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Substitution tilings with transcendental inflation factor

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SUBSTITUTION TILINGS WITH TRANSCENDENTAL INFLATION FACTOR

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Abstract. For any $\lambda > 2$, we construct a substitution on an infinite alphabet which gives rise to a substitution tiling with inflation factor λ. In particular, we obtain the first class of examples of substitutive systems with transcendental inflation factors. We also show that both the associated subshift and tiling dynamical systems are strictly ergodic, which is related to the quasicompactness of the underlying substitution operator.

1. Introduction

Substitutions on finite alphabets and associated dynamical systems generated by shifts are well studied from dynamical, topological, and geometrical points of view [2, 8, 10, 19]. Under certain conditions, such substitutions generate substitution tilings with finitely many building blocks called *prototiles*, with some algebraic inflation factor λ. In this setting, the algebraicity of λ follows immediately from the finiteness of the alphabet, and hence the finite-dimensionality of the associated substitution matrix.

Moreover, it was shown by Lagarias [14] that if a Delone set of finite type $\varLambda$ in $\mathbb{R}^d$ (i.e., one for which $\varLambda - \varLambda$ is a discrete closed subset of $\mathbb{R}^d$) has inflation symmetry, i.e., $\lambda\varLambda \subset \varLambda$ for some $\lambda > 1$, then λ must be an algebraic number.

This number-theoretic property of λ has also been exploited in the spectral analysis of the associated systems. For example, Solomyak showed in [22] that for one-dimensional self-affine tilings with (finite local complexity) FLC with inflation factor λ, the corresponding dynamical system admits non-trivial eigenfunctions if and only if λ is a Pisot number; see also [3, 13]. This uses the fact that $\alpha \in \mathbb{R}$ corresponds (in certain sense) to an eigenvalue if and only if $||\alpha\lambda^n|| \to 0$ as $n \to \infty$ where $||\cdot||$ denotes the distance to the closest integer. When λ is algebraic, this only holds when λ is a Pisot number. The still open Pisot–Vijayaraghavan problem asks whether there exist a transcendental λ and a non-zero real α for which such a Diophantine convergence result holds; see [4].

In addition, the inflation factor λ being a Pisot number allows for a canonical description of the Delone set $\varLambda$ obtained from the substitution tiling by a cut-and-project scheme: the Galois conjugates of λ yield the Minkowski embedding of $\varLambda$ as well as the star-map, which in turn yields the window for $\varLambda$, hence all components of the cut-and-project scheme for $\varLambda$. For details and terminology on cut-and-project schemes see [2].

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The Delone sets that we mentioned several times already are defined as follows. A point set $\varLambda \subseteq \mathbb{R}^d$ is a *Delone set* if it is

- uniformly discrete, so there is a positive radius r such that the balls of radius r centered at points of $\varLambda$ form a packing in $\mathbb{R}^d$, and
- relatively dense, so there is a positive radius R such that the balls of radius R centered at points of $\varLambda$ form a covering of $\mathbb{R}^d$.

In a specific case $d = 1$ which we mostly discuss in this paper, a set $\varLambda \subset \mathbb{R}$ is a Delone set if and only if the distance between two consecutive points of $\varLambda$ is bounded above and below by two positive numbers.

Delone sets arise in many topics related to mathematical crystallography, to distribution of points in Euclidean spaces, and in applications as well-spaced point sets. We refer to [6, 7, 11, 14, 15] and references therein for more detailed discussions of these sets and related properties. We want to emphasize their importance as point sets that can be used to model atomic structure of crystals or ordered matter.

One connection between Delone sets and substitutions on finite alphabets is the following. Under certain conditions on the substitution, one can associate to it a strictly positive, bounded length function ℓ that turns every bi-infinite word into a tiling $\mathcal{T}$ of $\mathbb{R}$ consisting of finitely many distinct prototiles transforming every letter a into segment of length $\ell(a)$ and preserving the order of letters. Additionally, the length function ℓ respects the substitution in a sense that every inflated segment can be sliced into segments corresponding to the initial letters, thus turning the substitution into an inflation rule; see Figure 1. We call such a length function a *natural length function*. One can then derive a Delone set $\varLambda \subset \mathbb{R}$ by identifying a tile with the location of its left endpoint; see Figure 2 for an example.

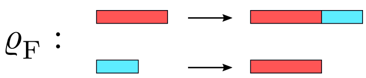

FIGURE 1. The geometric inflation rule associated to the Fibonacci substitution given by $a \mapsto ab, b \mapsto a$. Here the length function ℓ associates intervals of length $\lambda = (1 + \sqrt{5})/2$ and 1 to the letters a and b, respectively.

$$\ldots abaababa|abaababa \ldots$$

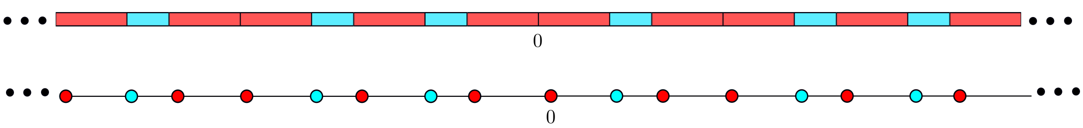

FIGURE 2. A bi-infinite word fixed under (the square of) the Fibonacci substitution, the corresponding tiling $\mathcal{T}$ of $\mathbb{R}$, and the derived Delone set $\varLambda$.

With a proper choice of bi-infinite word and the origin, the resulting Delone set $\varLambda \subset \mathbb{R}$ has inflation symmetry $\lambda\varLambda \subset \varLambda$ where λ is (an integer power of) the inflation factor of the initial substitution and hence algebraic.

It is not complicated to construct a Delone set $\varLambda \subset \mathbb{R}$ with a similar inclusion $\lambda\varLambda \subset \varLambda$ for a non-algebraic λ, but the additional structure of the family of Delone sets with the same "local structures" is not guaranteed as it is for dynamical properties of substitutions on finite alphabets.

Substitutions on infinite alphabets or infinitely many prototiles have long been considered before; see [9, 10]. However, most known examples are either of constant length (which means all associated tiles will just have the same length) or already have inherent pre-defined tile lengths. In [18], it was asked whether, for substitutions over compact alphabets, there are conditions which guarantee the existence of a natural length function, from which one can recover an associated geometric inflation system. The following question was posed in [18]:

Question 1.1. Does there exist a primitive substitution on a compact alphabet which admits a unique strictly positive (continuous) length function $\ell : \mathcal{A} \longrightarrow \mathbb{R}_{+}$ and a transcendental inflation factor λ?

We answer this question in the affirmative in this work. In this paper we present a framework to construct substitutions on infinite alphabets that result in subshift dynamical systems with many signature properties of subshifts associated with substitutions on finite alphabets. Particularly, the substitutions within our framework have well-defined length functions. Additionally, they give rise to substitution tilings with any prescribed inflation factor $\lambda > 2$ including transcendental numbers. Our main results are the following.

Theorem 1.2. *For every real number $\lambda > 2$ there exists a substitution $\varrho_{\mathbf{a}}$ on an (infinite) compact alphabet $\mathcal{A}$ such that the inflation factor of the associated inflation rule is λ and every element of the subshift (X_{ϱ}, σ) gives rise to a Delone set.*

Since algebraic numbers are countable the following result is immediate.

Corollary 1.3. *There is an substitution $\varrho_{\mathbf{a}}$ on an (infinite) compact alphabet $\mathcal{A}$ such that the inflation factor of the associated geometric substitution λ is a transcendental number, every element of the subshift (X_{ϱ}, σ) gives rise to a Delone set.*

The following result ensures that our substitutions obey the usual nice properties one expects from the finite alphabet case.

Theorem 1.4. *For the substitution $\varrho := \varrho_{\mathbf{a}} \colon \mathcal{A} \to \mathcal{A}^{+}$,*

(1) The subshift (X_{ϱ}, σ) is minimal and uniquely ergodic,
(2) There is a unique (up to scaling) strictly positive, continuous, natural tile length function which turns ϱ into a geometric substitution in $\mathbb{R}$ and,
(3) The associated tiling dynamical system $(\Omega_{\varrho}, \mathbb{R})$ is minimal and uniquely ergodic.

