$ is a test function, then supp $\varphi$ is open compact, and consequently there exist a finite number of integers $l_i$ and a finite number of points $z_i$ in $\mathbb{Q}_p^n$ such that

$$\text{supp } \varphi = \bigsqcup_{i=1}^{r} \left( z_i + p^{-l_i}\mathbb{Z}_p^n \right).$$

We set

$$k := \max_{1 \leq i \leq r} -l_i.$$

Then $z_i + p^{-l_i}\mathbb{Z}_p^n \supset z_i + p^k \mathbb{Z}_p^n$ (i.e. $B_{-k}^n(z_i) \subset B_{l_i}^n(z_i)$) and

$$\varphi \mid_{x + p^k \mathbb{Z}_p^n} \equiv \varphi(x) \text{ for any } x \in \text{supp } \varphi.$$

This means that $\varphi$ is constant on the cosets of $p^k\mathbb{Z}_p^n$ (i.e. on the cosets of $\mathbb{Q}_p^n / p^k \mathbb{Z}_p^n$). The integer $k$ is called the *index of local constancy* of $\varphi$. We now use the fact that supp $\varphi$ is compact, which means that it is closed and bounded, then there exists an integer $m$ such that

$$\text{supp } \varphi \subset p^m \mathbb{Z}_p^n.$$

Naturally, for any $x \in$supp $\varphi$, $x + p^k\mathbb{Z}_p^n \subset p^m\mathbb{Z}_p^n$, which implies that $k \geq m$.

In conclusion, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ if and only if there exist integers $k$, $m$, with $k \geq m$, such that $\varphi$ is constant on the cosets of $p^k\mathbb{Z}_p^n$ (i.e. on the cosets of $p^m\mathbb{Z}_p^n / p^k\mathbb{Z}_p^n$) and is supported on $p^m\mathbb{Z}_p^n$. These functions form $\mathbb{C}$-vector space denoted as $\mathcal{D}_k^m(\mathbb{Q}_p^n) := \mathcal{D}_k^m$. We fix a set of representatives $I$s of $p^m\mathbb{Z}_p^n / p^k\mathbb{Z}_p^n =: G_{m,k}$, then the characteristic functions of the balls $I + p^k\mathbb{Z}_p^n$, $I \in G_{m,k}$ span $\mathcal{D}_k^m$, i.e.

$$\left\{ \Omega \left( p^k \Vert x - I \Vert_p \right) \right\}_{I \in G_{m,k}}$$

is a basis for $\mathcal{D}_k^m$. Notice that the dimension of $\mathcal{D}_k^m$ is $\# G_{m,k} = p^{(m-k)n}$.

LEMMA 6.17. $\mathcal{F}(\mathcal{D}_k^m) \subset \mathcal{D}_{-m}^{-k}$.

PROOF. Take $\varphi \in \mathcal{D}_k^m$, since $\varphi(x) = \sum_{I \in G_{m,k}} c_I \Omega\left(p^k \|x - I\|_p\right)$, and $\mathcal{F}$ is a linear operator, we may assume that $\varphi(x) = \Omega\left(p^k \|x - I\|_p\right)$. Then

$$\begin{aligned}\mathcal{F}_{x \to \xi}\left(\Omega\left(p^k \|x - I\|_p\right)\right) &= \int_{I + p^k \mathbb{Z}_p^n} \chi_p(\xi \cdot x)\, d^n x = p^{-nk} \chi_p(\xi \cdot I) \int_{\mathbb{Z}_p^n} \chi_p\left(p^k \xi \cdot y\right) d^n y \\ &= p^{-nk} \chi_p(\xi \cdot I)\, \Omega\left(p^{-k} \|\xi\|_p\right).\end{aligned}$$

Which means that $\mathcal{F}_{x \to \xi}\left(\Omega\left(p^k \|x - I\|_p\right)\right)$ is supported in $p^{-k}\mathbb{Z}_p^n$. Finally, we verify that $-m$ is the index of local constancy of $\mathcal{F}_{x \to \xi}\left(\Omega\left(p^k \|x - I\|_p\right)\right)$. By using that $p^{-k} < \|I\|_p \le p^{-m}$ we have

$$\chi_p(\xi \cdot I)\,|_{\xi_0 + p^{-m}\mathbb{Z}_p^n} = \chi_p(\xi_0 \cdot I),$$

for any $\xi_0 \in p^{-k}\mathbb{Z}_p^n$, and consequently $p^{-nk} \chi_p(\xi \cdot I)\, \Omega\left(\left\|p^k \xi\right\|_p\right) \in \mathcal{D}_{-m}^{-k}$. □

THEOREM 6.18 (see e.g. [**1**, Theorem 4.8.2]). *The map*

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Q}_p^n) & \to & \mathcal{D}(\mathbb{Q}_p^n) \\ \varphi & \to & \widehat{\varphi}, \end{array} \tag{6.3}$$

*where* $\widehat{\varphi}(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\, \varphi(x)\, d^n x$, *is a well-defined linear operator, with inverse given by*

$$\varphi(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x)\, \widehat{\varphi}(\xi)\, d^n \xi.$$

*In other words, the mapping* (6.3) *is a isomorphism of* $\mathbb{C}$*-vector spaces on* $\mathcal{D}(\mathbb{Q}_p^n)$.

**6.4. The inverse Fourier transform.** One expects that the inverse Fourier transform be given by

$$f(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x)\, \widehat{f}(\xi)\, d^n \xi = \int_{\mathbb{Q}_p^n} \overline{\chi_p(\xi \cdot x)} \widehat{f}(\xi)\, d^n \xi.$$

This formula does not always make sense since $\widehat{f}$ is not necessarily in $L^1$ when $f \in L^1$.

EXERCISE 6.19. Show that the Fourier transform of $f(x) = \Omega\left(|x|_p\right) \ln\left(\frac{1}{|x|_p}\right)$ is

$$\widehat{f}(\xi) = \begin{cases} \frac{p^{-1} \ln p}{1 - p^{-1}} & \text{if } |\xi|_p \le 1 \\ \\ \left(\frac{\ln p}{1 - p^{-1}}\right) |\xi|_p^{-1} & \text{if } |\xi|_p > 1. \end{cases}$$

DEFINITION 6.20. If $g$ is locally integrable and $k \in \mathbb{Z}$, we define

$$A_k g = \int_{\mathbb{Q}_p^n} g(x)\, \Delta_k(x)\, d^n x = \int_{\|x\|_p \le p^k} g(x)\, d^n x.$$

Notice that if $g \in L^1$, then $A_k g \to \int_{\mathbb{Q}_p^n} g(x)\, d^n x$ as $k \to \infty$ (why?). Now, the limit $\lim_{k \to \infty} A_k g$ may exist even though $\int_{\mathbb{Q}_p^n} g(x)\, d^n x$ does not exist.

EXERCISE 6.21. To show the following fact: if $f \in L^1$, $k \in \mathbb{Z}$, then

$$\begin{aligned} A_k\left(\widehat{f}\,\overline{\chi_p\left(x\cdot\right)}\right) &= \int_{\mathbb{Q}_p^n} \widehat{f}\left(\xi\right)\ \overline{\chi_p\left(x\cdot\xi\right)}\ \Delta_k\left(\xi\right) d^n\xi \\ &= \int_{\mathbb{Q}_p^n} f\left(y\right)\delta_k\left(y-x\right) d^n y = p^{nk}\int_{\left\Vert x-y\right\Vert_p \leq p^{-k}} f\left(y\right) d^n y. \end{aligned}$$

DEFINITION 6.22. Let $f:\mathbb{Q}_p^n \to \mathbb{C}$ be locally integrable. A point $x \in \mathbb{Q}_p^n$ is called a regular point of $f$ if

$$f_k(x) := p^{nk}\int_{\left\Vert x-y\right\Vert_p \leq p^{-k}} f\left(y\right) d^n y \to f\left(x\right) \text{ as } k\to\infty.$$

THEOREM 6.23 ([**23**, Theorem 1.14]). *Let $f$ be a locally integrable function. There exists a zero measure subset $L = L(f)$ such that any $x \in \mathbb{Q}_p^n \smallsetminus L$ is a regular point of $f$.*

EXERCISE 6.24. Assume that $f$ is locally integrable and continuous at $x$. Show that

$$p^{nk}\int_{\left\Vert x-y\right\Vert_p \leq p^{-k}} f\left(y\right) d^n y \to f\left(x\right) \text{ as } k\to\infty.$$

COROLLARY 6.25. *If $f \in L^1$, then*

$$A_k\left(\widehat{f}\,\overline{\chi_p\left(x\cdot\right)}\right) = \int_{\mathbb{Q}_p^n} \widehat{f}\left(\xi\right)\chi_p\left(-x\cdot\xi\right)\Delta_k\left(\xi\right) d^n\xi \to f(x)$$

*almost everywhere. In particular, it converges at each point of continuity of $f$.*

