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# The Cartesian analytical solutions for the $N$-dimensional compressible Navier-Stokes equations with density-dependent viscosity

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**Abstract**

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In this paper, we prove the existence of general Cartesian vector solutions $\mathbf{u} = \mathbf{b}(t) + A(t)\mathbf{x}$ for the $N$-dimensional compressible Navier-Stokes equations with density-dependent viscosity, based on the matrix and curve integration theory. Two exact solutions are obtained by solving the reduced systems.

**MSC:** 35Q31, 35C05, 76B03, 76M60

**Key Words:** Compressible Navier-Stokes equations with density-dependent viscosity, Cartesian solutions, symmetric and anti-symmetric matrix, quadratic form, curve integration.

## 1 Introduction

We consider the initial value problem for a general $N$-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients [1]-[11]

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\rho[\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u}] - \mathrm{div}(h(\rho)\triangledown\mathbf{u}) - \triangledown(g(\rho)\mathrm{div}(\mathbf{u})) + \triangledown p = \mathbf{0}, \tag{1.2}$$

where $t \in (0, +\infty)$ is the time; $\mathbf{x} = (x_1, x_2, \cdots, x_N)^T \in R^N$ is the spatial coordinate; while $\mathbf{u} = (u_1, u_2, \cdots, u_N)^T$, $\rho(\mathbf{x}, t)$, $p(\mathbf{x}, t) = k\rho^\gamma$ denote respectively the velocity, density, and pressure of the fluid at a position $\mathbf{x}$. The $h(\rho)$ and $g(\rho)$ are Lame viscosity coefficients fulfilling

$$h(\rho) > 0, \quad h(\rho) + Ng(\rho) \geq 0,$$

and particularly in this paper, we consider the following two functions:

$$h(\rho) = k_1\rho^\gamma, \quad g(\rho) = k_2\rho^\gamma. \tag{1.3}$$

The equations (1.1)-(1.2) with (1.3) contains many physical equations. For example, if $k_2 = 0$, it is reduced to the compressible Navier-Stokes equations with density-dependent [12]-[16]

$$\begin{aligned} &\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ &\rho[\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u}] - \mathrm{div}(h(\rho)\triangledown\mathbf{u}) + \triangledown p = \mathbf{0}. \end{aligned} \tag{1.4}$$

If $k_2 = 0$ and $h(\rho)$ is constant, the system (1.1)-(1.2) is reduced to the classical compressible Navier-Stokes equations [17]-[19]

$$\begin{aligned} &\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ &\rho[\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u}] + \triangledown p - \tilde{k}_1 \triangle \mathbf{u} = \mathbf{0}, \end{aligned} \tag{1.5}$$

which include the compressible Euler equations as a special case ($\tilde{k}_1 = 0$) [19]-[22]

$$\begin{aligned} &\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ &\rho[\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u}] + \triangledown p = \mathbf{0}. \end{aligned} \tag{1.6}$$

If $k_2 = 0$, $\rho$ is a constant, and the system (1.1)-(1.2) is reduced to the classical incompressible Navier-Stokes equations [23]-[27]

$$\begin{aligned} &\mathrm{div}(\mathbf{u}) = 0, \\ &\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u} + \triangledown p - k_1 \triangle \mathbf{u} = \mathbf{0}, \end{aligned} \tag{1.7}$$

which include incompressible Euler equations as a special case ($k_1 = 0$) [10], [28]-[32]

$$\begin{aligned} &\mathrm{div}(\mathbf{u}) = 0, \\ &\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u} + \triangledown p = \mathbf{0}. \end{aligned} \tag{1.8}$$

The Navier-Stokes equations play a very important role in fluids, oceanography and atmospheric dynamics. The system has been investigated extensively and intensively. There are much progress made on the local strong solutions and the global weak solutions, for example [1]–[8]. There are interesting papers on analytical solutions of the Navier-Stokes equations for the special functions $h(\rho)$ and $g(\rho)$ [16, 19, 9, 10]. However, these known analytical solutions are not explicit. Based on the new matrix theory and decomposition technique, An, Fan and Yuen proved the existence of the Cartesian solutions for the compressible Euler equations (1.6) [22]. Then Chow, Fan and Yuen further generalized to the damped Euler equations [33].