The paper is organized as follows. In Section 2, we recall notions and properties of subshifts and substitutions on compact alphabets. We also recall the properties of the substitution operator (which is the infinite-alphabet analogue of the substitution matrix) and its dynamical consequences for the subshift.

In Section 3, given a bounded sequence $\mathbf{a} = (a_i)_i$ of nonnegative integers with an additional mild condition, we demonstrate how to construct a map $\varrho_{\mathbf{a}}$ from the set of letters $\mathbb{N}_0 = \{[0], [1], [2], [3], \ldots\}$ to the set of finite words over $\mathbb{N}_0$. Here $\mathbb{N}_0$ represents a (sub)set of letters of our alphabet that correspond to nonnegative integers and we use notation $[i]$ for the ith letter to avoid confusion with nonnegative integer i. We then define an appropriate embedding $\iota_{\mathbf{a}}(\mathbb{N}_0)$ of $\mathbb{N}_0$ into a shift space $\mathcal{S}$ over a finite alphabet and consider its compactification $\mathcal{A} := \overline{\iota_{\mathbf{a}}(\mathbb{N}_0)}$ (with respect to the topology on $\mathcal{S}$) as our alphabet. We extend the map $\varrho_{\mathbf{a}}$ to a continuous one on $\mathcal{A}$, which will be our substitution. This allows us to use main results of Mañibo, Rust, and Walton [18] for the associated dynamical system and geometric substitution in $\mathbb{R}$. Particularly, we prove Theorem 1.4 in this section.

In Section 4 we provide a closed form for the inflation factor λ of the substitution in terms of the sequence $\mathbf{a} = (a_i)_i$ and show that for every $\lambda > 2$, there is a suitable sequence $(a_i)_i$ that gives inflation factor λ. This allows us to establish Theorem 1.2. In Section 5 we show that all suitable periodic sequences yield algebraic inflation factors and provide a counterexample for the converse. We provide a concrete example of a substitution with transcendental inflation factor in Section 6. We end with some open questions and concluding remarks.

2. Substitutions on compact alphabets

In this section, we adapt the notations and definitions in [18].

2.1. **Subshifts and substitutions.**

Let $\mathcal{A}$ be a compact Hausdorff space, which will be our alphabet whose elements we call letters. Let $\mathcal{A}^n$ be the collection of all words of length n over $\mathcal{A}$, i.e., $w = w_1 w_2 \cdots w_n$, with $w_i \in \mathcal{A}$. We say that a word u is a subword of w (which we denote by $u \triangleleft w$) if there exist $1 \leqslant i, j \leqslant n$ such that $u = w_i w_{i+1} \cdots w_j = w_{[i,j]}$. Note that the relation of "being subword" can be easily generalized when w is one-sided infinite or bi-infinite. The set $\mathcal{A}^+ = \bigcup_{n \geqslant 1} \mathcal{A}^n$ of all (non-empty) finite words over $\mathcal{A}$ is also topological space (since the sets $\mathcal{A}^n$ are naturally endowed with the product topology). Let $\mathcal{A}^* = \mathcal{A}^+ \cup \{\varepsilon\}$, where ε is an empty word. This is a free monoid with concatenation as the binary operation.

Definition 2.1. A *substitution* ϱ is a continuous map from $\mathcal{A}$ to $\mathcal{A}^+$.

For every letter $a \in \mathcal{A}$, we call $\varrho^k(a)$ the *level-k supertile* of type a. We define the *language* $\mathcal{L}(\varrho)$ of the substitution to be the set

$$\mathcal{L}(\varrho) = \overline{\{u \in \mathcal{A}^* \mid u \triangleleft \varrho^k(a), k \in \mathbb{N}_0, a \in \mathcal{A}\}}.$$

This allows one to define a subshift associated to ϱ. Let $\mathcal{A}^{\mathbb{Z}}$ be the set of all bi-infinite sequences over $\mathcal{A}$, also called the *full shift* over $\mathcal{A}$, which is compact with respect to the Tychonoff topology. Note that the left shift map $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined pointwise via $\sigma(x)_n = x_{n+1}$, is a homeomorphism. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is called a *subshift* if it is closed and is shift-invariant, i.e., $\sigma(X) \subseteq X$. We now define the subshift associated to ϱ via its corresponding language, i.e., $X_\varrho := \big\{x \in \mathcal{A}^{\mathbb{Z}} \mid u \in \mathcal{L}(\varrho) \text{ for all } u \triangleleft x\big\}$; see [18]. It can be shown that $\varrho(X_\varrho) = X_\varrho$. A substitution ϱ is said to be *recognizable* if for every power $k \geqslant 1$, every element x of the subshift admits a unique decomposition into level-k supertiles of ϱ.

Definition 2.2. A substitution is called *primitive* if for every non-empty open set $U \subset \mathcal{A}$, there exists $k(U) \in \mathbb{N}$ such that $\varrho^{k(U)}(a)$ contains an element in U for all $a \in \mathcal{A}$.

Primitivity has a lot of immediate consequences for the corresponding subshift, some of which we mention in the following result, see [18, Thm. 3.30].

Proposition 2.3. *Let ϱ a primitive substitution over a compact Hausdorff alphabet. Then*

(1) The subshift X_ϱ is non-empty.
(2) The topological dynamical system (X_ϱ, σ) is minimal, i.e., every element $x \in X_\varrho$ has a dense orbit.

2.2. **The substitution operator.** Let $\varrho\colon \mathcal{A} \to \mathcal{A}^+$ be a substitution over a compact Hausdorff alphabet. One can associate to it a substitution operator $M := M_\varrho$ on the space $C(\mathcal{A})$ of real-valued continuous functions over $\mathcal{A}$ via

$$Mf(a) = \sum_{b \in \varrho(a)} f(b)$$

where $\varrho(a)$ is seen as a multiset. This is the compact alphabet analogue of the transpose of the substitution matrix in the finite alphabet setting. This operator is always positive and bounded. The generalisation of the Perron–Frobenius eigenvalue in this setting is the spectral radius $r(M)$ of M. Since M is positive, $r(M)$ is always in the spectrum of M but it is not guaranteed that it is an eigenvalue. We refer the reader to [20] for a background on positive operators on Banach lattices. For the substitution operator, one has the following bounds for $r(M)$.

Lemma 2.4 ([18])**.** *For all $k \in \mathbb{N}$, one has*

$$\min_{a \in \mathcal{A}} |\varrho^k(a)| \leqslant r(M)^k \leqslant \max_{a \in \mathcal{A}} |\varrho^k(a)|.$$

An operator T with $r(T) = 1$ over a Banach space is called *quasicompact* if there exists $k \in \mathbb{N}$ and a compact operator K such that $\|T^k - K\| < 1$.

Proposition 2.5 ([18])**.** *Let ϱ be a primitive substitution on a compact Hausdorff alphabet $\mathcal{A}$ for which there exists $k \in \mathbb{N}$ and a finite subset $F \subset \mathcal{A}$ of isolated points such that*

$$\operatorname{card}\big\{b \in \varrho^k(a) \mid b \notin F\big\} < r(M)^k$$

for all $a \in \mathcal{A}$. Then the normalised substitution operator $T\colon = M/r(M)$ is quasicompact.

Theorem 2.6 ([18]). *Let ϱ be a primitive, recognisable substitution on a compact Hausdorff alphabet. Suppose further that $T = M/r(M)$ is quasicompact. Then one has*

(i) *ϱ admits a unique natural tile length function which is strictly positive*
(ii) *The corresponding tiling dynamical system $(\Omega_{\varrho}, \mathbb{R})$ is uniquely ergodic.*
(iii) *The corresponding symbolic dynamical system (X_{ϱ}, σ) is uniquely ergodic.*

3. The construction and its dynamical properties

3.1. **Pre-substitution.** Let $\mathbf{a} = (a_i)_i = a_0, a_1, a_2, \ldots$ be a sequence of nonnegative integers that satisfies the following additional properties.