PROOF. The corollary follows from Exercise 6.21, Theorem 6.23 and Exercise 6.24. □

THEOREM 6.26. *If $f$ and $\widehat{f}$ are both integrable then $f$ is equal a.e. to a continuous function. With $f$ modified (on a set of measure zero) to be continuous, we have*

$$f(x) = \int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right)\widehat{f}\left(\xi\right) d^n\xi \quad \textit{for all } x \in \mathbb{Q}_p^n. \tag{6.4}$$

PROOF. If $\widehat{f}$ is integrable, then $A_k\left(\widehat{f}\,\chi_p\left(-\xi\cdot\right)\right)$ converges to a continuous function:

$$\int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right)\widehat{f}\left(\xi\right) d^n\xi. \tag{6.5}$$

By Corollary 6.25, $f\left(x\right)$ agrees with (6.5) a.e. and consequently $f$ is continuous almost everywhere. By modifying $f$ on a set of measure zero we obtain formula (6.4). □

EXERCISE 6.27. Define

$$\mathcal{L}(\mathbb{Q}_p^n) := \left\{f:\mathbb{Q}_p^n \to \mathbb{C}:\ f,\,\widehat{f} \in L^1 \text{ and } f,\,\widehat{f} \text{ are continuous.}\right\}$$

Then $\mathcal{L}(\mathbb{Q}_p^n) \underset{\rightarrow}{\mathcal{F}} \mathcal{L}(\mathbb{Q}_p^n)$ is an isomorphism of $\mathbb{C}$-vector spaces. In particular, the formula

$$f(0) = \int_{\mathbb{Q}_p^n} \widehat{f}\left(\xi\right) d^n\xi$$

holds true.

Corollary 6.28. *If $f,\ g \in L^1$ and $\widehat{f} = \widehat{g}$ a.e., then $f(x) = g(x)$ a.e.*

Proof. Since $\widehat{(f-g)} = 0$, by Theorem 6.26, $(f-g)(x) = 0$ a.e. $\square$

Remark 6.29. Monotone convergence lemma (Levi's monotone convergence theorem). Let $h_k : \mathbb{Q}_p^n \to [0,\infty]$, $k \in \mathbb{N}$, be a sequence of non-negative Borel measurable functions satisfying

$$0 \le h_k(x) \le h_{k+1}(x) \le \infty \quad \text{for any } x \in \mathbb{Q}_p^n.$$

Assume that $h : \mathbb{Q}_p^n \to [0,\infty]$ is the pointwise limit of the sequence $\{h_k\}_{k\in\mathbb{N}}$. Then $h$ is Borel measurable and

$$\lim_{k\to\infty} \int_{\mathbb{Q}_p^n} h_k(x)\, d^n x = \int_{\mathbb{Q}_p^n} h(x) d^n x.$$

The integral $\int_{\mathbb{Q}_p^n} h(x) d^n x$ could be equal to $+\infty$.

Corollary 6.30. *If $f \in L^1$, $\widehat{f} \ge 0$ and $f$ is continuous at zero, then $\widehat{f} \in L^1$ and $f(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x)\, \widehat{f}(\xi)\, d^n\xi$ at each regular point of $f$. In particular, $f(0) = \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\, d^n\xi$.*

Proof. We need only to show that $\widehat{f} \in L^1$. Since $f$ and $\Delta_k \in L^1$, by Lemma 6.16 and Exercise 6.6, we have

$$\int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\, \Delta_k(\xi)\, d^n\xi = \int_{\mathbb{Q}_p^n} f(\xi)\, \delta_k(\xi)\, d^n\xi.$$

By Theorem 6.23,

$$f(0) = \lim_{k\to\infty} \int_{\mathbb{Q}_p^n} f(\xi)\, \delta_k(\xi)\, d^n\xi = \lim_{k\to\infty} \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\, \Delta_k(\xi)\, d^n\xi.$$

Finally, by using the fact that $\widehat{f} \ge 0$, and monotone convergence lemma, we have $\widehat{f} \in L^1$. $\square$

## 7. The $L^2$-theory

Theorem 7.1. *If $f \in L^1 \cap L^2$, then $\left\| \widehat{f} \right\|_2 = \|f\|_2$.*

Proof. We set $g(x) := \overline{f}(-x)$, then $\widehat{g} = \overline{\widehat{f}}$. Since $f,\ g \in L^1$, $f * g \in L^1$ and

$$\widehat{f * g} = \widehat{f}\widehat{g} = \left| \widehat{f} \right|^2 \ge 0,$$

cf. Proposition 6.13 and Lemma 6.15.

Now, since $f,\ g \in L^2$, then $f * g$ is continuous, indeed, by using the Cauchy-Schwarz inequality,

$$|(f*g)(x+y) - (f*g)(x)| = \left| \int_{\mathbb{Q}_p^n} \{f(x+y-z) - f(x-z)\}\, g(z) d^n z \right|$$

$$\le \sqrt{\int_{\mathbb{Q}_p^n} |g(z)|^2\, d^n z}\ \sqrt{\int_{\mathbb{Q}_p^n} |f(x+y-z) - f(x-z)|^2\, d^n z}$$

$$= \|g\|_2 \sqrt{\int_{\mathbb{Q}_p^n} |f(u+y) - f(u)|^2\, d^n z} = \|g\|_2\, \|f(\cdot + y) - f(\cdot)\|_2 \to 0 \text{ as } y \to 0,$$

by the dominated convergence theorem and the fact that

$$|f(u+y)-f(u)|^2 \leq 4\max\left\{|f(u+y)|^2, |f(u)|^2\right\} \leq 4\left\{|f(u+y)|^2+|f(u)|^2\right\}.$$

We now apply Corollary 6.30 to $f*g$, with $\widehat{f*g}\geq 0$, to get that $\widehat{f*g}=\left|\widehat{f}\right|^2\in L^1$ and

$$(f*g)(0)=\int_{\mathbb{Q}_p^n}\left|\widehat{f}(\xi)\right|^2 d^n\xi=\int_{\mathbb{Q}_p^n} g(-z)\, f(z)d^nz=\int_{\mathbb{Q}_p^n}|f(z)|^2\, d^nz.$$

□

Remark 7.2. Let $(Y, \|\cdot\|_Y)$ be a Banach space, this means that $(Y, \|\cdot\|_Y)$ is a normed complex space such that $Y$ is a complete metric space for the distance induced by $\|\cdot\|_Y$. Let $(X, \|\cdot\|_X)$ be a complex normed space, and let $D(T)$ be a subspace of $(X, \|\cdot\|_X)$. Let $T: D(T)\rightarrow Y$ be a linear bounded operator, i.e. $T$ satisfies $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ for any $\alpha$, $\beta\in\mathbb{C}$, and any $x$, $y\in D(T)$, and

$$\|T\|:=\sup_{x\in D(T)}\frac{\|T(x)\|_Y}{\|x\|_X}<\infty.$$

Then $T$ has an extension $\widetilde{T}:\overline{D(T)}\rightarrow Y$, where $\overline{D(T)}$ denotes the closure of $D(T)$ in $(X, \|\cdot\|_X)$, with the same norm $\left\|\widetilde{T}\right\|=\|T\|$. If $\overline{D(T)}=X$, i.e. if $D(T)$ is dense in $X$, then $\widetilde{T}$ is unique.

Remark 7.3. Let $T: X\rightarrow Y$ be a linear operator. Then $T$ is continuous if and only if $T$ is bounded (i.e. $\|T\|:=\sup_{x\in X}\frac{\|T(x)\|_Y}{\|x\|_X}<\infty$).

From this theorem, it follows that the mapping

$$\begin{array}{ccc} L^1\cap L^2 & \rightarrow & L^2 \\ f & \rightarrow & \widehat{f} \end{array}$$

is an $L^2$-isometry on $L^1\cap L^2$, which is a dense subspace of $L^2$ (Why?). Thus, this mapping has an extension to an $L^2$-isometry from $L^2$ into $L^2$. We now extend the Fourier transform to $L^2$.

Definition 7.4. For $f\in L^2$, let

$$f_k := f\Delta_k,\ \text{ for } k\in\mathbb{N}, \tag{7.1}$$

and

$$\widehat{f}(\xi):=\lim_{k\rightarrow\infty}\widehat{f_k}(\xi)=\lim_{k\rightarrow\infty}\int_{\|x\|_p\leq p^k}\chi_p(\xi\cdot x)\, f(x)d^nx,$$

where the limit is taken in $L^2$.