In section 2, we show that the compressible Navier-Stokes equations with density-dependent viscosity has the Cartesian solutions if $A$ fulfills appropriate matrix equations. By solving the reduced systems, two solvable cases are provided in sections 3 and 4.

## 2 Existence of the Cartesian solutions

Before we construct exact solutions, we can simplify equation (1.2) into an easy form.

With the $\gamma$-law, we may consider the case where the density $\rho$ and pressure $p$ satisfy a relation

$$p(\rho) = k\rho^{\gamma}, \tag{2.1}$$

with $k > 0$. We may take $k = 1$ without loss of generality by using a simple transformation $\rho \to k^{-1/\gamma}\rho,\ k_1 \to k_1 k^{-1/\gamma},\ k_2 \to k_2 k^{-1/\gamma}$.

Substituting (1.3) and (2.1) into the equation (1.2), we obtain that

$$\begin{aligned} &\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - k_1\gamma\rho^{\gamma-2}(\nabla\rho\cdot\nabla)\mathbf{u} - k_1\rho^{\gamma-1}\Delta\mathbf{u} \\ &- k_2\gamma\rho^{\gamma-2}\mathrm{div}(\mathbf{u})\nabla\rho - k_2\rho^{\gamma-1}\nabla(\mathrm{div}(\mathbf{u})) + \gamma\rho^{\gamma-2}\nabla\rho = \mathbf{0}. \end{aligned} \tag{2.2}$$

Making a transformation

$$\bar{p} = \begin{cases} \ln\rho, & for \quad \gamma = 1, \\ \frac{\gamma}{\gamma-1}\rho^{\gamma-1}, & for \quad \gamma \neq 1, \end{cases} \iff \rho = \begin{cases} \exp(\bar{p}), & for \quad \gamma = 1, \\ \mu\bar{p}^{\frac{1}{\gamma-1}}, & for \quad \gamma \neq 1, \end{cases} \tag{2.3}$$

with $\mu = (\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}$, we can obtain equations (1.1) and (1.2) in the form

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{2.4}$$

$$\begin{aligned} &\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - k_1(\nabla\bar{p}\cdot\nabla)\mathbf{u} - \frac{k_1(\gamma-1)}{\gamma}\bar{p}\Delta\mathbf{u} \\ &- k_2\mathrm{div}(\mathbf{u})\nabla\bar{p} - \frac{k_2(\gamma-1)}{\gamma}\bar{p}\nabla(\mathrm{div}(\mathbf{u})) + \nabla\bar{p} = \mathbf{0}. \end{aligned} \tag{2.5}$$

Here for an appropriate function $\bar{p}(\mathbf{x})$, we are eager to find a sufficient condition on the existence of the following Cartesian solutions for the compressible Navier-Stokes equations (2.4)-(2.5)

$$\mathbf{u} = \mathbf{b}(t) + A\mathbf{x},$$

where the $N$-dimensional vector function $\mathbf{b}(t)$ and $N \times N$ matrix function $A$ are defined by

$$\mathbf{b}(t) = (b_1(t), b_2(t), \cdots, b_N(t))^T, \quad A = (a_{ij}(t))_{N\times N},$$

and elements $b_i(t)$ and $a_{ij}(t)$ $(i,j=1,2,\cdots,N)$ are functions about $t$. Due to the equivalent relation (2.3) between $\bar{p}$ and $\rho$, we mainly deal with $\bar{p}$ when solving the compressible equations (2.4)-(2.5).

