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Diederich-Fornæss index and global regularity in the $\bar{\partial}$-Neumann problem: domains with comparable Levi eigenvalues

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DIEDERICH–FORNÆSS INDEX AND GLOBAL REGULARITY IN THE $\overline{\partial}$–NEUMANN PROBLEM: DOMAINS WITH COMPARABLE LEVI EIGENVALUES

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Abstract. Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $1 \leq q_0 \leq (n-1)$. We show that if q_0–sums of eigenvalues of the Levi form are comparable, then if the Diederich–Fornæss index of Ω is 1, the $\overline{\partial}$–Neumann operators N_q and the Bergman projections P_{q-1} are regular in Sobolev norms for $q_0 \leq q \leq n$. In particular, for domains in $\mathbb{C}^2$, Diederich–Fornæss index 1 implies global regularity in the $\overline{\partial}$–Neumann problem.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with (C^∞) smooth boundary. Then there exists $\eta \in (0,1)$ and a defining function ρ_η such that $-(-\rho_\eta)^\eta$ is plurisubharmonic on Ω near $b\Omega$ ([17],[32]). The supremum over all such η has come to be known as the Diederich–Fornæss index of Ω. The relevance of the index stems from the fact that in general, Ω does not admit a defining function that is plurisubharmonic on Ω near the boundary, not even locally ([18], [2], [20]).

The index is known to be strictly less than one on the Diederich–Fornæss worm domains ([17]; see [29] for the exact value). It is known to be equal to one in the following cases: Ω admits a defining function that is plurisubharmonic *at* (not necessarily near) the boundary ([19]), there are 'good vector fields' and the set of infinite type points is 'well behaved' ([22]), $b\Omega$ satisfies Property(P) ([22]), Ω is strictly pseudoconvex except for a simply connected complex manifold in the boundary ([28]). There are two things to note about this list. First, index one does not imply that there is a defining function that is plurisubharmonic on Ω (near $b\Omega$). Indeed, domains with real analytic boundaries are of finite type, so satisfy property(P), yet need not admit even local plurisubharmonic defining functions ([2, 20]). Second, and perhaps more strikingly, all domains in the list with index one are known to have globally regular Bergman projections and $\overline{\partial}$–Neumann operators ([8, 9, 10, 34, 22]), while on the worm domains, these operators are regular only up to a finite Sobolev level that is closely related to their index (see [6, 1, 13, 29]).

The question what the implications of the Diederich–Fornæss index are for global regularity arose in earnest after the results on the worm domains in [6, 1, 13] and after [8], where the authors showed that if Ω admits a defining function that is plurisubharmonic at the boundary, then the Bergman projections and $\overline{\partial}$–Neumann operators (at all from levels) on Ω are globally exactly regular, i.e. are continuous in Sobolev–s norms for $s \geq 0$. A quantitative study of this question was initiated in [27] where it was shown that if the index is one, and there is some control on the defining functions ρ_η as $\eta \to 1^-$, then global regularity holds. More precisely, the condition is that $\liminf_{(\eta\to 1^-)}((1-\eta)^{1/3}\max_{b\Omega}|\nabla h_\eta|) = 0$, where h_η is given by $\rho_\eta = e^{h_\eta}\rho$ for some fixed defining function ρ. The exponent 1/3 was

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improved to 1/2 in [21]. In [5], it is shown that if the index is $\eta_0 \leq 1$, then regularity holds for $0 \leq s < \eta_0/2$ ($\leq 1/2$). In [31], the authors, among other things, generalize the ideas from [27] to q–convex domains, with a corresponding notion of the index. More recently, the first author showed in [30] that index one implies global regularity for domains with comparable eigenvalues of the Levi form without assuming any control on the defining functions ρ_η, but instead making a technical assumption that controls the geometry of the set of infinite type boundary points.

In this paper, we show, via a different proof, that this assumption is not needed. We also consider the assumption at the level of q–forms for $q > 1$. In this case, it says that sums of q eigenvalues of the Levi form should be comparable; see section 2 for a precise definition and discussion.