Lemma 7.5. *If $f$, $g\in L^2$, then*

$$\int_{\mathbb{Q}_p^n}\widehat{f}\ gd^ny=\int_{\mathbb{Q}_p^n} f\ \widehat{g}d^ny.$$

Proof. We first notice that $f_k \overset{L^2}{\rightarrow} f$ and $g_k \overset{L^2}{\rightarrow} g$, and that $f_k$, $g_k\in L^1\cap L^2$ for every $k$. Hence, by Lemma 6.16,

$$\int_{\mathbb{Q}_p^n} f_k\ \widehat{g_k}d^nx=\int_{\mathbb{Q}_p^n}\widehat{f_k}\ g_kd^nx.$$

Now, by using Theorem 7.1 and the Cauchy-Schwarz inequality,

$$\left|\int_{\mathbb{Q}_p^n} f_k\ \widehat{g_k} d^n x\right| \leq \|f_k\|_2 \|\widehat{g_k}\|_2 = \|f_k\|_2 \|g_k\|_2 ,$$

which means that the bilinear form

$$(f_k, g_k) \to \int_{\mathbb{Q}_p^n} f_k\ \widehat{g_k} d^n x$$

is bounded (and consequently continuous) in each variable in $L^2$, then

$$\int_{\mathbb{Q}_p^n} f_k\ \widehat{g_k} d^n x \underset{\to}{L^2} \int_{\mathbb{Q}_p^n} f\ \widehat{g} d^n x.$$

□

Theorem 7.6. *The Fourier transform is unitary in $L^2$.*

Proof. We have to show that the Fourier transform is a bijective linear mapping that preserves the $L^2$-norm. We already know that $f \underset{\to}{\mathcal{F}} \widehat{f}$ is a linear $L^2$-isometry. It remains to show that it is onto. By contradiction, we assume that $\mathcal{F}$ is not onto. Notice that $\mathcal{F}(L^2)$ is closed in $L^2$ because $\left\|\widehat{f}\right\|_2 = \|f\|_2$, if $\mathcal{F}(L^2) \neq L^2$, then by general theory of Hilbert spaces, $\mathcal{F}(L^2)$ has an orthogonal complement $\mathcal{F}(L^2)^\perp$ such that $L^2 = \mathcal{F}(L^2) \oplus \mathcal{F}(L^2)^\perp$, where for any $g \in \mathcal{F}(L^2)^\perp$ and any $\widehat{f} \in \mathcal{F}(L^2)$, $\left\langle \widehat{f}, g\right\rangle = 0$. Then, there exists $\overline{h} \in L^2$, $\left\|\overline{h}\right\|_2 \neq 0$ such that

$$\left\langle \widehat{f}, \overline{h}\right\rangle = \int_{\mathbb{Q}_p^n} \widehat{f} h d^n x = 0 \text{ for any } f \in L^2.$$

By using Lemma 7.5, $\widehat{h} = 0$, but $\left\|\overline{h}\right\|_2 = \|h\|_2 = \left\|\widehat{h}\right\|_2 = 0$, cf. Theorem 7.1, which contradicts $\left\|\overline{h}\right\|_2 \neq 0$. □

Exercise 7.7. Show that $\left\|\widehat{f}\right\|_2 = \|f\|_2$ for $f \in L^2$ is equivalent to for any $f$, $g \in L^2$,

$$\langle f, g\rangle = \left\langle \widehat{f}, \widehat{g}\right\rangle \text{ i.e. } \int_{\mathbb{Q}_p^n} f\ \overline{g} d^n x = \int_{\mathbb{Q}_p^n} \widehat{f}\ \overline{\widehat{g}} d^n x.$$

Exercise 7.8. For $f \in L^2$, we set $\widetilde{f}$ for the reflection of $f$ which is defined as $\widetilde{f}(x) = f(-x)$. Show that if $f \in L^2$, then

$$\mathcal{F}^{-1}(f) = \mathcal{F}\left(\widetilde{f}\right).$$

Theorem 7.9. *If $f \in L^2$, then*

$$\lim_{k\to\infty} \int_{\|x\|_p \leq p^k} \chi_p(\xi \cdot x) f(x) d^n x = \widehat{f}(\xi) \quad \textit{almost everywhere.}$$

PROOF. We use that $\Delta_k$, $f$, $\widehat{f} \in L^2$, jointly with Lemma 7.5 and Exercise 7.7 to get

$$\begin{aligned}\lim_{k\to\infty}\int_{\|x\|_p\leq p^k}\chi_p(\xi\cdot x)\,f(x)d^nx &= \lim_{k\to\infty}\int_{\mathbb{Q}_p^n} f(x)\overline{\chi_p(-\xi\cdot x)\,\Delta_k(x)}d^nx \\ = \lim_{k\to\infty}\int_{\mathbb{Q}_p^n}\widehat{f}(z)\overline{\delta_k(z-\xi)}d^nz &= \lim_{k\to\infty}p^{nk}\int_{\|z-\xi\|_p\leq p^{-k}}\widehat{f}(z)d^nz.\end{aligned}$$

Now, since $\widehat{f} \in L^2$, then $\widehat{f} \in L^1_{loc}$, the result follows from Theorem 6.23. □

REMARK 7.10. (i) The Fourier transform can be extended to $L^1 + L^2$, which means that if $f = f_1 + f_2$, with $f_1 \in L^1$, $f_2 \in L^2$, then $\widehat{f} = \widehat{f_1} + \widehat{f_2}$, where the Fourier transforms are defined in $L^1$ and $L^2$ respectively. The function $\widehat{f}$ is well-defined in $L^1_{loc}$.

(ii) If $f \in L^1$, $g \in L^\rho$, $\rho \in [1, 2]$, then $\widehat{f * g} = \widehat{f}\,\widehat{g}$ a.e., cf. [**23**, Theorem 2.7].

## 8. $\mathcal{D}$ as a topological vector space

We define a topology on $\mathcal{D}$ as follows. We say that a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero, if the two following conditions hold:

(C1) there are two fixed integers $k$ and $m$ such that each $\varphi_j$ is constant on the cosets of $p^k\mathbb{Z}_p^n$ and is supported on $p^m\mathbb{Z}_p^n$, i.e. $\varphi_j \in \mathcal{D}_k^m$;

(C2) $\varphi_j \to 0$ uniformly.

$\mathcal{D}$ endowed with the above topology becomes a topological vector space.

We recall that $\mathcal{D}_k^m$ is $\mathbb{C}$-vector space of dimension $N_{m.k} := \#\left(p^m\mathbb{Z}_p^n/p^k\mathbb{Z}_p^n\right)$. Given $c = (c_1, \ldots, c_{N_{m.k}}) \in \mathbb{C}^{N_{m.k}}$, we set $\|c\|_{\mathbb{C}} = \max_i |c_i|$. Then $\left(\mathbb{C}^{N_{m.k}}, \|\cdot\|_{\mathbb{C}}\right)$ is a Banach space and

$$\left(\mathbb{C}^{N_{m.k}}, \|\cdot\|_{\mathbb{C}}\right) \simeq \mathcal{D}_k^m \text{ as topological spaces, (why?).}$$

A key fact is that uniform convergence in $\mathcal{D}_k^m$ agrees with the convergence in the supremum norm ($L^\infty$-norm), which in turn agrees with the convergence in the $\|\cdot\|_{\mathbb{C}}$-norm.

EXERCISE 8.1. Show that $\mathcal{D}$ is a complete and separable topological vector space.

THEOREM 8.2. *The map*

$$\begin{array}{ccc}\mathcal{D}(\mathbb{Q}_p^n) & \to & \mathcal{D}(\mathbb{Q}_p^n) \\ \varphi & \to & \widehat{\varphi},\end{array} \tag{8.1}$$

*where* $\widehat{\varphi}(\xi) = \int_{\mathbb{Q}_p^n}\chi_p(\xi\cdot x)\,\varphi(x)\,d^nx$, *is a homeomorphism of topological vector spaces , with inverse given by*

$$\varphi(x) = \int_{\mathbb{Q}_p^n}\chi_p(-\xi\cdot x)\,\widehat{\varphi}(\xi)\,d^n\xi.$$

PROOF. We already know that (8.1) is an isomorphism of vector spaces. It remains to show that the continuity of $\mathcal{F}$ and $\mathcal{F}^{-1}$. Let $\varphi_j \xrightarrow{\mathcal{D}} \varphi$, i.e. $\varphi_j, \varphi \in \mathcal{D}_k^m$ for some integers $m$, $k$, and $\varphi_j \xrightarrow{\text{unif.}} \varphi$. Since $\widehat{\varphi_j}, \widehat{\varphi} \in \mathcal{D}_{-m}^{-k}$, in order to show that

$\widehat{\varphi_j} \underrightarrow{\mathcal{D}} \widehat{\varphi}$, it is sufficient to establish that $\widehat{\varphi_j} \xrightarrow{\text{unif.}} \widehat{\varphi}$, i.e. $\|\widehat{\varphi_j} - \widehat{\varphi}\|_\infty \to 0$ as $j \to \infty$. By using that

$$\|\widehat{\varphi_j} - \widehat{\varphi}\|_\infty \leq \int_{\mathbb{Q}_p^n} |\varphi_j - \varphi|\, d^n x \leq \|\varphi_j - \varphi\|_\infty \int_{p^m \mathbb{Z}_p^n} d^n x$$
$$= p^{-mn} \|\varphi_j - \varphi\|_\infty \to 0 \text{ as } j \to \infty.$$

The continuity of the inverse Fourier transform is established by using the same argument. □

## 9. The space of distributions on $\mathbb{Q}_p^n$

The $\mathbb{C}$-vector space $\mathcal{D}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}$ is called the *Bruhat-Schwartz space of distributions* . We endow $\mathcal{D}'$ with the weak topology, i.e. a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'$ converges to $T$ if

$$\lim_{j\to\infty} T_j(\varphi) = T(\varphi) \text{ for any } \varphi \in \mathcal{D}.$$

Exercise 9.1. Define the map

$$\begin{array}{ccc} \mathcal{D}' \times \mathcal{D} & \to & \mathbb{C} \\ (T, \varphi) & \to & T(\varphi). \end{array}$$

Then $(T, \varphi)$ is a bilinear form which is continuous in $T$ and $\varphi$ separately. We call this map the pairing between $\mathcal{D}'$ and $\mathcal{D}$. From now on we will use $(T, \varphi)$ instead of $T(\varphi)$.