**Theorem 1** *Let*

$$B = I - k_2\mathrm{tr}(A)I - k_1 A, \tag{2.6}$$

$$C = B^{-1}(A_t + A^2)/2, \tag{2.7}$$

*where $I$ denotes an unitary matrix, $B^{-1}$ denotes an inverse matrix of $B$. If $A$ and $C$ satisfy the following matrix differential equations*

$$C^T = C, \tag{2.8}$$

$$C_t + (\gamma-1)\mathrm{tr}(A)C + CA + A^T C = 0, \tag{2.9}$$

*then the compressible Navier-Stokes equations (2.4)-(2.5) have explicit solutions in the form*

$$\mathbf{u} = \mathbf{b}(t) + A\mathbf{x}, \tag{2.10}$$

$$\bar{p} = -\mathbf{x}^T(B^{-1}\mathbf{b}_t + B^{-1}A\mathbf{b}) - \mathbf{x}^T C\mathbf{x} + c(t), \tag{2.11}$$

*where the vector function $\mathbf{b}(t)$ and scalar function $c(t)$ satisfy ordinary differential equations*

$$(B^{-1}\mathbf{b}_t + B^{-1}A\mathbf{b})_t + [(\gamma-1)\mathrm{tr}(A)I + A^T](B^{-1}\mathbf{b}_t + B^{-1}A\mathbf{b}) + 2C\mathbf{b} = \mathbf{0}, \tag{2.12}$$

$$c_t + (\gamma-1)\mathrm{tr}(A)c - \mathbf{b}^T(B^{-1}\mathbf{b}_t + B^{-1}A\mathbf{b}) = 0. \tag{2.13}$$

*Proof.* We first prove how to get the solution (2.11) through solving the equation

(2.5). Substituting (2.10) into (2.5) produces

$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\triangledown)\mathbf{u} - k_1(\triangledown\bar{p}\cdot\triangledown)\mathbf{u} - \frac{k_1(\gamma-1)}{\gamma}\bar{p}\Delta\mathbf{u} - k_2\mathrm{div}(\mathbf{u})\triangledown\bar{p}\\
&- ck_2(\gamma-1)\gamma\bar{p}\triangledown(\mathrm{div}(\mathbf{u})) + \triangledown\bar{p}\\
&= \mathbf{b}_t + A_t\mathbf{x} + [(\mathbf{b}+A\mathbf{x})\cdot\triangledown](\mathbf{b}+A\mathbf{x}) - k_1(\triangledown\bar{p}\cdot\triangledown)(\mathbf{b}+A\mathbf{x}) - \frac{k_1(\gamma-1)}{\gamma}\bar{p}\Delta(\mathbf{b}+A\mathbf{x})\\
&- k_2\mathrm{div}(\mathbf{b}+A\mathbf{x})\triangledown\bar{p} - \frac{k_2(\gamma-1)}{\gamma}\bar{p}\triangledown(\mathrm{div}(\mathbf{b}+A\mathbf{x})) + \triangledown\bar{p}\\
&= \mathbf{b}_t + A\mathbf{b} + (A_t + A^2)\mathbf{x} + (I - k_2\mathrm{tr}(A)I - k_1A)\triangledown\bar{p} = \mathbf{0}.
\end{aligned} \tag{2.14}$$

By using (2.6), we get

$$-B^{-1}(\mathbf{b}_t + A\mathbf{b}) - B^{-1}(A_t + A^2)\mathbf{x} = \triangledown\bar{p}.$$

For convenience, we introduce an auxiliary matrix

$$B^{-1} = (b_{ij})_{N\times N}, \quad C = \frac{1}{2}B^{-1}(A_t + A^2) = (c_{ij})_{N\times N}, \tag{2.15}$$

where

$$c_{ij} = \frac{1}{2}\left(\sum_{k=1}^{N} b_{ik}a_{kj,t} + \sum_{k=1}^{N}\sum_{l=1}^{N} a_{kl}a_{lj}b_{ik}\right), \tag{2.16}$$

and rewrite the equation (2.14) into the form of components

$$\begin{aligned}
Q_i(x_1,\cdots,x_N) &\equiv -\sum_{k=1}^{N} b_{ik}b_{kt} - \sum_{k=1}^{N}\sum_{j=1}^{N} b_{ij}a_{jk}b_k - 2\sum_{k=1}^{N} c_{ik}x_k\\
&= \frac{\partial p}{\partial x_i}, \quad i = 1,2,\cdots,N.
\end{aligned} \tag{2.17}$$

In order to solve $\bar{p}(\mathbf{x})$ from (2.17), these $N$ equations should be compatible with each other, that is, the vector functions $(Q_1, Q_2, \cdots, Q_N)$ should constitute a potential field of $\bar{p}(\mathbf{x})$, whose sufficient and necessary conditions are

$$\frac{\partial Q_j(x_1,\cdots,x_N)}{\partial x_i} = \frac{\partial Q_i(x_1,\cdots,x_N)}{\partial x_j}, \quad i,j = 1,2,\cdots,N, \tag{2.18}$$

which hold if and only if

$$c_{ji} = c_{ij}, \quad i,j = 1,2,\cdots,N,$$

which implies that $C$ is a symmetric matrix, that is, the condition (2.8) is satisfied.