(A1) The sequence $\mathbf{a}$ is bounded and we set $N := \max a_i$;
(A2) $a_0 \neq 0$;
(A3) The stretches of zeros in $\mathbf{a}$ are bounded. So, there exists an integer $C > 0$ such that if $a_i = a_{i+1} = \ldots = a_{i+k} = 0$, then $k < C$.

Definition 3.1. Let $\mathbf{a} \in \{0, 1, \ldots, N\}^{\mathbb{N}_0}$ be given and let $[i] \in \mathbb{N}_0$ be a letter in our pre-alphabet. We define the *pre-substitution* $\varrho_{\mathbf{a}} \colon \mathbb{N}_0 \to \mathbb{N}_0^+$ via

$$\begin{aligned} &\varrho_{\mathbf{a}}([0]) = [0]^{a_0}[1], \text{ and} \\ &\varrho_{\mathbf{a}}([i]) = [0]^{a_i}[i-1][i+1] \text{ for } i > 0. \end{aligned}$$

We call the rule $\varrho_{\mathbf{a}}$ a pre-substitution because we will later extend it to a full substitution on a compact alphabet, which is possible if one embeds $\mathbb{N}_0$ properly. To this pre-substitution, one can associate the corresponding matrix

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1+1 & a_2 & a_3 & a_4 & \ldots \\ 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

whose entries are defined by $\mathbf{A}_{ij} = \#[i]$ in $\varrho_{\mathbf{a}}([j])$. This matrix will become relevant later when we compute for statistical and geometric properties associated to ϱ.

This construction can be seen as a generalization of [17, Subsect. 3.5]. This example uses a specific case of this substitution with $a_i = 1$ for all $i \geqslant 0$.

3.2. **Compactification and its properties.** In order to use the results of [18], the alphabet has to be compact in some topological space. Here, we use the boundedness of the entries of $\mathbf{a}$ to define an appropriate compactification of $\mathbb{N}_0$.

Consider the full shift $\mathcal{S} = \{*, 0, 1, 2, \ldots, N\}^{\mathbb{Z}}$ over $N + 2$ letters. This is a compact topological space under the local topology. More specifically, for every $i \geqslant 0$, the ith nontrivial neigborhood (also called a cylinder set) of $\mathbf{b}$ in $\mathcal{S}$ consists of those elements of $\mathcal{S}$

that have the same subsequence $b_{-i} \dots b_{-1} \boxed{b_0} b_1 \dots b_i$ around the origin. In what follows, the boxed term will always specify the location of the origin.

It is well known that $\mathcal{S}$ is a metric space where the metric is given by $d_{\mathcal{S}}(\mathbf{b}, \mathbf{b}')$ between $\mathbf{b}, \mathbf{b}' \in \mathcal{S}$ to be $\frac{1}{2^i}$ if i is the smallest number in $\mathbb{N}_0$ such that the ith nontrivial neighborhoods of $\mathbf{b}$ and $\mathbf{b}'$ are different. Particularly, if $\mathbf{b}$ and $\mathbf{b}'$ have different letters at the origin then $i = 0$ and $d_{\mathcal{S}}(\mathbf{b}, \mathbf{b}') = 1$.

Fix a sequence $\mathbf{a} \in \{0, 1, \dots, N\}^{\mathbb{N}_0}$. We now define an embedding $\iota_{\mathbf{a}} \colon \mathbb{N}_0 \to \mathcal{S}$ depending on $\mathbf{a}$, which we do first by defining $\mathbf{0} := \iota_{\mathbf{a}}([0])$. The image of $[0]$ is defined pointwise via the rule

$$\iota_{\mathbf{a}}([0])_m := \begin{cases} a_m, & \text{for } m \geqslant 0 \\ *, & \text{otherwise.} \end{cases}$$

As a bi-infinite sequence, one has

$$\mathbf{0} := \dots * * * \boxed{a_0} a_1 a_2 a_3 \dots.$$

For every $i > 0$, we define $\iota_{\mathbf{a}}([i]) := \sigma^i(\mathbf{0})$, i.e.,

$$\mathbf{i} := \dots * * * \, a_0 a_1 \dots a_{i-1} \boxed{a_i} a_{i+1} a_{i+2} \dots.$$

Here σ is the left shift map as before.

Then the closure $\mathcal{A} := \overline{\iota_{\mathbf{a}}(\mathbb{N}_0)} \subset \mathcal{S}$ is the alphabet that we use in our further construction. For an element $\mathbf{b} \in \mathcal{A} \setminus \iota_{\mathbf{a}}(\mathbb{N}_0)$, we will denote the corresponding letter of the alphabet as $[\infty_{\mathbf{b}}]$. Thus the total *compact alphabet* $\mathcal{A}$ can be seen as the union of all isolated letters $[i]$ and all relevant infinities $[\infty_{\mathbf{b}}]$, so we can identify

$$\mathcal{A} = \{[i] \mid i \in \mathbb{N}_0\} \cup \{[\infty_{\mathbf{b}}] \mid \mathbf{b} \in \mathcal{A} \setminus \iota_{\mathbf{a}}(\mathbb{N}_0)\}.$$

Before defining the substitution on the letters $[\infty_{\mathbf{b}}]$, we prove two properties of the corresponding sequences in $\mathcal{S}$.

Lemma 3.2. *Let* $\mathbf{b} \in \mathcal{A}$. *Then* $\mathbf{b} \in \mathcal{A} \setminus \iota_{\mathbf{a}}(\mathbb{N}_0)$ *if and only if no entry of* $\mathbf{b}$ *is* $*$.

Proof. This follows directly from the definition of $\iota_{\mathbf{a}}(\mathbb{N}_0)$ as $\iota_{\mathbf{a}}(\mathbb{N}_0) = \{\sigma^n(\mathbf{0}), n \in \mathbb{N}_0\}$. For every n, only the entries to the left of $(-n)$th position of $\sigma^n(\mathbf{0})$ are $*$s. This means that every accumulation point of $\iota_{\mathbf{a}}(\mathbb{N}_0)$ cannot be in the σ-orbit of $\mathbf{0}$ and must be a bi-infinite sequence over $\{0, 1, \dots, N\}$. □

Lemma 3.3. *The shift map is invertible on the accumulation points. That is, if* $\mathbf{b} \in \mathcal{A} \setminus \iota_{\mathbf{a}}(\mathbb{N}_0)$, *then* $\sigma(\mathbf{b}), \sigma^{-1}(\mathbf{b}) \in \mathcal{A} \setminus \iota_{\mathbf{a}}(\mathbb{N}_0)$.

Proof. Since $\mathbf{b}$ is an accumulation point, there exists an increasing sequence $\{i_n\}_{n \geqslant 0}$ of natural numbers such that the $\mathbf{i}_n := \sigma^{i_n}(\mathbf{0})$ converges to $\mathbf{b}$ in the local topology. Without loss of generality, assume that $i_n > 1$ for all n. Note that the sequence $\{\sigma^{i_n - 1}(\mathbf{0})\}_{n \geqslant 0}$ converges to $\sigma^{-1}(\mathbf{b})$ and all elements of the sequence are in $\iota_{\mathbf{a}}(\mathbb{N}_0)$ being shifts of $\mathbf{0}$, which implies $\sigma^{-1}(\mathbf{b}) \in \overline{\iota_{\mathbf{a}}(\mathbb{N}_{\mathbf{0}})}$. The claim for $\sigma(\mathbf{b})$ can be proved using similar arguments. □

We then extend the substitution in Definition 3.1 with the image of all $[\infty_{\mathbf{b}}]$ under $\varrho_{\mathbf{a}}$. For

$$\mathbf{b} = \dots b_{-2}b_{-1}\boxed{b_0}b_1b_2\dots \in \mathcal{A}\setminus \iota_{\mathbf{a}}(\mathbb{N}_0)$$

we define

$$\varrho_{\mathbf{a}}([\infty_{\mathbf{b}}]) = [0]^{b_0}[\infty_{\sigma^{-1}(\mathbf{b})}][\infty_{\sigma(\mathbf{b})}] \tag{1}$$

Lemmas 3.2 and 3.3 ensure that this substitution is well defined as b_0 must be a number from $\{0,1,\dots,N\}$ and not $*$, and that the letters $[\infty_{\sigma(\mathbf{b})}]$ and $[\infty_{\sigma^{-1}(\mathbf{b})}]$ exist in $\mathcal{A}$.