Exercise 9.2. If $f \in L^\rho$, $1 \leq \rho \leq \infty$, then $f$ induces a distribution. More precisely,

$$(f, \varphi) = \int_{\mathbb{Q}_p^n} f \varphi d^n x$$

Remark 9.3. If $T$ is a distribution and $g$ is a locally integrable function such that

$$(T, \varphi) = \int_{\mathbb{Q}_p^n} g \varphi d^n x \text{ for all } \varphi \in \mathcal{D},$$

we identify $T$ with function $g$. In this case, some authors say that $T$ is a regular distribution.

Example 9.4. (i) The distribution $(\delta, \varphi) = \varphi(0)$ is called the Dirac distribution.

(ii) $(1, \varphi) = \int_{\mathbb{Q}_p^n} \varphi d^n x$.

Lemma 9.5. *Every linear functional on $\mathcal{D}$ is continuous, i.e. $\mathcal{D}'$ agrees with the algebraic dual of $\mathcal{D}$.*

Proof. Let $T : \mathcal{D} \to \mathbb{C}$ be a linear functional, and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence of test functions converging to 0. Then $\varphi_j \in \mathcal{D}_k^m$, for all $j$, consequently

$$\varphi_j(x) = \sum_{I \in G_{m,k}} \varphi_j(I)\, \Omega\left(p^k \|x - I\|_p\right),$$

and $\max_{I\in G_{m,k}} |\varphi_j(I)| \to 0$ as $j \to \infty$. Then

$$T\varphi_j(x) = \sum_{I \in G_{m,k}} \varphi_j(I)\, T\left(\Omega\left(p^k \|x - I\|_p\right)\right) = \sum_{I \in G_{m,k}} c_{k,I}\, \varphi_j(I),$$

and

$$|T\varphi_j(x)| \leq \sum_{I\in G_{m,k}} |c_{k,I}| \; |\varphi_j(I)| \leq \left\{ \sum_{I\in G_{m,k}} |c_{k,I}| \right\} \max_{I\in G_{m,k}} |\varphi_j(I)| \to 0,$$

as $j\to\infty$. $\square$

We set

$$\Gamma_p^{(n)}(\alpha) := \frac{1-p^{\alpha-n}}{1-p^{-\alpha}}, \text{ for } \alpha\in\mathbb{C}\smallsetminus\{0\}.$$

The function

$$k_\alpha(x) = \frac{\|x\|_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, \quad \text{for } x\in\mathbb{Q}_p^n, \; \operatorname{Re}(\alpha)>0, \, \alpha\neq n,$$

is called *the multi-dimensional Riesz Kernel.* It determines a distribution from $\mathcal{D}'$, which admits a meromorphic continuation to all $\alpha\in\mathbb{C}\smallsetminus\left\{n+\frac{2\pi\sqrt{-1}}{\ln p}\mathbb{Z}\right\}$ given by

$$(k_\alpha(x),\varphi(x)) = \frac{1-p^{-n}}{1-p^{\alpha-n}}\varphi(0) + \frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\int_{\|x\|_p>1} \|x\|_p^{\alpha-n}\varphi(x)\,d^nx$$

$$+\frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\int_{\|x\|_p\leq 1} \|x\|_p^{\alpha-n}(\varphi(x)-\varphi(0))\,d^nx. \tag{9.1}$$

In the case $\alpha=0$, by passing to the limit in (9.1), we obtain

$$(k_0(x),\varphi(x)) := \lim_{\alpha\to 0}(k_\alpha(x),\varphi(x)) = \varphi(0),$$

i.e., $k_0(x)=\delta(x)$, the Dirac delta function.

EXERCISE 9.6. With the above notation, for $\operatorname{Re}(\alpha)>0$, the following formulas hold true:

$$(k_{-\alpha}(x),\varphi(x)) = \frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n} (\varphi(x)-\varphi(0))\frac{d^nx}{\|x\|_p^{\alpha+n}}, \tag{9.2}$$

and

$$(k_\alpha(x),\varphi(x)) = \frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\int_{\mathbb{Q}_p^n} \|x\|_p^{\alpha-n}\varphi(x)\,d^nx, \text{ for } \alpha\in\mathbb{C}\smallsetminus\left\{n+\frac{2\pi\sqrt{-1}}{\ln p}\mathbb{Z}\right\}. \tag{9.3}$$

## 10. The Fourier transform on $\mathcal{D}'$

DEFINITION 10.1. For $T\in\mathcal{D}'$ its Fourier transform, denoted as $\mathcal{F}(T)$ or $\widehat{T}$, is the distribution defined as

$$(\mathcal{F}(T),\varphi) = (T,\mathcal{F}(\varphi)) \text{ for all } \varphi\in\mathcal{D}.$$

Notice that since $\varphi\to\mathcal{F}(\varphi)$ is a homeomorphism of $\mathcal{D}$, $\mathcal{F}(T)$ is well-defined.

EXAMPLE 10.2. $\widehat{\delta}=1$. Indeed,

$$\left(\widehat{\delta},\varphi\right) = (\delta,\widehat{\varphi}) = \widehat{\varphi}(0) = \int_{\mathbb{Q}_p^n}\varphi d^nx = (1,\varphi) \text{ for all } \varphi\in\mathcal{D}.$$

DEFINITION 10.3. The inverse Fourier transform of $T\in\mathcal{D}'$, denoted as $\mathcal{F}^{-1}(T)$ or $\check{T}$, is defined as

$$\left(\mathcal{F}^{-1}(T),\varphi\right) = \left(T,\mathcal{F}^{-1}(\varphi)\right) \text{ for all } \varphi\in\mathcal{D}.$$

Remark 10.4. Notice that $\mathcal{F}^{-1}(T) \in \mathcal{D}'$ and that $\mathcal{F}^{-1}(\mathcal{F}(T)) = T$ for any $T \in \mathcal{D}'$.

Lemma 10.5. *The map $T \to \mathcal{F}(T)$ is a homeomorphism of $\mathcal{D}'$ onto $\mathcal{D}'$.*

Proof. The onto part follows from Remark 10.4. For the continuity, we proceed as follows. Let $T_j \underset{\to}{\mathcal{D}'} T$, i.e. $(T_j, \varphi) \to (T, \varphi)$ for all $\varphi \in \mathcal{D}$. Then

$$\left(\widehat{T_j}, \varphi\right) = (T_j, \widehat{\varphi}) \to (T, \widehat{\varphi}) \text{ for all } \varphi \in \mathcal{D},$$

i.e.

$$\left(\widehat{T_j}, \varphi\right) \to \left(\widehat{T}, \varphi\right) \text{ for all } \varphi \in \mathcal{D}.$$

□

Definition 10.6. Linear change of variables for distributions. For $A \in GL_n\left(\mathbb{Q}_p^n\right)$ and $b \in \mathbb{Q}_p^n$, we define

$$(T(Ax+b), \varphi) = \frac{1}{|\det A|_p}\left(T, \varphi\left(A^{-1}(y-b)\right)\right).$$

Example 10.7. Recall that $\mathcal{T}_b(\varphi(x)) = \varphi(x-b)$ for $b \in \mathbb{Q}_p^n$ and $\varphi \in \mathcal{D}$. Then, for $G \in \mathcal{D}'$,

$$(\mathcal{T}_b G, \varphi) = (G, \mathcal{T}_{-b}\varphi).$$

Example 10.8. The reflection operator, denoted as $\widetilde{\bullet}$, acting on $\varphi \in \mathcal{D}$ is defined as $\widetilde{\varphi}(x) = \varphi(-x)$. Then for $G \in \mathcal{D}'$, $\left(\widetilde{G}, \varphi\right) = (G, \widetilde{\varphi})$.