It follows from the condition (2.18), the function $\bar{p}(\mathbf{x})$ is a complete differential, that is,

$$d\bar{p}(\mathbf{x}) = \sum_{i=1}^{N} \frac{\partial \bar{p}(\mathbf{x})}{\partial x_i} dx_i = \sum_{i=1}^{N} Q_i(x_1, \cdots, x_N) dx_i. \tag{2.19}$$

Therefore the second kind of curvilinear integral of $p(x)$ is independent of its integration route. In this way, we may take a special integration route and directly obtain

$$\begin{aligned}
p(\mathbf{x}) &= \sum_{i=1}^{N} \int_{(0,0,\cdots,0)}^{(x_1,x_2,\cdots,x_N)} Q_i(x_1, x_2, \cdots, x_N) dx_i \\
&= \int_0^{x_1} Q_1(x_1, 0, \cdots, 0) dx_1, + \int_0^{x_2} Q_2(x_1, x_2, 0, \cdots, 0) dx_2 \\
&+ \cdots + \int_0^{x_N} Q_N(x_1, x_2, \cdots, x_N) dx_N \\
&= -\sum_{i=1}^{N} \left[ \sum_{k=1}^{N} b_{ik} b_{kt} - \sum_{k=1}^{N} \sum_{j=1}^{N} b_{ij} a_{jk} b_k \right] x_i - \sum_{i=1}^{N} c_{ii} x_i^2 - 2 \sum_{i,k=1,\ i<k}^{N} c_{ik} x_i x_k + c(t) \\
&= -\mathbf{x}^T (B^{-1}\mathbf{b}_t + B^{-1} A \mathbf{b}) - \mathbf{x}^T C \mathbf{x} + c(t).
\end{aligned}$$

Next, we prove that the functions (2.10)-(2.11) satisfy the equation (2.4) under the conditions (2.12) and (2.13). For $\gamma > 1$, by using (2.9), (2.12) and (2.13), we have

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho \mathbf{u}) = \rho_t + \rho \operatorname{tr}(A) + \mathbf{u} \cdot \nabla \rho \\
&= -\frac{\mu}{\gamma - 1} \bar{p}^{\frac{1}{\gamma-1}-1} \{ \mathbf{x}^T [C_t + (\gamma - 1)\operatorname{tr}(A) C + 2A^T C] \mathbf{x} \\
&+ \mathbf{x}^T (B^{-1}\mathbf{b}_t + B^{-1} A \mathbf{b})_t + [(\gamma - 1)\operatorname{tr}(A) I + A^T](B^{-1}\mathbf{b}_t + B^{-1} A \mathbf{b}) + 2C\mathbf{b}] \\
&- [c_t + (\gamma - 1)\operatorname{tr}(A) c - \mathbf{b}^T (B^{-1}\mathbf{b}_t + B^{-1} A \mathbf{b})] \} = 0,
\end{aligned} \tag{2.20}$$

where we have used the condition

$$\mathbf{x}^T [C_t + (\gamma - 1)\operatorname{tr}(A) C + 2A^T C] \mathbf{x} = 0,$$

which is equivalent to

$$[C_t + (\gamma - 1)\operatorname{tr}(A) C + 2A^T C]^T = -[C_t + (\gamma - 1)\operatorname{tr}(A) C + 2A^T C],$$

that is,

$$C_t + (\gamma - 1)\mathrm{tr}(A)C + CA + A^T C = 0.$$

The case with $\gamma = 1$ can be proved in a similar way, by replacing $\mu\bar{p}^{\frac{1}{\gamma-1}}$ by $\exp(\bar{p})$ in the proof procedure of equation (2.20). Thus, we proved the existence of the solutions (2.10)-(2.11) for the $N$-dimensional Navier-Stokes equations (2.4)-(2.5). □

The condition (2.9) is a complicated matrix differential equation with $N^2$ scalar equations. It is very difficult to obtain its general solutions of the reduced system. Therefore, some special techniques are needed to solve the reduced systems.