Proposition 3.4. *The map $\sigma_{\mathbf{a}}\colon \mathcal{A}\to\mathcal{A}^+$ defined in Def. 3.1 and Eq.* (1) *is continuous and hence a substitution.*

Proof. This follows immediately from construction. In particular, if $d_{\mathcal{S}}(\sigma^i(\mathbf{0}),\mathbf{b}) \leqslant \frac{1}{2^m}$ for some $m\geqslant 1$ and $i>0$, then $\mathbf{i}$ and $\mathbf{b}$ coincide at least within m letters from the origin. Thus $d_{\mathcal{S}}(\sigma^{i-1}(\mathbf{0}),\sigma^{-1}(\mathbf{b})) \leqslant \frac{1}{2^{m-1}}$. Similarly, $d_{\mathcal{S}}(\sigma^{i+1}(\mathbf{0}),\sigma(\mathbf{b})) \leqslant \frac{1}{2^{m-1}}$. Additionally, the condition on $\sigma^i(\mathbf{0})$ implies $a_i = b_0$. This means that the words $\varrho_{\mathbf{a}}([i]) = [0]^{a_i}[i-1][i+1]$ and $\varrho_{\mathbf{a}}([\infty_{\mathbf{b}}]) = [0]^{b_0}[\infty_{\sigma^{-1}(\mathbf{b})}][\infty_{\sigma(\mathbf{b})}]$ both belong to $\mathcal{A}^{b_0+2}$ and are close in $\mathcal{A}^+$ in the disjoint union topology. □

Theorem 3.5. *Let* $\mathbf{a}$ *be a sequence of nonnegative integers satisfying conditions* (A1)–(A3) *and $\varrho_{\mathbf{a}}$ be the substitution on $\mathcal{A}$ corresponding to* $\mathbf{a}$. *The substitution $\sigma_{\mathbf{a}}$ satisfies the following:*

(i) *$\varrho_{\mathbf{a}}$ is primitive;*
(ii) *$\varrho_{\mathbf{a}}$ is recognizable; and*
(iii) *the associated (normalised) substitution operator is quasicompact.*

Proof. We first prove that every letter $a\in\mathcal{A}$ contains $[0]$ in its level-$(C+1)$ supertile $\varrho_{\mathbf{a}}^{C+1}(a)$ where C is the constant from (A3). To do that we first notice that $\varrho_{\mathbf{a}}([0])$ contains $[0]$ because $a_0\neq 0$ due to (A2). Thus if a supertile contains $[0]$ then every further application of $\varrho_{\mathbf{a}}$ to that supertile will contain $[0]$ as well.

If $a\in\iota_{\mathbf{a}}(\mathbb{N}_0)$, then $a=[i]$ for some i and among integers $a_i, a_{i+1},\dots,a_{i+C}$ at least one is positive due to restriction (A3). Suppose $a_{i+k}>0$ where $0\leqslant k\leqslant C$. Then $\varrho_{\mathbf{a}}^k([i])$ contains the letter $[i+k]$ and since $\varrho_{\mathbf{a}}^{k+1}([i])$ contains $[0]$ so does $\varrho_{\mathbf{a}}^{C+1}([i])$.

If $a\in\mathcal{A}\setminus\iota_{\mathbf{a}}(\mathbb{N}_0)$, then $a=[\infty_{\mathbf{b}}]$ for an appropriate $\mathbf{b}=\dots b_{-2}b_{-1}\boxed{b_0}b_1b_2\dots$. We also notice that among numbers $b_0,b_1,\dots,b_C$ at least one is positive because $\mathbf{b}$ is a limit of some subsequence of $(\sigma^i(\mathbf{0}))_i$. After that the arguments are similar to the previous case as $\varrho_{\mathbf{a}}^k([\infty_{\mathbf{b}}])$ contains $[\infty_{\sigma^k(\mathbf{b})}]$.

For primitivity, we need to show that for every open set U there exists $k(U)\in\mathbb{N}$ such that for every letter $a\in\mathcal{A}$, $\varrho_{\mathbf{a}}^{k(U)}(a)$ contains a letter in U. We consider two types of open sets: the singletons consisting of isolated points $[j], j\in\mathbb{N}_0$ and balls $B_\varepsilon([\infty_{\mathbf{b}}])$ for some accumulation point $\mathbf{b}$.

If $[j]$ is an isolated point, the claim is straightforward for $U = \{[j]\}$ since $[j] \triangleleft \varrho_{\mathbf{a}}^{j}([0])$ and hence $[j] \triangleleft \varrho_{\mathbf{a}}^{j+C+1}(a)$ for any $a \in \mathcal{A}$.

Now let $\mathbf{b}$ be an accumulation point and $U = B_\varepsilon([\infty_{\mathbf{b}}])$ for some $\varepsilon > 0$. Following the proof of Lemma 3.3, there exists a sequence of isolated points $\{\mathbf{i}_n\}_{n\geqslant 0}$ such that for every $\varepsilon > 0$, there is a T such that $d_{\mathcal{S}}(\mathbf{i}_n, \mathbf{b}) < \varepsilon$ for all $n > T$. For the embedding, this means $[i_n] \in U$. Same as above, we pick any $n > T$ and then for every $a \in \mathcal{A}$, we get that $[i_n] \in \varrho_{\mathbf{a}}^{i_n+C+1}(a)$, which proves the claim.

Next, we show that the substitution $\varrho_{\mathbf{a}}$ is recognizable. Here, we use the fact that if we see a (maximal) block of zeros, then a level-1 supertile starts with that block and the whole block belongs to that one supertile. This is true because zeros are *never* preceded by non-zero letters in level-1 supertile and every level-1 supertile ends with a non-zero letter.

Given $x \in X_\varrho$, we show that there is a unique way to decompose it as a concatenation of level-1 supertiles. First, we look to the right of the origin and look for the minimal $n \in \mathbb{N}$ such that $x_{n-1} \neq [0]$ and $x_n = [0]$. This means x_n is a beginning of a level-1 supertile. We iterate the process to the right and to the left of x_n, placing a cut each time we are in such a situation. Between each two consecutive cuts we have a block of zeroes followed by some number of non-zero letters. We note that every level-1 supertile without initial zeroes has exactly two non-zero letters so the inital block of zeroes form level-1 supertile with one or two non-zero letters depending on the parity of the number of non-zero letters between two consecutive cuts. From this, one gets the unique decomposition into level-1 supertiles.

The process for level-k decomposition for $k > 1$ is similar, where now one has to look at positions where $x_{[n,n+|\varrho_{\mathbf{a}}^{k-1}([0])|-1]} = \varrho_{\mathbf{a}}^{k-1}([0])$ and $x_{[n-|\varrho_{\mathbf{a}}^{k-1}([0])|,n-1]} \neq \varrho_{\mathbf{a}}^{k-1}([0])$. Since the substitution is injective, the subwords between such every two consecutive positions can be unambiguously cut into level-k supertiles.

Lastly, we show that the associated (normalised) substitution operator is quasicompact. Here, we want to show that $\varrho_{\mathbf{a}}$ satisfies the conditions of Proposition 2.5. We show that this is satisfied for $k = C + 2$ where C is the constant from property (A3) and for $F = \{[0], [1], \ldots, [C + 1]\}$.