Exercise 10.9. Show that

$$G \to \mathcal{F}(G),\ G \to \mathcal{F}^{-1}(G),\ G \to \mathcal{T}_b G,\ G \to \widetilde{G}$$

are homeomorphisms of $\mathcal{D}'$ onto $\mathcal{D}'$.

Definition 10.10. For $G \in \mathcal{D}'$ and $\theta \in \mathcal{D}$, we define the distribution $\theta G$ by

$$(\theta G, \varphi) = (G, \theta\varphi) \text{ for all } \varphi \in \mathcal{D}.$$

Exercise 10.11. For $\psi$, $\varphi \in \mathcal{D}$, $G \in \mathcal{D}'$, the maps

$$G \to \varphi G, \quad \psi \to \psi\varphi$$

are continuous maps from $\mathcal{D}'$ into $\mathcal{D}'$ and from $\mathcal{D}$ into $\mathcal{D}$, respectively.

Exercise 10.12. Show that $\mathcal{F}_{x\to\xi}(\chi_p(x \cdot \xi_0)) = \delta(\xi - \xi_0)$ in $\mathcal{D}'$.

Example 10.13. We want to compute $\mathcal{F}^{-1}_{\xi\to x}\left(\frac{1}{\|\xi\|_p^\alpha + m^2}\right)$, where $\alpha > 0$ and $m > 0$. Notice that if $\alpha > n$, then $\frac{1}{\|\xi\|_p^\alpha + m^2} \in L^1$ (Why?). In the general case, $\frac{1}{\|\xi\|_p^\alpha + m^2} \notin L^1$. If we use the notion of improper integral, we have

$$\int_{\mathbb{Q}_p^n} \frac{\chi(-\xi \cdot x)}{\|\xi\|_p^\alpha + m^2} d^n\xi = \sum_{j=-\infty}^{\infty} \int_{\|\xi\|_p = p^j} \frac{\chi(-\xi \cdot x)}{\|\xi\|_p^\alpha + m^2} d^n\xi.$$

The exact mathematical meaning of this formula is

$$\sum_{j=-\infty}^{N} \int_{\|\xi\|_p = p^j} \frac{\chi(-\xi \cdot x)}{\|\xi\|_p^\alpha + m^2} d^n\xi \underset{\to}{\mathcal{D}'} \mathcal{F}^{-1}_{\xi\to x}\left(\frac{1}{\|\xi\|_p^\alpha + m^2}\right) \text{ as } N \to \infty.$$

Indeed, for any test function $\varphi$, we have

$$\sum_{j=-\infty}^{N}\int_{\mathbb{Q}_p^n}\int_{\|\xi\|_p=p^j}\frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^{\alpha}+m^2}\varphi\left(x\right)d^n\xi d^n x=\int_{\mathbb{Q}_p^n}\int_{\|\xi\|_p\leq p^N}\frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^{\alpha}+m^2}\varphi\left(x\right)d^n\xi d^n x$$
$$=\int_{\|\xi\|_p\leq p^N}\frac{1}{\|\xi\|_p^{\alpha}+m^2}\left\{\int_{\mathbb{Q}_p^n}\chi\left(-\xi\cdot x\right)\varphi\left(x\right)d^n x\right\}d^n\xi=\int_{\|\xi\|_p\leq p^N}\frac{\mathcal{F}^{-1}(\varphi)\left(\xi\right)}{\|\xi\|_p^{\alpha}+m^2}d^n\xi$$
$$=\int_{\mathbb{Q}_p^n}\frac{\mathcal{F}^{-1}(\varphi)}{\|\xi\|_p^{\alpha}+m^2}d^n\xi=\left(\frac{1}{\|\xi\|_p^{\alpha}+m^2},\mathcal{F}^{-1}(\varphi)\right)\text{ for } N \text{ sufficiently large.}$$

Proposition 10.14. *For* $\alpha\in\mathbb{C}\smallsetminus\left\{n+\frac{2\pi\sqrt{-1}}{\ln p}\mathbb{Z}\right\}$, $\mathcal{F}_{x\rightarrow\xi}\left(k_{\alpha}(x)\right)=\|\xi\|_p^{-\alpha}$ *in* $\mathcal{D}'$.

Proof. In $\mathcal{D}'$, by using Exercise 5.3, we have

$$\begin{aligned}\mathcal{F}_{x\rightarrow\xi}\left(k_{\alpha}(x)\right)&=\lim_{N\rightarrow\infty}\sum_{j=-\infty}^{N}\int_{\|x\|_p=p^j}k_{\alpha}(x)\chi_p\left(x\cdot\xi\right)d^n x\\&=\lim_{N\rightarrow\infty}\sum_{j=-\infty}^{N}\frac{p^{j(\alpha-n)}}{\Gamma_p^{(n)}\left(\alpha\right)}\int_{\|x\|_p=p^j}\chi_p\left(x\cdot\xi\right)d^n x\\&=\lim_{N\rightarrow\infty}\sum_{j=-\infty}^{N}\frac{p^{j(\alpha-n)}}{\Gamma_p^{(n)}\left(\alpha\right)}\begin{cases}p^{jn}\left(1-p^{-n}\right) & \text{if } \ ord\left(\xi\right)\geq j\\ -p^{jn-n} & \text{if } \ ord(\xi)=j-1\\ 0 & \text{if } \ ord(\xi)\leq j-2.\end{cases}\end{aligned}$$

For $\xi\neq 0$ there is $N>ord\left(\xi\right)$, then

$$\mathcal{F}_{x\rightarrow\xi}\left(k_{\alpha}(x)\right)=\sum_{j=-\infty}^{ord(\xi)}\frac{\left(1-p^{-n}\right)p^{j\alpha}}{\Gamma_p^{(n)}\left(\alpha\right)}-\frac{p^{-n}p^{(ord(\xi)+1)\alpha}}{\Gamma_p^{(n)}\left(\alpha\right)}=\|\xi\|_p^{-\alpha}.$$

□

Lemma 10.15. *If* $\alpha>0$, *then*

$$\|x\|_p^{\alpha}=\frac{1-p^{\alpha}}{1-p^{\alpha-n}}\int_{\mathbb{Q}_p^n}\|y\|_p^{\alpha-n}\left\{\chi_p\left(x\cdot y\right)-1\right\}d^n y\ \textit{in}\ \mathcal{D}'.$$

Proof. Take $\psi$, $\varphi\in\mathcal{D}$ with $\psi=\mathcal{F}^{-1}\left(\varphi\right)$. Then by Proposition 10.14 and formula (9.2), we have

$$\begin{aligned}\left(\|x\|_p^{\alpha},\varphi\right)&=\Gamma_p^{(n)}(\alpha+n)\left(k_{\alpha+n},\varphi\right)=\Gamma_p^{(n)}(\alpha+n)\left(\mathcal{F}_{x\rightarrow\xi}(k_{\alpha+n}),\psi\left(\xi\right)\right)\\&=\Gamma_p^{(n)}(\alpha+n)\left(\|\xi\|_p^{-\alpha-n},\psi\left(\xi\right)\right)=\left(k_{-\alpha},\psi\right)\\&=\frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\left(\psi(x)-\psi(0)\right)\frac{d^n x}{\|x\|_p^{\alpha+n}}\\&=\frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\frac{1}{\|x\|_p^{\alpha+n}}\left\{\int_{\mathbb{Q}_p^n}\left(\chi_p\left(-\xi\cdot x\right)-1\right)\varphi\left(\xi\right)d^n\xi\right\}d^n x.\end{aligned}$$

Now if $\varphi(\xi) = 0$ for $\|\xi\|_p > p^N$, then

$$\int_{\mathbb{Q}_p^n} (\chi_p(-\xi \cdot x) - 1)\, \varphi(\xi)\, d^n\xi = 0 \text{ for } \|x\|_p \leq p^N,$$

and consequently we may use Fubini's theorem to get

$$\left(\|x\|_p^\alpha, \varphi\right) = \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \varphi(\xi) \left\{ \int_{\mathbb{Q}_p^n} \frac{\chi_p(\xi \cdot x) - 1}{\|x\|_p^{\alpha+n}}\, d^n x \right\} d^n\xi.$$

□

Definition 10.16. We say that a locally constant function $f$ belongs to $\mathcal{U}_{loc}\left(\mathbb{Q}_p^n\right) := \mathcal{U}_{loc}$ if and only if $f$ is constant on the cosets of $p^k\mathbb{Z}_p^n$ for some $k \in \mathbb{Z}$. Equivalently, there is an integer $k$ such that for any $x \in \mathbb{Q}_p^n$,

$$f(x + x') = f(x) \text{ for any } x' \in B_{-k}^n. \tag{10.1}$$

The largest $-k$ such that (10.1) is verified is called the exponent (or parameter) of local constancy of $f$.