# 3 First reduction: constant matrix A

We can rewrite (2.9) in the form

$$C_t + (\gamma - 1)\mathrm{tr}(A)C + [C, A] + (A + A^T)C = 0, \tag{3.1}$$

where $[C, A] = CA - AC$ denotes the Lie bracket between $A$ and $C$. This form (3.1) makes us easily know how to find conditions on the matrix $A$.

**Theorem 2** *If $k_1 = 0$ and $A$ is an anti-symmetric constant matrix, then the compressible Navier-Stokes equations (1.1)-(1.2) admit a general explicit solution*

$$\mathbf{u} = \mathbf{b}(t) + A\mathbf{x}, \tag{3.2}$$

$$\bar{p} = -\mathbf{x}^T(B^{-1}\mathbf{b}_t + B^{-1}A\mathbf{b}) - \mathbf{x}^T C\mathbf{x} + c(t), \tag{3.3}$$

*where the vector function $\mathbf{b}(t)$ and scalar function $c(t)$ are given by*

$$\mathbf{b}(t) = \mathbf{b}_1 t, \tag{3.4}$$

$$c = \frac{1}{3}\mathbf{b}_1^T A\mathbf{b}_1 t^3 + \frac{1}{2}\mathbf{b}_1^T\mathbf{b}_1 t^2, \tag{3.5}$$

*where $\mathbf{b}_1$ is an arbitrary constant vector.*

*Proof.* We just need to verify that the conditions (2.8), (2.9) are satisfied under the Theorem 2.

If $A$ is an anti-symmetric constant matrix, then

$$A + A^T = 0, \quad \mathrm{tr}(A) = 0. \tag{3.6}$$

Again noting that $k_1 = 0$, from (2.6) and (2.7), we have

$$B = I, \quad C = \frac{1}{2}A^2, \tag{3.7}$$

which implies that the condition (2.8) is satisfied, that is, $C$ is a symmetric matrix such that

$$C^T = \frac{1}{2}A^T A^T = \frac{1}{2}(-A)(-A) = \frac{1}{2}A^2 = C. \tag{3.8}$$

The direct calculation shows that the Lie bracket between $A$ and $C$ is commutative:

$$[A, C] = AC - CA = 0. \tag{3.9}$$

Finally by using (3.1), (3.6) and (3.9), we have

$$C_t + (\gamma - 1)\mathrm{tr}(A)C + CA + A^T C = 0,$$

which implies that (2.9) is satisfied.

Making use of (3.6) and (3.7), we reduce equations (2.12) and (2.13) to

$$\mathbf{b}_{tt} = \mathbf{0}, \quad c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b}) = 0.$$

These two equations admit the solutions (3.4) and (3.5) respectively. □

We can obtain many special solutions for the compressible Navier-Stokes equations by Theorem 2 if we can solve the reduced system. Here, we can give an example.

**Example 1** *For the 3D compressible Navier-Stokes equations with $k_1 = 0$, by taking*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix},$$

$$C = \frac{1}{2}B^{-1}(A_t + A^2) = \frac{1}{2}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

*we get a solution*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} t \\ t \\ t \end{pmatrix},$$

$$\bar{p} = -\mathbf{x}^T(\mathbf{b}_t + A\mathbf{b}) - \mathbf{x}^T C\mathbf{x} + c(t)$$

$$= -\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} t \\ t \\ t \end{pmatrix}\right)$$

$$- \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \frac{3}{2}t^2$$

$$= -\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -2x_1 + x_2 + x_3 & x_1 - 2x_2 + x_3 & x_1 + x_2 - 2x_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \frac{3}{2}t^2$$

$$= -x_1 - x_2 - x_3 - \frac{1}{2}\left[(-2x_1 + x_2 + x_3)x_1 + (x_1 - 2x_2 + x_3)x_2 + (x_1 + x_2 - 2x_3)x_3\right] + \frac{3}{2}t^2$$

$$= x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_1x_3 + x_2x_3) - (x_1 + x_2 + x_3) + \frac{3}{2}t^2.$$

# 4 Second reduction: time-dependent matrix A

In the following section, we can consider another interesting exact solution of Theorem 1.