Theorem 1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q_0 \leq (n-1)$. Assume that q_0–sums of the eigenvalues of the Levi from are comparable. Then, if the Diederich-Fornæss index of Ω is 1, the Bergman projections P_{q-1} and the $\overline{\partial}$–Neumann operators N_q, $q_0 \leq q \leq n$, are continuous in Sobolev–s norms for $s \geq 0$.*

When $q = (n-1)$, there is only one q–sum, and the comparability condition is trivially satisfied.

Corollary 1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. If the Diederich-Fornæss index of Ω is 1, the Bergman projection P_{n-2} and the $\overline{\partial}$–Neumann operators N_{n-1} are continuous in Sobolev–s norms for $s \geq 0$.*

For emphasis, we single out the case $n = 2$, obtaining regularity for P_0 and N_1, the cases usually considered the most important.

Corollary 2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. If the Diederich-Fornæss index of Ω is 1, the Bergman projection P_0 and the $\overline{\partial}$–Neumann operator N_1 are continuous in Sobolev–s norms for $s \geq 0$.*

Remark: Two smooth bounded pseudoconvex domains in $\mathbb{C}^n$ which are biholomorphic via a biholomorphism that extends to a diffeomorphism of the closures have the same Diederich–Fornæss index. An immediate consequence of Corollary 2 is therefore that if two smooth bounded pseudoconvex domains in $\mathbb{C}^2$ are biholomorphic, then if one has index 1, so does the other (the biholomorphism extends, by [3]).

The main ingredients in our proof of Theorem 1 are first recent work in [29], as reformulated in [35], where it is shown how to express the Diederich–Fornæss index in terms of estimates on D'Angelo forms $\alpha_\eta := \alpha^{\rho_\eta}$ associated with a defining function ρ_η of the domain (see section 2). The resulting estimates on α_η then involve $\sum_{j,k}(\partial^2 h_\eta/\partial z_j \partial\overline{z_k})u_j\overline{u_k}$, where h_η is as above (see (9) below). The second important point is the observation that one does not need pointwise estimates on $|\alpha_\eta|$, much weaker L^2–type estimates suffice (compare [33]). So instead of controlling the Hessian of h_η pointwise, which seems hopeless, one only has to deal with $\int_\Omega \sum_{j,k}(\partial^2 h_\eta/\partial z_j \partial\overline{z_k})u_j\overline{u_k}$. This latter expression is familiar in the L^2 theory of the $\overline{\partial}$–Neumann problem, and one can tweak known machinery to obtain a strong estimate on $\int_\Omega |\alpha_\eta(L_u)|^2$ (Proposition 1 below). From the point of view of exploiting the Diederich–Fornæss index, this is the central estimate. Once this estimate is in hand, the proof of Theorem 1 initially follows [33], but then uses the formulas for the commutators of $\overline{\partial}$ and $\overline{\partial}^*$ with the 'usual' transversal vector fields found in [23] (Lemmas 3 and 4 below).

The rest of the paper is organized as follows. Section 2 contains definitions and notation, as well as some preliminary lemmas. Section 3 contains Proposition 1 and its proof. The proof of Theorem 1 is given in section 4.

2. Notation and Preliminaries

Let Ω be a bounded domain in $\mathbb{C}^n$ with (C^∞) smooth boundary. We denote by P_q the Bergman projection on $(0,q)$–forms, $0 \leq q \leq n$, and by N_q the $\overline{\partial}$–Neumann operator on $(0,q)$–forms, $1 \leq q \leq n$. We refer the reader to [12] and [34] for background on these topics, as well as for standard notation.

We use the following notation for weighted Sobolev norms: for a real valued function g so that $e^{-g} \in C^\infty(\overline{\Omega})$, we set $\|u\|_{k,g} = \left(\sum_{s=0}^{k} \int_\Omega |\nabla^s u|^2 e^{-g}\right)^{1/2}$, where ∇^s denotes the vector of all derivatives of order s. If u is a form, the derivatives act coefficients as usual (in Euclidean coordinates, unless stated otherwise).