Example 10.17. (i) $\mathcal{D} \subset \mathcal{U}_{loc}$. (ii) Consider $f \in \mathcal{U}_{loc}$ such that $|f(y)|\, e^{-t\|y\|_p^\alpha} \in L^1$ for $t > 0$. Then

$$h(x) = \int_{\mathbb{Q}_p^n} \{f(x-y) - f(x)\}\, e^{-t\|y\|_p^\alpha} d^n y \in \mathcal{U}_{loc}.$$

Proposition 10.18. *$f \in \mathcal{U}_{loc}$ if and only if $f \in \mathcal{D}'$ and there is $k \in \mathbb{Z}$ such that $\mathcal{T}_x f = f$ for any $x \in p^k\mathbb{Z}_p^n$.*

Proof. Since $\mathcal{U}_{loc} \subset L^1_{loc}$, any $f \in \mathcal{U}_{loc}$ gives rise to an element from $\mathcal{D}'$ satisfying $(\mathcal{T}_x f, \varphi) = (f, \varphi)$ for any $x \in p^k\mathbb{Z}_p^n$.

Conversely, assume that $f \in \mathcal{D}'$, and $\mathcal{T}_x f = f$, for all $x \in p^k\mathbb{Z}_p^n$. We set

$$h(y) = p^{kn}\left(f, \mathcal{T}_y \Delta_{-k}\right).$$

Then $h(y) \in \mathcal{U}_{loc}$. Indeed, since for all $x \in p^k\mathbb{Z}_p^n$,

$$\left(\mathcal{T}_x h\right)(y) = p^{kn}\left(f, \mathcal{T}_{y-x}\Delta_{-k}\right) = p^{kn}\left(\mathcal{T}_x f, \mathcal{T}_y\Delta_{-k}\right) = p^{kn}\left(f, \mathcal{T}_y\Delta_{-k}\right) = h(y).$$

Now we show that $f = h$ in $\mathcal{D}'$. It is sufficient to verify

$$(h, \varphi) = (f, \varphi) \text{ i.e. } p^{kn}\int_{\mathbb{Q}_p^n} \left(\mathcal{T}_{-y} f, \Delta_{-k}\right) \varphi(y)\, d^n y = (f, \varphi),$$

for $\varphi$ equals to the characteristic function of a sufficiently small ball $J + p^l\mathbb{Z}_p^n$. Thus, we may assume that $J + p^l\mathbb{Z}_p^n \subset J + p^k\mathbb{Z}_p^n$, with $l \geq k$. Then

$$\left(h, 1_{J+p^l\mathbb{Z}_p^n}\right) = h(J)p^{-nl}$$

and consequently,

$$\begin{aligned}
\left(h, 1_{J+p^l\mathbb{Z}_p^n}\right) &= p^{kn}\int_{J+p^l\mathbb{Z}_p^n} \left(\mathcal{T}_{-y} f, \Delta_k\right) d^n y = p^{kn-nl}\int_{\mathbb{Z}_p^n} \left(\mathcal{T}_{-J-p^l u} f, \Delta_k\right) d^n u \\
&= p^{kn-nl}\int_{\mathbb{Z}_p^n} \left(\mathcal{T}_{-p^l u} f, \mathcal{T}_J\Delta_k\right) d^n u = p^{kn-nl}\int_{\mathbb{Z}_p^n} \left(f, \mathcal{T}_J\Delta_k\right) d^n u \\
&= p^{kn-nl}\left(f, \mathcal{T}_J\Delta_k\right) \\
&= h(J)p^{-nl}.
\end{aligned}$$

□

DEFINITION 10.19. For $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, $T * \varphi$ is defined by $\widehat{T * \varphi} = \widehat{\varphi}\, \widehat{T}$.

EXERCISE 10.20. Show that for $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, the map $T \rightarrow T * \varphi$ is a continuous map from $\mathcal{D}'$ into $\mathcal{D}'$. See Exercise 10.11.

EXERCISE 10.21. To show the following formulas for $\varphi$, $\psi \in \mathcal{D}$, $G \in \mathcal{D}'$ and $x \in \mathbb{Q}_p^n$:

(i) $\widetilde{(\mathcal{T}_{-x}\varphi)} = \mathcal{T}_x\widetilde{\varphi}$;
(ii) $\widetilde{(\mathcal{T}_{-x}G)} = \mathcal{T}_x\widetilde{G}$;
(iii) $\widetilde{(\varphi * \psi)} = \widetilde{\varphi} * \widetilde{\psi}$;
(iv) $(\widetilde{\widetilde{\varphi} * \psi)} = \varphi * \widetilde{\psi}$.

PROPOSITION 10.22 ([**23**, Theorem 3.15]). *The following three characterizations of $\varphi * G \in \mathcal{D}'$ for $\varphi \in \mathcal{D}$, $G \in \mathcal{D}'$ are equivalent:*
(i) $\widehat{\varphi * G} = \widehat{\varphi}\, \widehat{G}$*;*
(ii) $(\varphi * G, \psi) = (G, \widetilde{\varphi} * \psi)$ *for* $\psi \in \mathcal{D}$*;*
(iii) $\varphi * G$ *belongs to* $\mathcal{U}_{loc}$ *and agrees with function*

$$g(x) = (G, T_x\widetilde{\varphi}) = (G(y), \varphi\,(x-y)).$$

EXERCISE 10.23. Assume that $\varphi \in \mathcal{U}_{loc}$ such that

$$\int_{\|y\|_p > 1} \|y\|_p^{\alpha-n}\, |\varphi(y)\,|\, d^n y < \infty.$$

Then $k_\alpha * \varphi$ is a function given by

$$\begin{aligned}(k_\alpha * \varphi)\,(x) = \frac{1-p^{-n}}{1-p^{\alpha-n}}\varphi(x) &+ \frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\int_{\|y\|_p > 1} \|y\|_p^{\alpha-n}\varphi(x-y)\, d^n y \\ &+ \frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\int_{\|y\|_p \leq 1} \|y\|_p^{\alpha-n}(\varphi(x-y)-\varphi(x))\, d^n y.\end{aligned}$$

**Hint**. Recall that $\varphi$ is necessarily continuous.

EXERCISE 10.24. Assume that $\operatorname{Re}(\alpha) > 0$, then for $\varphi \in \mathcal{D}$, following formulas hold true:

$$\begin{aligned}(k_{-\alpha} * \varphi)\,(x) &= \frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n} (\varphi(x-y)-\varphi(x))\,\frac{d^n y}{\|y\|_p^{\alpha+n}} \\ &= \frac{1-p^{\alpha}}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n} (\varphi(y)-\varphi(x))\,\frac{d^n x}{\|y-x\|_p^{\alpha+n}},\end{aligned}$$

and

$$(k_\alpha * \varphi)\,(x) = \frac{1-p^{-\alpha}}{1-p^{\alpha-n}}\int_{\mathbb{Q}_p^n} \|y\|_p^{\alpha-n}\varphi(x-y)\, d^n y, \text{ for } \alpha \in \mathbb{C}\smallsetminus\left\{n + \frac{2\pi\sqrt{-1}}{\ln p}\mathbb{Z}\right\}.$$

EXERCISE 10.25. For $f \in \mathcal{U}_{loc}$, $G \in \mathcal{D}'$, we define $fG \in \mathcal{D}'$ as $(fG, \varphi) = (G, f\varphi)$ for $\varphi \in \mathcal{D}$. Show that $G \rightarrow fG$ is a continuous map of $\mathcal{D}'$ into $\mathcal{D}'$.

DEFINITION 10.26. A distribution $G \in \mathcal{D}'$ has compact support if there is a $k \in \mathbb{Z}$ such that $\Delta_k G = G$.

We denote by $\mathcal{D}'_{comp}$ the $\mathbb{C}$-vector space of distributions with compact support.

THEOREM 10.27. *$f \in \mathcal{U}_{loc}$ if and only if $\widehat{f}$ has compact support. In particular, if $f \in \mathcal{D}'$ then $\mathcal{T}_x f = f$ for all $x \in p^k \mathbb{Z}_p^n$ if and only if $\Delta_k \widehat{f} = \widehat{f}$.*

PROOF. Assume that $f \in \mathcal{U}_{loc}$ and that $\mathcal{T}_x f = f$ (i.e. $f\mid_{x+p^k\mathbb{Z}_p^n} \equiv f(x)$). By using Proposition 10.22-(iii), we have

$$\mathcal{F}^{-1}\left(\Delta_k \widehat{f}\right) = \mathcal{F}^{-1}(\Delta_k) * f = \mathcal{F}(\Delta_k) * f = \delta_k * f$$
$$= p^{nk} \int_{\|x-y\|_p \leq p^{-k}} f(y) d^n y = p^{nk} \int_{x+p^k\mathbb{Z}_p^n} f(y) d^n y = p^{nk} f(x) \int_{x+p^k\mathbb{Z}_p^n} d^n y = f(x),$$

i.e. $\Delta_k \widehat{f} = \widehat{f}$.