**Theorem 3** *If*

$$A = f(t)I, \tag{4.1}$$

*then $C$ is a symmetric matrix and matrix differential equation (2.9) reduces to*

$$(1 - Nk_2 f - k_1 f)f_{tt} + 2ff_t + [N(\gamma - 1) + 2](ff_t + f^3) + (Nk_2 + k_1)(f_t^2 - f^2 f_t)$$
$$- (Nk_2 + k_1)[N(\gamma - 1) + 2](f^2 f_t + f^4) = 0. \tag{4.2}$$

*In this way, we get a general solution*

$$\mathbf{u} = \mathbf{b}(t) + f\mathbf{x}, \tag{4.3}$$

$$\bar{p} = -\frac{f_t + f^2}{2(1 - Nk_2 f - k_1 f)}(x_1^2 + \cdots + x_N^2) - \frac{\mathbf{x}^T(\mathbf{b}_t + f\mathbf{b})}{1 - Nk_2 f - k_1 f} + c(t), \tag{4.4}$$

*where $\mathbf{b}(t)$ and $c(t)$ satisfy*

$$\left(\frac{\mathbf{b}_t + f\mathbf{b}}{1 - Nk_2 f - k_1 f}\right)_t + [N(\gamma - 1) + 1]f\frac{\mathbf{b}_t + f\mathbf{b}}{1 - Nk_2 f - k_1 f} + \frac{f_t + f^2}{1 - Nk_2 f - k_1 f}\mathbf{b} = 0, \tag{4.5}$$

$$c_t + N(\gamma - 1)fc - \mathbf{b}^T\frac{\mathbf{b}_t + f\mathbf{b}}{1 - Nk_2 f - k_1 f} = 0, \tag{4.6}$$

*while $f$ satisfies the equation (4.2).*

*Proof.* From (2.6), (2.7) and (4.1), we have

$$B = (1 - Nk_2 f - k_1 f)I,$$
$$C = \frac{1}{2}B^{-1}(A_t + A^2) = \frac{f_t + f^2}{2(1 - Nk_2 f - k_1 f)}I. \tag{4.7}$$

Substituting (4.1) and (4.7) into (2.9), we get (4.2).

The original solutions (2.10) and (2.11) become (4.3) and (4.4). The original equations (2.12) and (2.13) reduce to (4.5) and (4.6). □

The differential equations (4.2) is still complicated and hard to solve. We continue to discuss its solvability.

In the case $Nk_2 + k_1 = 0$, the system (4.2) can be reduced to

$$f_{tt} + [N(\gamma-1)+4]ff_t + [N(\gamma-1)+2]f^3 = 0,$$

which has an explicit solution

$$f = \alpha t^{-1}, \tag{4.8}$$

where

$$\alpha = \frac{2}{N(\gamma-1)+2}.$$

This kind of exact solutions have no limit on the dimensional number $N$ and the density power $\gamma$.

We obtain from (4.5) that

$$(\mathbf{b}_t + f\mathbf{b})_t + [N(\gamma-1)+1]f(\mathbf{b}_t + f\mathbf{b}) + (f_t + f^2)\mathbf{b} = \mathbf{0}. \tag{4.9}$$

We have

$$\mathbf{b}(t) = \mathbf{p}t^{-1}, \tag{4.10}$$

where $\mathbf{p}$ is a vector constant.

Solving (4.6) gives

$$c(t) = e^{-\int N(\gamma-1)f dt}\left(\int \mathbf{b}^T(\mathbf{b}_t + f\mathbf{b})e^{\int N(\gamma-1)f dt}dt + \beta\right).$$

Let us see an illustrative example.