We look at the images of letters of $\mathcal{A}$ under $\varrho_{\mathbf{a}}^{C+2}$. Every application of $\varrho_{\mathbf{a}}$ at least doubles the number of letters and therefore for all $a \in \mathcal{A}$, $|\varrho_{\mathbf{a}}^{C+2}(a)| \geqslant 2^{C+2}$. However, if we apply $\varrho_{\mathbf{a}}$ that many times, then at least once one letter will be substituted with three or more. Indeed, if $a = [j]$ for some $j \geqslant 0$, then after first $C + 1$ applications of $\varrho_{\mathbf{a}}$ we will see letters $[j + 1], [j + 2], \ldots, [j + C + 1]$ and one of $a_{j+1}, a_{j+2}, \ldots, a_{j+C+1}$ must be positive by property (A3) and the corresponding letter is substituted by at least three letters. The case for $a = [\infty_{\mathbf{b}}]$ is similar. That means that $|\varrho_{\mathbf{a}}^{C+2}(a)| \geqslant 2^{C+2} + 1$ and by Lemma 2.4, $r(M)^{C+2} \geqslant 2^{C+2} + 1$.

On the other hand we note that for every $n \leqslant C + 1$, the supertile $\varrho_{\mathbf{a}}^{n}([0])$ does not contain letters not from F, and the supertile $\varrho_{\mathbf{a}}^{C+2}([0])$ contains only one letter not from F. For every other letter $a \neq [0]$, we will repeatedly apply $\varrho_{\mathbf{a}}$ $C + 2$ times to a but we will color some letters with red in the process. The goal is to show that red letters cannot lead to letters outside of F and there are not too many non-red letters.

Here are the rules how we color letters with red. The initial letter is not colored with red. If one applies $\varrho_{\mathbf{a}}$ to a non-red letter $[i]$, then the initial zeros of $\varrho_{\mathbf{a}}([i])$, if any, are colored with red. Additionally, if we apply $\varrho_{\mathbf{a}}$ to a red letter, then all resulting letters are red.

Note that no red zero can create a letter outside of F because we apply the substitution to red zeros at most $C+1$ times. Also, all red letters originate from red zeros so all letters outside of F will be non-red in the end. But the number of non-red letters at most doubles because every letter is substituted with at most two non-zeros. Thus

$$\operatorname{card}\left\{[j] \in \varrho_{\mathbf{a}}^{C+2}([i]) \mid [j] \notin F\right\} \leqslant 2^{C+2} < 2^{C+2} + 1 \leqslant r(M)^{C+2},$$

which by Lemma 2.5 implies the quasicompactness after normalisation. □

Proof of Theorem 1.4. According to Theorem 3.5, the substitution $\varrho_{\mathbf{a}}$ is primitive, recognizable, and the corresponding normalised substitution operator is quasicompact. Applying Theorem 2.6 to our substitution we immediately obtain Theorem 1.4. □

4. Inflation factor

The main goal of this section is to give an explicit formula for the inflation factor of the substitution $\varrho_{\mathbf{a}}$ provided $\mathbf{a}$ satisfies conditions (A1)–(A3). This is done in Proposition 4.3 below but first we introduce an auxiliary parameter $\mu = \mu(\mathbf{a})$ which is the unique number in $(0, 1)$ that satisfies

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} a_i \mu^i. \qquad (2)$$

With that μ we set

$$\lambda := \mu + \frac{1}{\mu} > 2.$$

Our first goal is to show that we can find an appropriate sequence $\mathbf{a}$ to get every possible λ.

Lemma 4.1. *For every prescribed $\lambda > 2$ we can find a suitable $\mathbf{a}$ satisfying conditions* (A1)–(A3) *such that $\lambda = \mu + \frac{1}{\mu}$ and μ satisfies Eq.* (2).

Proof. Let $0 < \mu < 1$ be such that $\mu + \dfrac{1}{\mu} = \lambda$.

First we choose positive integer C such that $\mu + \mu^C \leqslant 1$. Then

$$\frac{1}{\mu} \geqslant \frac{1}{1-\mu^C} = \sum_{i=0}^{\infty} \mu^{iC}.$$

Let $\mu' := \frac{1}{\mu} - \frac{1}{1-\mu^C} \geqslant 0$ be the difference. We will show that it is possible to represent μ' as a sum of powers of μ with bounded nonnegative coefficients.

Let N be such a positive integer that

$$\frac{1}{\mu} \leqslant \frac{N}{1-\mu} = \sum_{i=0}^{\infty} N\mu^i.$$

We consider the following family F^0 that represent series with sum not greater than μ',

$$F^0 := \{(b_i)_i \in \{0, 1, \ldots, N\}^{\mathbb{N}_0} \mid \sum_{i=0}^{\infty} b_i\mu^i \leqslant \mu'\}.$$

The family F^0 contains infinitely many sequences. We recursively choose the sequence of subfamilies

$$F^0 \supseteq F^1 \supseteq F^2 \ldots$$

such that the ith term of every sequence in F^{i+1} is the maximal number c_i in $\{0, 1, \ldots, N\}$ such that F^i contains infinitely many sequences with the ith term equal to c_i.

We claim that $\mu' = \sum_{i=0}^{\infty} c_i\mu^i$. Indeed, the right-hand side cannot be greater than μ' because for every i, the sequence $(c_0, c_1, \ldots, c_i, 0, 0, \ldots)$ belongs to F^i.

If $\mu' > \sum_{i=0}^{\infty} c_i\mu^i$ then we consider two cases. If there are infinitely many c_is that are less than N, then for one sufficiently large i we can swap c_i with $c_i + 1$ and keep the sum of the series less than μ'. This contradicts with the choice of c_i since in that case there are infinitely many sequences in F^i with ith term equal to $c_i + 1 \leqslant N$.

In the second case, the sequence $(c_i)_i$ stabilizes at the value N. Let j be the index such that

$$c_j < N = c_{j+1} = c_{j+2} = \ldots.$$

We claim that the sequence $(c_0, c_1, \ldots, c_{j-1}, c_j + 1, 0, 0, \ldots)$ gives the series with sum less than μ' and therefore c_j was not chosen to be maximal. Indeed, by our assumtion $\mu' > \sum_{i=0}^{\infty} c_i\mu^i = \sum_{i=0}^{j} c_i\mu^i + \sum_{i=j+1}^{\infty} N\mu^i$ and by the choice of N, $\mu^j \leqslant \sum_{i=j+1}^{\infty} N\mu^i$. Combining these two inequalities we get the claim and a contradiction with our assumption that μ' is greater than the sum of the series.

To finish the proof we write

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} \mu^{iC} + \mu' = \sum_{i=0}^{\infty} \mu^{iC} + \sum_{i=0}^{\infty} c_i\mu^i.$$

The first series ensures that properties (A2) and (A3) hold, and the second series guarantees property (A1) for the sum. □

In the previous section we established that we can apply Theorem 3.5 to our substitution. In particular, it means there exists a natural tile length function associated with this substitution, which is strictly positive. We next show that $\lambda = r(M)$ and give explicit natural lengths for all letters in the subalphabet $\iota_{\mathbf{a}}(\mathbb{N}_0) = \{[i] \colon i \in \mathbb{N}_0\}$ of isolated points. Recall that

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1+1 & a_2 & a_3 & a_4 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Define $\ell : \iota_{\mathbf{a}}(\mathbb{N}_0) \longrightarrow \mathbb{R}^+$ to be $\ell([0]) := 1$, and for every $k > 0$.

$$\ell([k]) = \mu^k + \sum_{j=1}^{k} \sum_{i=j}^{\infty} a_i \mu^{i+k+1-2j}. \tag{3}$$

Lemma 4.2. *The function $\ell \colon \iota_{\mathbf{a}}(\mathbb{N}_0) \longrightarrow \mathbb{R}^+$ in Eq. (3) is uniformly continuous on $\iota_{\mathbf{a}}(\mathbb{N}_0)$ with respect to the topology on $\mathcal{S}$.*