Now suppose that $\widehat{f} \in \mathcal{D}'$ satisfying $\Delta_k \widehat{f} = \widehat{f}$. Then for any $\varphi \in \mathcal{D}$ and for all $x \in p^k \mathbb{Z}_p^n$,

$$(\mathcal{T}_x f, \varphi) = (f, \mathcal{T}_{-x}\varphi) = \left(\mathcal{F}(f), \mathcal{F}^{-1}(\mathcal{T}_{-x}\varphi)\right)$$
$$= \left(\mathcal{F}(f)(\xi), \chi_p(x \cdot \xi) \mathcal{F}^{-1}(\varphi)(\xi)\right) = \left(\Delta_k(\xi) \widehat{f}(\xi), \chi_p(x \cdot \xi) \mathcal{F}^{-1}(\varphi)(\xi)\right)$$
$$= \left(\widehat{f}(\xi), \Delta_k(\xi) \chi_p(x \cdot \xi) \mathcal{F}^{-1}(\varphi)(\xi)\right) = \left(\widehat{f}, \mathcal{F}^{-1}(\varphi)\right) = (f, \varphi).$$

□

DEFINITION 10.28. Given $F \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ and $G \in \mathcal{D}'\left(\mathbb{Q}_p^m\right)$, their *direct product* $F \times G$ is defined by the formula

$$(F(x) \times G(y), \varphi(x,y)) = (F(x), (G(y), \varphi(x,y))) \text{ for } \varphi(x,y) \in \mathcal{D}\left(\mathbb{Q}_p^{n+m}\right).$$

The direct product of distributions is well defined due to the fact that any $\varphi(x,y) \in \mathcal{D}\left(\mathbb{Q}_p^{n+m}\right)$ can be expressed as

$$\varphi(x,y) = \sum_{i=1}^{L} \theta_i(x) \psi_i(y) \text{ where } \theta_i(x) \in \mathcal{D}\left(\mathbb{Q}_p^n\right), \psi_i(y) \in \mathcal{D}\left(\mathbb{Q}_p^m\right).$$

Which in turn is a consequence of the fact that

$$B_r^l(a) = B_r(a_1) \times \cdots \times B_r(a_l)$$

for any $l \in \mathbb{N} \smallsetminus \{0\}$, $a \in \mathbb{Q}_p^l$ and $r \in \mathbb{Z}$. Furthermore, the direct product is commutative: $F \times G = G \times F$. Notice that in the case $G = 1$, we have

$$\left(F(x), \int_{\mathbb{Q}_p^m} \varphi(x,y) d^m y\right) = \int_{\mathbb{Q}_p^m} (F(x), \varphi(x,y)) d^m y.$$

EXAMPLE 10.29. $\delta(x_1) \times \delta(x_2) \times \cdots \times \delta(x_n) = \delta(x)$, $x = (x_1, \ldots, x_n)$.

EXERCISE 10.30. Show that the direct product of distributions is continuous with respect to the joint factors.

PROPOSITION 10.31. *Let $f \in \mathcal{D}'$ with $\Delta_k f = f$. Then $\widehat{f}$ is a locally constant function given by*

$$\widehat{f}(\xi) = (f(\xi), \Delta_k(\xi) \chi_p(x \cdot \xi)).$$

Proof. By using the commutativity of the direct product of distributions with $G=1$ and Theorem 10.27,

$$\begin{aligned}\left(\widehat{f},\varphi\right) &= (f,\widehat{\varphi}) = (\Delta_k f,\widehat{\varphi}) = (f,\Delta_k\widehat{\varphi}) \\ &= \left(f\left(\xi\right), \int_{\mathbb{Q}_p^n} \Delta_k\left(\xi\right)\chi_p\left(x\cdot\xi\right)\varphi\left(x\right)d^n x\right) \\ &= \int_{\mathbb{Q}_p^n} \left(f\left(\xi\right),\Delta_k\left(\xi\right)\chi_p\left(x\cdot\xi\right)\right)\varphi\left(x\right)d^n x,\end{aligned}$$

$\square$

Definition 10.32. If $T$, $W\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ and $W$ has compact support, then $T*W\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is defined by

$$\widehat{T*W}=\widehat{T}\,\widehat{W}.$$

Exercise 10.33. If $W\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ has compact support, then the map $T\to T*W$ is a continuous map from $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ into $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$.

**Hint**. Notice that the following mappings are continuous:

$$\begin{array}{lll}\mathcal{D}'\left(\mathbb{Q}_p^n\right) & \to & \mathcal{D}'\left(\mathbb{Q}_p^n\right)\\ \widehat{T} & \to & \widehat{T}\,\widehat{W},\end{array}$$

and

$$\begin{array}{lll}\mathcal{D}'\left(\mathbb{Q}_p^n\right) & \xrightarrow{\text{mult. }\widehat{W}} & \mathcal{D}'\left(\mathbb{Q}_p^n\right)\\ \widehat{T} & \to & \widehat{T}\,\widehat{W}\\ \updownarrow\mathcal{F}^{-1} & & \updownarrow\mathcal{F}^{-1}\\ \mathcal{D}'\left(\mathbb{Q}_p^n\right) & \xrightarrow{*W} & \mathcal{D}'\left(\mathbb{Q}_p^n\right).\\ T & \to & T*W\end{array}$$

Theorem 10.34. *(i) Let $T_1,\ldots,T_s\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ all but (at most) one being in $\mathcal{U}_{loc}$. Then $T_1\cdots T_s\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is well-defined as a commutative and associative product.*

*(ii) Let $W_1,\ldots,W_s\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ all but (at most) one being in $\mathcal{D}'_{comp}$. Then $W_1*\cdots*W_s\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is well-defined as a commutative and associative convolution product.*

Example 10.35. (i) Take $\phi\in\mathcal{D}$, then the associated distribution has compact support.

(ii) $\delta\in\mathcal{D}'$ has compact support (Why?).

(iii) Let $T\in\mathcal{D}'$, then $\delta*T=T$. Indeed,

$$\widehat{\delta*T}=\widehat{\delta}\,\widehat{T}=1\widehat{T}=\widehat{T}.$$

(iv) Let $T\in\mathcal{D}'$, then $\delta_k*T\in\mathcal{D}'$, and

$$\lim_{k\to\infty}\delta_k*T=T\text{ in }\mathcal{D}'.$$

Indeed,

$$\lim_{k\to\infty}\widehat{\delta_k*T}=\lim_{k\to\infty}\widehat{\delta_k}\widehat{T}=\lim_{k\to\infty}\Delta_k\widehat{T}=\widehat{T}\text{ in }\mathcal{D}'.$$

Theorem 10.36. *$\mathcal{D}$ is dense in $\mathcal{D}'$. More precisely, every distribution is the weak limit of a sequence of test functions.*

PROOF. Take $T \in \mathcal{D}'$, then $\delta_k * T \in \mathcal{U}_{loc}$, $\Delta_l(\delta_k * T) \in \mathcal{D}$ for any $l$, $k$. We now show that

$$\lim_{l\to\infty}\left(\lim_{k\to\infty}\left(\Delta_l\widehat{(\delta_k * T)}\right)\right) = \widehat{T} \text{ in } \mathcal{D}'.$$

Indeed, $\Delta_l\widehat{(\delta_k * T)} = \delta_l * \widehat{(\delta_k * T)} = \delta_l * \Delta_k\widehat{T}$. For a fixed $\Delta_k$, the map

$$\begin{array}{ccc} \mathcal{D}' & \to & \mathcal{D}' \\ G & \to & \delta_l * G \end{array}$$

is continuous. Then $\lim_{l\to\infty}\delta_l * \Delta_k\widehat{T} = \delta * \Delta_k\widehat{T} = \Delta_k\widehat{T}$. Now the map

$$\begin{array}{ccc} \mathcal{D}' & \to & \mathcal{D}' \\ G & \to & \Delta_k G \end{array}$$

is continuous. Consequently $\lim_{k\to\infty}\Delta_k\widehat{T} = \widehat{T}$, and

$$\lim_{l\to\infty}\left(\lim_{k\to\infty}\left(\Delta_l\widehat{(\delta_k * T)}\right)\right) = \lim_{l\to\infty}\delta_l * \widehat{T} = \widehat{T} \text{ in } \mathcal{D}'.$$

Now by using the fact that the Fourier transform is continuous in $\mathcal{D}'$, we have

$$\lim_{l\to\infty}\left(\widehat{\lim_{k\to\infty}\left(\Delta_l(\delta_k * T)\right)}\right) = \widehat{T}.$$

Finally, by using that the Fourier transform is an isomorphism in $\mathcal{D}'$,

$$\lim_{l\to\infty}\left(\lim_{k\to\infty}\left(\Delta_l(\delta_k * T)\right)\right) = T.$$

□

**10.1. Operations on distributions.** The multiplication and the convolution between arbitrary distributions is a subtle concept. Here review these operations following [**24**, Sections 6, 7].