It will be important that certain constants do not depend on η. We adopt the convention to denote such constants with C and we allow the actual value to change from one occurrence to the next. On the other hand, constants that do depend on η will be denoted by C_η, with the same convention about the actual value. We will similarly use subscripts to denote dependence on the form level, or on both form level and η. When it becomes cumbersome to write $C(\cdots)$, we will use the symbol $\lesssim$ instead.

It is by now a standard fact that derivatives of type $(0,1)$ and complex tangential derivatives of either type are benign for the $\overline{\partial}$–Neumann problem and the Bergman projection ([8], [34], Lemma 5.6). The proof of Theorem 1 requires a version for weighted norms, which we formulate here for the reader's convenience; it follows readily form the unweighted version.

Lemma 1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $k \in \mathbb{N}$, and Y a vector field of type $(1,0)$ with coefficients in $C^\infty(\overline{\Omega})$ that is tangential on the boundary. There exists a constant C such that when the weight $e^{-h/2} \in C^\infty(\overline{\Omega})$, there exists a constant C_h so that for $u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)$ we have the estimates*

$$\sum_{j,J} \left\| \frac{\partial u_J}{\partial \overline{z_j}} \right\|^2_{k-1,h} \leq C \left(\|\overline{\partial} u\|^2_{k-1,h} + \|\overline{\partial}^* u\|^2_{k-1,h} \right) + C_h \|u\|^2_{k-1,h} \,, \tag{1}$$

and

$$\|Yu\|^2_{k-1,h} \leq C \left(\|\overline{\partial} u\|^2_{k-1,h} + \|\overline{\partial}^* u\|^2_{k-1,h} \right) + C_h \|u\|_{k-1,h} \|u\|_{k,h} \,. \tag{2}$$

Writing weights as exponentials is convenient in the context of Kohn–Morrey–Hörmander type formulas, but somewhat artificial in the lemma, as only the smoothness of $e^{-h/2}$ matters. We keep the convention so we do not need to introduce additional notation.

Proof of Lemma 1. Apply the unweighted version ([34], Lemma 5.6) to the form $e^{-h/2}u \in dom(\overline{\partial}^*)$ and observe that if a derivative D hits $e^{-h/2}$, the resulting term is of the form $(Dh)e^{-h/2}u$ and is thus dominated by $C_h\|u\|^2_{k-1,h}$ and $C_h\|u\|_{k-1,h}\|u\|_{k,h}$, respectively. □

There are two points in the proof of Theorem 1 where we need an (unweighted) estimate that is stronger than (2), namely (with Y as in Lemma 1)

$$\|Yu\|^2_{k-1} \leq C \left(\|\overline{\partial} u\|^2_{k-1} + \|\overline{\partial}^* u\|^2_{k-1} \right) + C_h \|u\|^2_{k-1} \,. \tag{3}$$

Such estimates, referred to as maximal estimates, play an important role in the theory of the $\overline{\partial}$–Neumann problem (see for example the introduction in [25]). We will however use Lemma 1 where possible, so as to minimize the use of these stronger estimates.

The necessary and sufficient condition for maximal estimates to hold is that q–sums of eigenvalues of the Levi eigenvalues dominate the trace of the Levi form ([15], Théorème 3.1 for $q = 1$, [4], Théorème 3.7 for $q > 1$). More precisely, this condition means that there is a constant C such that for any point $p \in b\Omega$ and $j_1, \cdots, j_q$ with $1 \le j_1 < \cdots < j_q \le (n-1)$, it holds that $C(\lambda_1(p) + \lambda_2(p) + \cdots + \lambda_{n-1}(p)) \le \lambda_{j_1}(p) + \cdots + \lambda_{j_q}(p)$, where $\lambda_1(p), \cdots, \lambda_{(n-1}(p)$ denote the eigenvalues of the Levi from at p, listed with multiplicity These eigenvalues are independent of the basis as long as they are taken with respect to an orthonormal basis[1]. Because Ω is pseudoconvex, all Levi eigenvalues are non negative, and the previous condition is easily seen to be equivalent to the following one: there exists a constant C such that for any pair of q–tuples $(j_1, \cdots, j_q)$ and $(k_1, \cdots, k_q)$, it holds that $\lambda_{j_1}(p) + \cdots + \lambda_{j_q}(p) \le C(\lambda_{k_1}(p) + \cdots + \lambda_{k_q}(p))$ (by an argument similar to the one in the proof of Lemma 2 below). We say for short that q–sums of the Levi eigenvalues of Ω are comparable. Note that for $q = (n-1)$, there is only one q–sum, and the comparability condition trivially holds. In particular, the condition trivially holds for domains in $\mathbb{C}^2$ and $q = 1$.