Proof. Let $[k], [k+t] \in \iota_{\mathbf{a}}(\mathbb{N}_0)$ such that $d([k],[k+t]) = \frac{1}{2^n} < \frac{1}{2^{n-1}}$ for some $n \in \mathbb{N}$. We show that $|\ell([k]) - \ell([k+t])| < \varepsilon(n)$. First, note that $\sigma^k(\mathbf{0})$ and $\sigma^{k+t}(\mathbf{0})$ agree on $[-n, n]$, where $n \leqslant k$. This means

$$a_{k+r} = a_{k+t+r} \quad \text{for} \quad -n \leqslant r \leqslant n. \tag{4}$$

We can then split the double sum on the right hand-side of Eq. (3) into three parts:

$$\ell([k]) = \mu^k + \underbrace{\sum_{j=1}^{k-n-1} \sum_{i=j}^{\infty} a_i \mu^{i+k+1-2j}}_{\text{(I)}} + \underbrace{\sum_{j=k-n}^{k} \sum_{i=k+n+1}^{\infty} a_i \mu^{i+k+1-2j}}_{\text{(II)}} + \underbrace{\sum_{j=k-n}^{k} \sum_{i=j}^{k+n} a_i \mu^{i+k+1-2j}}_{\text{(III)}}.$$

Note that the component (III) is the same for both $\ell([k])$ and $\ell([k+t])$ because of Eq. (4), and hence this component vanishes in $|\ell([k]) - \ell([k+t])|$. We now determine n-dependent bounds for (I).

$$\begin{aligned} \sum_{j=1}^{k-n-1} \sum_{i=j}^{\infty} a_i \mu^{i+k+1-2j} &\leqslant N \sum_{j=1}^{k-n-1} \sum_{i=j}^{\infty} \mu^{i+k+1-2j} = N \sum_{j=1}^{k-n-1} \sum_{i=0}^{\infty} \mu^{i+k+1-j} \\ &= N \sum_{j=1}^{k-n-1} \frac{\mu^{k+1-j}}{1-\mu} = \frac{N}{(1-\mu)} (\mu^n + \cdots \mu^k) \leqslant \frac{N\mu^n}{(1-\mu)^2}. \end{aligned}$$

Here, the first inequality follows from (A1). One can carry out an analogous calculation for (II), which yields

$$\sum_{j=k-n}^{k} \sum_{i=k+n+1}^{\infty} a_i \mu^{i+k+1-2j} \leqslant \frac{N\mu^{n+2}}{(1-\mu)^2}.$$

Combining these estimates with the fact that $\mu^k - \mu^{k+t} < \mu^n$ gives us

$$|\ell([k]) - \ell([k+t])| < \left(1 + \frac{2N}{(1-\mu)^2} + \frac{2\mu^2 N}{(1-\mu)^2}\right)\mu^n,$$

from which the claim is immediate. □

In what follows, we let K be the positive cone of $C(\mathcal{A})$, i.e., all functions with $f(a) \geqslant 0$ for all $a \in \mathcal{A}$.

Proposition 4.3. *The function ℓ in Eq.* (3) *is a left eigenvector of $\mathbf{A}$ to λ. Moreover, it extends to a strictly positive, continuous natural length function on $\mathcal{A}$ and $\lambda = r(M)$ is the associated inflation factor.*

Proof. We first show that the (infinite) vector $\ell_{\mathcal{A}} := (\ell([0]), \ell([1]), \ell([2]), \ldots)$ of natural tile lengths is a left eigenvector of the matrix $\mathbf{A}$ with eigenvalue $\lambda = \mu + \frac{1}{\mu}$ so $\ell_{\mathcal{A}}\mathbf{A} = \lambda \ell_{\mathcal{A}}$.

We consider the product in the left-hand side column by column. Particularly, the first column says that $a_0\ell([0]) + \ell([1]) = \lambda\ell([0])$. And since $\ell([0]) = 1$, then

$$\ell([1]) = \lambda - a_0 = \mu + \frac{1}{\mu} - a_0 = \mu + \sum_{i=0}^{\infty} a_i\mu^i - a_0 = \mu + \sum_{i=1}^{\infty} a_i\mu^i$$

which satisfies the statement of this lemma.

After that, looking at the kth column for $k > 1$ we get

$$a_{k-1}\ell([0]) + \ell([k-2]) + \ell([k]) = \lambda\ell([k-1])$$

or

$$\ell([k]) = \lambda\ell([k-1]) - \ell([k-2]) - a_{k-1}.$$

Particularly, for $k = 2$

$$\begin{aligned}\ell([2]) = \lambda\ell([1]) - \ell([0]) - a_1 &= \left(\mu + \frac{1}{\mu}\right)\left(\mu + \sum_{i=1}^{\infty} a_i\mu^i\right) - 1 - a_1 = \\ &= \mu^2 + 1 + \sum_{i=1}^{\infty} a_i\mu^{i+1} + \sum_{i=1}^{\infty} a_i\mu^{i-1} - 1 - a_1 = \\ &= \mu^2 + \sum_{i=1}^{\infty} a_i\mu^{i+1} + \sum_{i=2}^{\infty} a_i\mu^{i-1} = \mu^2 + \sum_{j=1}^{2}\sum_{i=j}^{\infty} a_i\mu^{i+3-2j}.\end{aligned}$$

For all other k the computations can be carried out by induction. Using that $\lambda = \mu + \frac{1}{\mu}$ and expanding the product $\lambda\ell([k-1])$ we get

$$\ell([k]) = \lambda\ell([k-1]) - \ell([k-2]) - a_{k-1} =$$

$$= \left(\mu + \frac{1}{\mu}\right)\left(\mu^{k-1} + \sum_{j=1}^{k-1}\sum_{i=j}^{\infty} a_i\mu^{i+k-2j}\right) - \left(\mu^{k-2} + \sum_{j=1}^{k-2}\sum_{i=j}^{\infty} a_i\mu^{i+k-1-2j}\right) - a_{k-1} =$$

$$= \mu^k + \mu^{k-2} + \sum_{j=1}^{k-1}\sum_{i=j}^{\infty} a_i\mu^{i+k+1-2j} + \sum_{j=1}^{k-1}\sum_{i=j}^{\infty} a_i\mu^{i+k-1-2j} - \mu^{k-2} - \sum_{j=1}^{k-2}\sum_{i=j}^{\infty} a_i\mu^{i+k-1-2j} - a_{k-1}.$$

Here we can cancel μ^{k-2} and also cancel the double sum with negative sign leaving only the term with $j = k-1$ from the second double sum. Continuing the expression we get

$$= \mu^k + \sum_{j=1}^{k-1}\sum_{i=j}^{\infty} a_i\mu^{i+k+1-2j} + \sum_{i=k-1}^{\infty} a_i\mu^{i+k-1-2(k-1)} - a_{k-1}.$$

Note that the first term of the second sum is a_{k-1} so we can cancel that as well. Additionally, the power of μ in the second sum is $i + k - 1 - 2(k-1) = i + k + 1 - 2k$. These adjustments give us the final expression

$$= \mu^k + \sum_{j=1}^{k-1}\sum_{i=j}^{\infty} a_i\mu^{i+k+1-2j} + \sum_{i=k}^{\infty} a_i\mu^{i+k+1-2k} = \mu^k + \sum_{j=1}^{k}\sum_{i=j}^{\infty} a_i\mu^{i+k+1-2j}$$

as needed.

We now show that $\lambda = r(M)$. Since $\varrho_{\mathbf{a}}$ is primitive and $M/r(M)$ is quasicompact, the spectral radius $r(M)$ of the the substitution operator M is an eigenvalue with a strictly positive, continuous eigenfunction ℓ; see [18, Thm. 4.23 (D)] and [20, Thm. V.5.2 and Prop. V.5.6]. Moreover, $r(M)$ is the only eigenvalue of M with an eigenfunction in K; see [20, Thm. V.5.2(iv)].