DEFINITION 10.37. Given $F, G \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$, their convolution $F * G$ is defined by

$$(F * G, \varphi) = \lim_{k\to\infty}\left(F(x) \times G(y), \Delta_k(x)\varphi(x+y)\right) \tag{10.2}$$

if the limit exists for all $\varphi \in \mathcal{D}\left(\mathbb{Q}_p^n\right)$.

THEOREM 10.38. *(i) If $F * G$ exists, then $G * F$ exists and $F * G = G * F$. (ii) If $F, G \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ and $suppF \subset B_N^n$ (i.e. if $\Delta_N F = F$), then the convolution $F * G$ exists, and it is given by the formula*

$$(F * G, \varphi) = \left(F(x) \times G(y), \Delta_N(x)\varphi(x+y)\right) \text{ for } \varphi \in \mathcal{D}\left(\mathbb{Q}_p^n\right).$$

*(iii) In the case in which $F = \psi \in \mathcal{D}\left(\mathbb{Q}_p^n\right)$, $\psi * G$ is a locally constant function given by*

$$(\psi * G)(z) = \left(G(y), \psi(z-y)\right).$$

PROOF. (i) We notice that without loss of generality in (10.2), we can assume that $\varphi(x) = \Omega\left(p^{-l}\|x-j\|_p\right)$ for some integer $l$ and some $j \in \mathbb{Q}_p^n$.

Claim. For $k$ sufficiently large, we have $\Omega\left(p^{-k}\left\|x-j\right\|_p\right)=\Omega\left(p^{-k}\left\|x\right\|_p\right)$ (i.e. $\Delta_k\left(x-j\right)=\Delta_k(x)$) and

(10.3)
$$\Omega\left(p^{-k}\left\|x-j\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)=\Omega\left(p^{-k}\left\|y\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right).$$

Indeed,

$$\begin{aligned}(F*G,\varphi)&=\lim_{k\to\infty}\left(F\left(x\right)\times G\left(y\right),\Omega\left(p^{-k}\left\|x\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)\right)\\&=\lim_{k\to\infty}\left(F\left(x\right)\times G\left(y\right),\Omega\left(p^{-k}\left\|x-j\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)\right)\\&=\lim_{k\to\infty}\left(G\left(y\right)\times F\left(x\right),\Omega\left(p^{-k}\left\|y\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)\right)\\&=\lim_{k\to\infty}\left(G\left(y\right)\times F\left(x\right),\Delta_k(y)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)\right)=\left(G*F,\varphi\right).\end{aligned}$$

In order to establish (10.3), we may assume without loss of generality that $k>l$. Now $\Omega\left(p^{-k}\left\|x-j\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)=1$ if and only if

$$\left\|x-j\right\|_p\leq p^k\quad\text{and}\quad\left\|x+y-j\right\|\leq p^l. \tag{10.4}$$

Now, if $\left\|y\right\|_p>p^k$, then by the ultrametric property of $\left\|\cdot\right\|_p$, $\left\|x+y-j\right\|=\left\|y\right\|_p\leq p^l$. Which is impossible. Consequently (10.4) implies that

$$\left\|y\right\|_p\leq p^k\quad\text{and}\quad\left\|x+y-j\right\|\leq p^l. \tag{10.5}$$

Now $\Omega\left(p^{-k}\left\|y\right\|_p\right)\Omega\left(p^{-l}\left\|x+y-j\right\|_p\right)$ if and only if (10.5) holds true. If $\left\|x-j\right\|_p>p^k$, then since $\left\|y\right\|_p\leq p^k$, $\left\|x+y-j\right\|=\left\|x-j\right\|_p\leq p^l$. Which is impossible. Then (10.4) holds true.

(ii) It follows from

$$\begin{aligned}(F*G,\varphi)&=\lim_{k\to\infty}\left(F\left(x\right)\times G\left(y\right),\Delta_k\left(x\right)\varphi\left(x+y\right)\right)\\&=\lim_{k\to\infty}\left(\Delta_N\left(x\right)F\left(x\right)\times G\left(y\right),\Delta_k\left(x\right)\varphi\left(x+y\right)\right)\\&=\lim_{k\to\infty}\left(F\left(x\right)\times G\left(y\right),\Delta_k\left(x\right)\Delta_N\left(x\right)\varphi\left(x+y\right)\right)\\&=\left(F\left(x\right)\times G\left(y\right),\Delta_N\left(x\right)\varphi\left(x+y\right)\right).\end{aligned}$$

(iii) By using (ii), with $\Delta_N\psi=\psi$, we have

$$\begin{aligned}(\psi*G,\varphi)&=\left(\psi\left(x\right)\times G\left(y\right),\Delta_N\left(x\right)\varphi\left(x+y\right)\right)\\&=\int_{\mathbb{Q}_p^n}\left(G\left(y\right),\Delta_N\left(x\right)\varphi\left(x+y\right)\right)\psi\left(x\right)d^nx\\&=\left(G(y),\int_{\mathbb{Q}_p^n}\psi\left(x\right)\varphi\left(x+y\right)d^nx\right)=\left(G(y),\int_{\mathbb{Q}_p^n}\psi\left(z-y\right)\varphi\left(z\right)d^nz\right)\\&=\int_{\mathbb{Q}_p^n}\left(G(y),\psi\left(z-y\right)\right)\varphi\left(z\right)d^nz.\end{aligned}$$

The fact that $(G(y),\psi\left(y-z\right))\in\mathcal{U}_{loc}$ was established in Proposition 10.22. □

Definition 10.39. Given $T,G\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$, their product $T\cdot G$ is defined by

$$\left(FT\cdot G,\varphi\right)=\lim_{k\to\infty}\left(G,\left(T*\delta_k\right)\varphi\right)$$

if the limit exists for all $\varphi \in \mathcal{D}\left(\mathbb{Q}_p^n\right)$. Alternatively,

$$T \cdot G = \lim_{k\to\infty} \left(T * \delta_k\right) G \text{ in } \mathcal{D}'\left(\mathbb{Q}_p^n\right).$$

THEOREM 10.40. *The existence of the product $T \cdot G$ is equivalent to the existence of $\mathcal{F}[T] * \mathcal{F}[G]$. In addition, $\mathcal{F}[T \cdot G] = \mathcal{F}[T] * \mathcal{F}[G]$ and $\mathcal{F}[T * G] = \mathcal{F}[T] \cdot \mathcal{F}[G]$. If the product $T \cdot G$ exists then the product $G \cdot T$ exists and they are equal.*

PROOF. The result follows from the following facts: $\left(T * \delta_k\right) G \in \mathcal{D}'$ (because $T * \delta_k \in \mathcal{U}_{loc}$), $\Delta_k \widehat{T} * \widehat{G}$ (because $\Delta_k \widehat{T}$ has compact support), and that

$$\mathcal{F}\left(\left(T * \delta_k\right) G\right) = \Delta_k \widehat{T} * \widehat{G}.$$

By taking the limit $k \to \infty$, and using the continuity of the Fourier transform, we have

$$\lim_{k\to\infty} \mathcal{F}\left(\left(T * \delta_k\right) G\right) = \mathcal{F}\left(\lim_{k\to\infty} \left(T * \delta_k\right) G\right) = \lim_{k\to\infty} \Delta_k \widehat{T} * \widehat{G}.$$

Which implies that the existence of the product $T \cdot G$ is equivalent to the existence of $\mathcal{F}[T] * \mathcal{F}[G]$. Finally, if $T \cdot G$ exists, by uisng the fcat that the convolution of dsitributions is commutative, we have

$$\widehat{T \cdot G} = \widehat{T} * \widehat{G} = \widehat{G} * \widehat{T} = \widehat{G \cdot T}.$$

$\square$

EXERCISE 10.41. Given $f(x)$ a non-constant polynomial in $\mathbb{Q}_p[x_1, \ldots, x_n]$, the complex power attached to it (also called the Igusa local zeta function of $f$) is the distribution $|f|_p^s$, $s \in \mathbb{C}$, defined as

$$\left(|f|_p^s, \varphi\right) = \int_{\mathbb{Q}_p \smallsetminus f^{-1}(0)} |f(x)|_p^s \, \varphi(x) \, d^n x, \text{ for } \operatorname{Re}(s) > 0,$$

where $a^s := e^{s \ln a}$ for $a > 0$. Show that for $\operatorname{Re}(\alpha)$, $\operatorname{Re}(\beta) > 0$,

$$|f|_p^\alpha \, |f|_p^\beta = |f|_p^{\alpha+\beta} \text{ in } \mathcal{D}'.$$

**Hint**. If $f(x) \neq 0$, then for a $k$ sufficiently large,

$$|f(x+y)|_p = |f(x)|_p \text{ for any } y \in p^k \mathbb{Z}_p^n.$$

This fact follows from the Taylor formula for a polynomial.

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