We will need that the comparability condition on the Levi eigenvalues percolates up the Cauchy–Riemann complex:

Lemma 2. *Ω as above. Assume the comparability condition for q–sums of eigenvalues of the Levi form is satisfied for some level q. Then it is satisfied at level $(q+1)$.*

Proof. The (standard) argument is to observe that for $\{j_1, \cdots, j_{q+1}\}$ fixed, $\lambda_{j_1}(p) + \cdots \lambda_{j_{q+1}}(p) = (1/q) \sum_{\{k_1,\cdots,k_q\} \subset \{1,\cdots,q+1\}} (\lambda_{j_{k_1}}(p) + \cdots + \lambda_{j_{k_q}})$. ☐

Let ρ be a defining function for Ω. Near $b\Omega$, set $L_n^\rho = (1/\sum_j |\partial\rho/\partial z_j|^2) \sum_{j=1}^n (\partial\rho/\partial\overline{z_j})\partial/\partial z_j$, and continue it smoothly to all of Ω. Then $L_n^\rho \rho \equiv 1$ near $b\Omega$ (so L_n^ρ is in general normalized differently from L_n). We also define $T^\rho := L_n^\rho - \overline{L_n^\rho}$, and $\sigma^\rho := (1/2)(\partial\rho - \overline{\partial}\rho)$; note that $\sigma^\rho(T^\rho) \equiv 1$ near $b\Omega$. The D'Angelo 1–from is defined as $\alpha^\rho = -\mathcal{L}_{T^\rho}\sigma^\rho$, where $\mathcal{L}_{T^\rho}$ denotes the Lie derivative in the direction of T^ρ. Although α^ρ depends on the defining function ρ, all its relevant properties are intrinsic (that is, do not depend on the choice of ρ). We will use various of these properties, all of which can be found in [34]. In particular, if $L = \sum_{k=1}^n w_k(\partial/\partial z_k)$ is a local section of $T^{(1,0)}(b\Omega_\varepsilon)$, where $\Omega_\varepsilon = \{z \in \Omega \mid \rho(z) < -\varepsilon\}$, $0 \le \varepsilon$ small, then

$$\alpha^\rho(\overline{L}) = \partial\rho([L_n^\rho, \overline{L}]) = \frac{1}{\sum_j |\partial\rho/\partial z_j|^2} \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j \partial\overline{z_k}} \frac{\partial\rho}{\partial\overline{z_j}} \overline{w_k} \; ; \tag{4}$$

(see [34], (5.76) and (5.85)). Moreover, if $\rho_h = e^h\rho$ is another defining function and α_h is the associated D'Angelo form, then

$$\alpha_h(L) = \alpha^\rho(L) + \partial h(L) \tag{5}$$

([34], (5.84)).

The forms α^ρ play a crucial role for Sobolev estimates in the $\overline{\partial}$–Neumann problem in that they control commutators of certain vector fields with $\overline{\partial}$ and $\overline{\partial}^*$(compare (4) above, [34], [23],[14]). We recall two lemmas from [23] that give explicit formulas for these commutators, modulo benign errors. These formulas are stated for forms supported in a special boundary chart, with the usual orthonormal boundary frame $L_1, \cdots, L_n$ and dual frame $\omega_1, \cdots, \omega_n$ (note that L_n differs from L_n^ρ by its normalization). Vector fields (differential operators) on forms are acting coefficientwise in this special chart. Lemmas 3 and 4 are for forms $v = {\sum}'_J v_J \overline{\omega_J} \in C^\infty_{(0,q)}(\overline{\Omega})$, supported in a special boundary chart, $0 \le q \le n$.