Since the subalphabet $\iota_{\mathbf{a}}(\mathbb{N}_0)$ is dense in $\mathcal{A}$ and since ℓ is uniformly continuous on $\iota_{\mathbf{a}}(\mathbb{N}_0)$ by Lemma 4.2, it follows that ℓ extends to a unique continuous function on $\mathcal{A}$, which must necessarily be in the positive cone K, and must be an eigenfunction of M to the eigenvalue λ. It then follows that $\lambda = \mu + \frac{1}{\mu}$ must be the spectral radius $r(M)$ of the substitution operator. □

Remark. The second claim in the previous result is an infinite-alphabet analogue of Perron eigenvalue condition by Lagarias and Wang for inflation functional equations admitting weak Delone set solutions in $\mathbb{R}^d$, which states $\det(A) = \lambda(S)$ where $\lambda(S)$ is the Perron–Frobenius eigenvalue of the subdivision matrix S and A is the inflation map; see [15].

Lemma 4.4. *Let* $\mathbf{d} = (1, \mu, \mu^2, ...)^T$ *and* $\lambda = \mu + \dfrac{1}{\mu}$, *then*

$$\mathbf{A}\mathbf{d} = \lambda\mathbf{d}.$$

Remark. Here $(\dots)^T$ denotes the transpose vector (also infinite) in order to make it compatible with matrix-vector product.

Proof. Writing the equation $\mathbf{A}\mathbf{d} = \lambda\mathbf{d}$ row by row we obtain

$$\mu + \sum_{i=0}^{\infty} a_i \mu^i = \mu + \frac{1}{\mu} = \lambda \cdot 1$$

from the first row. This equation is satisfied by Equation (2). The i^{th} row (for $i \geq 2$) is

$$\mu^{i-2} + \mu^i = (\mu + \mu^{-1})\mu^{i-1} = \lambda \mu^{i-1},$$

and the claim follows. □

Before we proceed, we emphasize that $\mathbf{A}$ is the restriction of the *dual* M' of the substitution operator to the subalphabet $\iota_{\mathbf{a}}(\mathbb{N}_0)$ of isolated points. Here $M' \colon C(\mathcal{A})' \to C(\mathcal{A})'$, where $C(\mathcal{A})'$ is the space of continuous real-valued linear functionals on $C(\mathcal{A})$. Thus, right eigenvectors of $\mathbf{A}$ are related to the eigenfunctions of the dual operator M'.

We now get the following consequence regarding frequencies of isolated points for any element x in the subshift X_{ϱ} generated by $\varrho_{\mathbf{a}}$.

Proposition 4.5. *Let* $\mathbf{a}$ *be a sequence that satisfies* (A1)–(A3)*. Then the vector* $(1-\mu)\mathbf{d}$ *is the vector of frequencies of the isolated points.*

Proof. From primitivity and quasicompactness, it follows that the dual operator M' also has a one-dimensional eigenspace corresponding to $r(M)$, which is spanned by a strictly positive linear form φ in the dual space $C(\mathcal{A})'$ [20, Prop. III.8.5(c)]. Here, strict positivity means φ assigns a positive number to elements of the positive cone K of $C(\mathcal{A})$, i.e., $\varphi(f) > 0$ for $f \in K$. From Lemma 4.4, $\mathbf{d}$ is a right eigenvector of $\mathbf{A}$ to λ. We now show that the eigenvector $\mathbf{d}$ extends to a unique strictly positive continuous linear form.

Consider the functions of the form $\mathbb{1}_{[i]} \in K$ given by $\mathbb{1}_{[i]}(a) = 1$ if $a = [i]$ and 0 otherwise. Suppose φ is a continuous linear form with $\varphi(\mathbb{1}_{[i]}) > 0$ for all $[i]$. We show that $\varphi(f) > 0$ for all $f \in K$. Let $f \in K$. By a similar argument in the proof of Proposition 4.3, the only function with $f([i]) = 0$ for all $[i] \in \iota_{\mathbf{a}}(\mathbb{N}_0)$ is the constant function $f = 0$. Suppose $0 \neq f \in K$. Then $f([i]) = c > 0$ for some $[i] \in \iota_{\mathbf{a}}(\mathbb{N}_0)$. This means $f = g + c \cdot \mathbb{1}_{[i]}$ with $g \in K$. This implies $\varphi(f) = \varphi(g) + c\varphi(\mathbb{1}_{[i]}) > 0$ and hence φ is a strictly positive linear form.

Now, note that the Banach subalgebra $\mathcal{B}$ generated by the set $\{\mathbb{1}_{[i]}\}$ is dense in $C(\mathcal{A})$ by the Stone–Weierstrass theorem. Set $\varphi(\mathbb{1}_{[i]}) := \mathbf{d}_i$. By continuity and the denseness of $\mathcal{B}$ in $C(\mathcal{A})$, this extends to a unique $\varphi \in C(\mathcal{A})'$, which is strictly positive by the argument above, and hence must correspond to the unique eigenfunction of M' corresponding to the spectral radius, which completes the proof. The last statement regarding the frequencies follow from the normalization $\sum_{i \in \mathbb{N}_0} \varphi(\mathbb{1}_{[i]}) = 1$. □

Proof of Theorem 1.2. For a given λ, we can find an appropriate sequence using the results of Lemma 4.1. The resulting alphabet is infinite and compact. The only remaining part of the Theorem is to show that every element of the subshift gives rise to a Delone set. This follows automatically from the continuity and strict positivity of ℓ, which means there exist constants C_1 and C_2 such that for every $a \in \mathcal{A}$, $C_1 \leqslant \ell(a) \leqslant C_2$ where ℓ is the natural length function.

In fact, we get an explicit lower bound from property (A3), which states that for every $k > 0$ at least one of coefficients $a_k, \ldots, a_{k+C}$ is not zero. This means the internal sum in equation (3) for $j = k$ is at least

$$a_k \mu + a_{k+1}\mu^2 + \ldots + a_{k+C}\mu^{C+1}.$$

Thus for every letter $[k], k \in \mathbb{N}_0$, $\ell([k]) \geqslant \mu^{C+1}$. This completes the proof. □

Remark. It is worth noting that the substitution on the alphabet $\mathcal{A}$ is always on infinitely many letters but the resulting Delone set may have finite local (geometric) complexity.

For example, if $a_0 = 2$ and $a_1 = a_2 = a_3 = \ldots = 1$, then $\varrho_{\mathbf{a}}$ is a constant length substitution with $\lambda = 3$. In that case natural lengths of all tiles in $\mathcal{A}$ are 1 and the resulting Delone set coincides with $\mathbb{Z}$.

Moreover, if $\mathbf{a}$ is such that λ is an integer, then the lengths of all tiles are integers. This can be seen from the proof of Proposition 4.3 for tiles $[k]$ and from continuity of the natural length function for the accumulation points of the alphabet. Therefore the resulting Delone set is a subset of $\mathbb{Z}$ and is of finite local complexity as well.

Example 4.6. For every suitable sequence $\mathbf{a}$ we can construct a Delone set $\varLambda$ with the corresponding geometric inflation symmetry defined by λ or its integer power even when λ is transcendental.

Let k be an integer such that $\varrho_{\mathbf{a}}^k([0])$ contains letter $[0]$ inside. Once we turn the word $\varrho_{\mathbf{a}}^k([0])$ into a patch of length $\lambda^k \ell([0]) = \lambda^k$ there is an internal segment corresponding to $[0]$ of length 1. If $a_0 = 1$ we can take $k = 2$ and if $a_0 > 1$, then we can take $k = 1$. We the place the origin strictly inside this unit segment so that the λ^k-inflation turns the unit segment into the segment of length λ^k that correspond to $\varrho_{\mathbf{a}}^k([0])$. With this choice of the origin the substition $\varrho_{\mathbf{a}}^k$ applied to $[0]$ repeatedly will give a Delone set $\varLambda$ such that $\lambda^k \varLambda \subset \varLambda$ even when λ (and λ^k) is transcendental.

Note that this does not require one to start with a bi-infinite symbolic fixed point of ϱ (which in general need not exist). Moreover, due to Theorem 1.4, the tiling associated to this special Delone set $\varLambda$ generates the geometric hull $(\Omega_{\varrho}, \mathbb{R})$ by minimality.