**Example 2** *For the 3D compressible Navier-Stokes equations, by taking $k_1$ and $k_2$ satisfying $k_1 + 3k_2 = 0$,*

$$A = \frac{2t^{-1}}{3(\gamma-1)+2}I, \quad B = I, \quad C = \frac{-3(\gamma-1)t^{-2}}{(3(\gamma-1)+2)^2}I,$$

*by (4.10), we can take*

$$\mathbf{b}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t^{-1}. \tag{4.11}$$

*The equation (4.6) becomes*

$$c_t + \frac{6(\gamma-1)t^{-1}}{3(\gamma-1)+2}c = \frac{-9(\gamma-1)t^{-3}}{3(\gamma-1)+2} = \frac{-9(\gamma-1)t^{-3}}{3\gamma-1}, \tag{4.12}$$

*which has a solution*

$$c = \frac{9(\gamma-1)}{4}t^{-2}$$

*via checking*

$$\begin{aligned}
& c_t + \frac{6(\gamma-1)}{3(\gamma-1)+2}t^{-1}c \\
&= \left(\frac{9(\gamma-1)}{4}t^{-2}\right)_t + \frac{6(\gamma-1)t^{-1}}{3(\gamma-1)+2}\cdot\left(\frac{9(\gamma-1)}{4}t^{-2}\right) \\
&= -\frac{9(\gamma-1)}{2}t^{-3} + \frac{6(\gamma-1)}{3\gamma-1}\frac{9(\gamma-1)}{4}t^{-3} \\
&= -\frac{9(\gamma-1)}{2}t^{-3} + \frac{27(\gamma-1)^2}{2(3\gamma-1)}t^{-3} \\
&= -\frac{9(\gamma-1)}{2}\frac{(3\gamma-1)}{3\gamma-1}t^{-3} + \frac{27(\gamma-1)^2}{2(3\gamma-1)}t^{-3} \\
&= \frac{-9(3\gamma^2-\gamma-3\gamma+1)}{2(3\gamma-1)}t^{-3} + \frac{27(\gamma^2-2\gamma+1)}{2(3\gamma-1)}t^{-3} \\
&= \frac{-27\gamma^2+36\gamma-9+27\gamma^2-54\gamma+27}{2(3\gamma-1)}t^{-3} \\
&= \frac{-18\gamma+18}{2(3\gamma-1)}t^{-3} \\
&= \frac{-9(\gamma-1)}{3\gamma-1}t^{-3}.
\end{aligned}$$

*Finally, according to (4.3) and (4.4), we can directly obtain a solution*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{2t^{-1}}{3(\gamma-1)+2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t^{-1},$$

$$\begin{aligned}
\bar{p} &= -\frac{f_t + f^2}{2(1-3k_2 f - k_1 f)}(x_1^2+x_2^2+x_3^2) - \mathbf{x}^T(\mathbf{b}_t + f\mathbf{b}) + c(t) \\
&= -\frac{1}{2}\left(-\frac{2}{3(\gamma-1)+2}t^{-2} + (\frac{2}{3(\gamma-1)+2})^2 t^{-2}\right)(x_1^2+x_2^2+x_3^2) \\
&- \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \left( -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t^{-2} + \frac{2}{3(\gamma-1)+2} t^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t^{-1} \right) + \frac{9(\gamma-1)}{4} t^{-2} \\
&= \frac{(3\gamma-1)-2}{(3\gamma-1)^2} t^{-2}(x_1^2+x_2^2+x_3^2) - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \begin{pmatrix} \frac{-(3\gamma-1)+2}{3\gamma-1} \\ \frac{-(3\gamma-1)+2}{3\gamma-1} \\ \frac{-(3\gamma-1)+2}{3\gamma-1} \end{pmatrix} t^{-2} + \frac{9(\gamma-1)}{4} t^{-2} \\
&= \frac{3\gamma-3}{(3\gamma-1)^2} t^{-2}(x_1^2+x_2^2+x_3^2) + \left(\frac{3\gamma-3}{3\gamma-1}\right)(x_1+x_2+x_3)t^{-2} + \frac{9(\gamma-1)}{4} t^{-2}.
\end{aligned}$$

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