[1] In [11], footnote 1, the authors point out that thanks to a theorem of Ostrowski ([24], Theorem 4.5.9) the comparability condition remains independent of the basis even when the orthonormality requirement is dropped.

Lemma 3 ([23], Lemma 3.1)**.** *Let $k \in \mathbb{N}$, $\psi \in C^\infty(\overline{\Omega})$, and ρ a defining function for Ω. Then*

$$
\begin{aligned}
(6)\quad & [\overline{\partial}, (e^{-\psi}(L_n - \overline{L_n}))^k]v \\
&= -k \sum_j {\sum_J}' (\overline{L_j}\psi + \alpha^\rho(\overline{L_j}))(e^{-\psi}(L_n - \overline{L_n}))^k v_J\, \overline{\omega_j} \wedge \overline{\omega_J} + A_\psi(v) + B_\psi(v)\ ,
\end{aligned}
$$

where $A_\psi(v)$ consists of terms of order k with at least one of the derivatives being barred or complex tangential, and $B_\psi(v)$ consist of terms of order at most $(k-1)$.

We also have

Lemma 4 ([23], Lemma 3.2)**.** *With the assumptions as in Lemma 3, but additionally $v \in dom(\overline{\partial}^*)$ and $q \geq 1$. Then*

$$
\begin{aligned}
(7)\quad & [\overline{\partial}^*, (e^{-\psi}(L_n - \overline{L_n}))^k]v \\
&= \sum_{j<n} {\sum_S}' (L_j(\psi) + \alpha^\rho(L_j))(e^{-\psi}(L_n - \overline{L_n})^k u_{jS}\, \overline{\omega_S} + A_\psi(v) + B_\psi(v) + C_\psi(v_N),
\end{aligned}
$$

where $A_\psi(v)$ and $B_\psi(v)$ are of the same type as in Lemma 3 (but different operators), and $C_\psi(v_N)$ is of order k on the normal component $v_N = {\sum}'_S v_{Sn}\, (\overline{\omega_S} \wedge \overline{\omega_n})$ of v.

Note that if ψ is real valued (this not required in the lemma) and $j < n$, then $\overline{L_j}\psi + \alpha^\rho(\overline{L_j})$ equals $\alpha^{\rho(\psi)}(\overline{L_j})$, where $\alpha^{\rho(\psi)}$ is the D'Angelo form associated with the defining function $e^\psi\rho$ (in view of (5)). Likewise, $L_j(\psi) + \alpha^\rho(L_j) = \alpha^{\rho(\psi)}(L_j)$ when $j < n$. The error terms $A_\psi(v)$, $B_\psi(v)$, and $C_\psi(v_N)$ are normally under control ($B_\psi(v)$ is lower order, for $A_\psi(v)$ use Lemma 1, and the normal component acts as a subelliptic multiplier, see estimate (2.96) in [34]; Proposition 4.7, part (G), in [26]). In fact, in [23] the error terms are given only in this estimated form. On the other hand, for the proof of the lemmas, which results from computing the case $k = 1$ in the boundary frame, and then using induction on k, it is convenient to have an expression for the terms.

If the Diederich–Fornæss index of Ω is 1, then there exist $\eta \in (0,1)$, arbitrarily close to 1, and defining functions $\rho_\eta = e^{h_\eta}\rho$ such that $-(-\rho_\eta)^\eta)$ is plurisubharmonic near the boundary in Ω. It easily follows that this statement then holds for all $\eta \in (0,1)$ (when $0 < \eta < \eta_1$, compose the plurisubharmonic function $-(-\rho_{\eta_1})^{\eta_1}$ with the convex increasing function $-(-x)^{\eta/\eta_1}$, $x < 0$, to obtain that $-(-\rho_{\eta_1})^\eta$ is plurisubharmonic). We fix a defining function ρ for Ω that near the boundary agrees with the signed boundary distance. For $\eta \in (0,1)$, denote by h_η a function in $C^\infty(\overline{\Omega})$ so that $-(-\rho_\eta)^\eta$ is plurisubharmonic near the boundary in Ω (recall that $\rho_\eta := e^{h_\eta}\rho$).