5. Periodic sequences

The main goal of this section is to show that if $\mathbf{a}$ is eventually periodic, then the associated parameter μ and the inflation factor λ are algebraic. The converse statement is false and we provide an explicit example of a rational inflation factor that cannot be achieved in this framework using eventually periodic $\mathbf{a}$.

Proposition 5.1. *If* $\mathbf{a}$ *is eventually periodic, then* μ *and* λ *are algebraic.*

Proof. We assume that the sequence $(a_i)_i$ is periodic starting with index j with period k. So if $i \geqslant j$, then $a_{i+k} = a_i$. Then we can write

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} a_i\mu^i = a_0 + \ldots + a_{j-1}\mu^{j-1} + \sum_{i=j}^{\infty} a_i\mu^i =$$
$$= a_0 + \ldots + a_{j-1}\mu^{j-1} + (a_j\mu^j + \ldots + a_{j+k-1}\mu^{j+k-1}) \sum_{m=0}^{\infty} \mu^{mk} =$$
$$= a_0 + \ldots + a_{j-1}\mu^{j-1} + \frac{a_j\mu^j + \ldots + a_{j+k-1}\mu^{j+k-1}}{1-\mu^k}. \quad (5)$$

Multiplying by $\mu(1-\mu^k)$ we get that μ is a root of a polynomial with integer coefficients and hence algebraic.

Since $\lambda = \mu + \frac{1}{\mu}$, it is algebraic as well. □

Example 5.2. Let us take $\mu = \frac{2}{5}$ and $\lambda = \frac{29}{10}$.

We note that Equation (5) results in a polynomial with integer coefficients and the constant term 1. For our chosen $\mu = \frac{2}{5}$, no polynomial with integer coefficients and root μ can have constant term 1. This means that every sequence $\mathbf{a}$ that results in $\lambda = \frac{29}{10}$ as the inflation factor is non-periodic.

Another topic that shows a much clearer distinction between periodic and non-periodic sequences $\mathbf{a}$ is the structure of the accumulation points in the alphabet. If $\mathbf{a}$ is eventually periodic, then one can establish that the set $\mathcal{A} \setminus \iota_{\mathbf{a}}(\mathbb{N}_0)$ is finite. On the other hand, if $\mathbf{a}$ is not eventially periodic, the family of accumulation points is infinite and in particular, can be uncountable.

6. Example with transcendental inflation factor

In this section, we give an explicit substitution whose inflation factor λ is transcendental. Consider the Thue–Morse sequence $\mathtt{t}(n) := (-1)^{s_2(n)} \in \{-1, 1\}$ where $s_2(n)$ is the number of ones in the binary expansion of n, see [21, `A010060`]. Consider the generating function $T(z) := \sum_{n\geqslant 0} \mathtt{t}(n)z^n$, which is a transcendental power series over $\mathbb{Q}(z)$. Mahler proved the following result regarding values of T on algebraic parameters.

Theorem 6.1 ([16])**.** *Let* $\alpha \neq 0$ *be an algebraic number with* $|\alpha| < 1$*. Then the number* $T(\alpha)$ *is transcendental.*

We would like to leverage this result and use a modified version of $\mathbf{t}(n)$ as our sequence which will define a substitution. To this end, let

$$\mathbf{a}(n) := (3 + \mathbf{t}(n))/2, \tag{6}$$

which yields the (strictly positive) Thue–Morse sequence where 1 is replaced by 2 and -1 is replaced by 1. One can easily verify that the generating function $A(z)$ for the sequence $\mathbf{a}(n)$ satisfies

$$A(z) = \frac{3}{2} \cdot \frac{1}{1-z} + \frac{1}{2} T(z) \tag{7}$$

We now have the following result.

Theorem 6.2. *Let* $\mathbf{a}$ *be the sequence given in Eq.* (6) *and consider the corresponding substitution* $\varrho_{\mathbf{a}}$*. One then has*

(1) The subshift (X_ϱ, σ) *and the tiling dynamical system* $(\Omega_\varrho, \mathbb{R})$ *are strictly ergodic.*
(2) There exists a unique natural tile length function for ϱ *which is strictly positive*
(3) The inflation factor λ *corresponding to* $\varrho_{\mathbf{a}}$ *is transcendental.*

Proof. The first two claims follow directly from Theorem 1.4 since $\varrho_{\mathbf{a}}$ is primitive, recognisable and has a quasicompact substitution operator (after normalisation). It remains to show that λ is transcendental.

We know that $\lambda = \mu + \frac{1}{\mu}$, where $\mu \in (0,1)$ is the unique solution to Eq. (2), which means $\frac{1}{\mu} = A(\mu)$. It suffices to show that μ is transcendental. From Eq. (7), one has

$$\frac{1}{\mu} = A(\mu) = \frac{3}{2} \cdot \frac{1}{1-\mu} + \frac{1}{2} T(\mu).$$

Suppose on the contrary μ is algebraic. Then, by Theorem 6.1 $T(\mu)$, is transcendental because $\mu \in (0,1)$.

On the other hand, $T(\mu) = \frac{2}{\mu} - \frac{3}{1-\mu}$ is algebraic which is a contradiction. Thus μ is transcendental and hence is λ, as μ is the solution of the equation $x^2 - \lambda x + 1 = 0$. $\square$

In fact, adding a nonnegative periodic sequence $\mathbf{b} = (b_n)_{n \geqslant 0}$ to $\mathbf{a}$ gives rise to another sequence whose corresponding λ which is also transcendental. To see this, suppose the smallest period of $\mathbf{b}$ is p. The generating function for $\mathbf{b}$ is then

$$B(z) = \frac{b_0 + b_1 z + \cdots + b_{p-1} z^{p-1}}{1 - z^p}$$

where b_i's are integers. Considering $\mathbf{a} + \mathbf{b}$, and inserting μ in the generating function, we get

$$\frac{1}{\mu} = (A+B)(\mu) = \frac{3}{2} \cdot \frac{1}{1-\mu} + \frac{1}{2} T(\mu) + \frac{b_0 + b_1 \mu + \cdots + b_{p-1} \mu^{p-1}}{1 - \mu^p},$$

where the last term is clearly algebraic whenever μ is algebraic. We arrive at the same contradiction, and hence λ must again be transcendental. The same argument extends when

b is eventually periodic. This provides a way to construct an explicit infinite subfamily with transcendental λ.

Remark. We note that in general the transcendence of the generating function $A(z)$ over the field $\mathbb{Q}(z)$ does not suffice to conclude that $A(\mu)$ is a transcendental number whenever $\mu \in (0,1)$ is algebraic. The Thue–Morse power series $T(z)$ satisfies the functional equation $T(z) = (1-z)T(z^2)$. For power series satisfying similar functional equations (called Mahler-type functional equations), dichotomy results regarding the transcendence of number of the form $A(\mu)$ are available; see [1, Thm. 5.3]. We refer the reader to [1] for a survey on Mahler's method on transcendence and linear independence results.

7. Concluding remarks

There is an alternative definition of quasicompactness which involves the *essential spectrum*. For the substitution operator M, this is equivalent to the essential spectral radius $r_{\text{ess}}(M)$ being strictly less than $r(M)$. We conjecture that $r_{\text{ess}}(M) = 2$ for all substitutions we constructed in this work which satisfy the conditions (A1)–(A3). This has a variety of implications in discrepancy estimates, in particular to questions regarding bounded distance equivalence, which we plan to address in forthcoming work [12].

It also remains to find the right conditions on the inflation factor λ which guarantee or exclude the existence of measurable or continuous eigenvalues for either dynamical system (X_ϱ, σ) or $(\Omega_\varrho, \mathbb{R})$. Note that convergence of points $x_n \to x$ in X_ϱ is more subtle than in shifts over finite alphabets, where converging points must necessarily agree on a large patch around the origin. In the infinite local complexity regime, there are more mechanisms for convergence which do not require exact agreement. There are results on weaker versions of the Pisot–Vijayaraghavan problem; see [5] for instance. It would be interesting to see whether such numbers with biased distribution mod 1 give rise to substitutions with interesting spectral features.

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