3. Diederich-Fornæss index and estimates on D'Angelo forms

Our proof of Theorem 1 depends on recent work in [29], as reformulated in [35], where it is shown that the Diederich–Fornæss index of a domain can be expressed in terms of the form α_η ($= \alpha^{\rho_\eta}$). Denote by ω_η the $(1,0)$–part of α_η; then $\alpha_\eta = \omega_\eta + \overline{\omega_\eta}$ (there is no ambiguity as to the these parts for the real one form α_η on the boundary, because $\alpha_\eta(T) = 0$, see also the discussion in Section 4.3 of [14]). The formulation most convenient for us is in Theorem 1.1 in [35]:

$$
(8)\quad DF(\Omega) = \sup\left\{0 < \eta < 1 : \left(\frac{\eta}{1-\eta}(\omega_\rho \wedge \overline{\omega_\rho} - \overline{\partial}\omega_\rho)\right)(L \wedge \overline{L}) \leq 0\ ;\ p \in \Sigma\ ,\ L \in \mathcal{N}_p\right\}\ ,
$$

where $\Sigma \subset b\Omega$ denotes the set of weakly pseudoconvex boundary points, $\mathcal{N}_p$ denotes the null space of the Levi from at $p \in b\Omega$, and the supremum is over all η so that there exists a defining function ρ_η such that the inequality holds. Because $|\alpha_\eta(L)|^2 = (\omega_\eta \wedge \overline{\omega_\eta})(L \wedge \overline{L})$ and $\overline{\partial}\omega_\eta(L \wedge \overline{L}) = \overline{\partial}\alpha_\eta(L \wedge \overline{L})$, (8) implies $|\alpha_\eta(L(p))|^2 \lesssim (1-\eta)\overline{\partial}\alpha_\eta(L(p) \wedge \overline{L(p)})$; $p \in \Sigma$, $L(p) \in \mathcal{N}_p$. Computing α_η from its definition as a Lie derivative (see section 5.9 in [34]) shows that $\alpha_\eta = \alpha + dh_\eta + (Th_\eta)\sigma$. Therefore $\overline{\partial}\alpha_\eta = \overline{\partial}\alpha + \overline{\partial}dh_\eta + \overline{\partial}((Th_\eta)\sigma)$. The last term vanishes on $\mathcal{N}_p$: $\overline{\partial}((Th_\eta)\sigma)(L \wedge \overline{L}) = (\overline{\partial}(Th_\eta) \wedge \sigma)(L \wedge \overline{L}) + (Th_\eta)\overline{\partial}(\partial\rho - \overline{\partial}\rho)(L \wedge \overline{L})$; the first term vanishes because σ annihilates L and $\overline{L}$, the second because $\overline{\partial}\partial\rho(L \wedge \overline{L}) = 0$ when $L \in \mathcal{N}_p$. Since also $\overline{\partial}dh_\eta = \overline{\partial}\partial h_\eta = -\partial\overline{\partial}h_\eta$, we have

$$(9) \qquad |\alpha_\eta(L(p))|^2 \lesssim (1-\eta)\left(\sum_{j,k=1}^{n} \frac{\partial^2(-h_\eta)}{\partial z_j \partial \overline{z_k}} L_j \overline{L_k} + |L|^2\right) \ ; \ p \in \Sigma\ ,\ L(p) \in \mathcal{N}_p\ ;$$

the $|L|^2$ term comes from $\overline{\partial}\alpha$.

We also need a bound when $L(p)$ is merely complex tangential at the boundary. Specifically, we need a bound for $\sum_J{}' |\alpha_\eta(L_u^J)|^2$, where $u = \sum_K{}' u_K \overline{dz_K} \in C^\infty_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)$ and $L_u^J = \sum_j u_{jJ}(\partial/\partial z_j)$. Denote by A the (closed) set of all boundary points where the Levi form vanishes identically. A equals the set where every sum of q eigenvalues vanishes (by pseudoconvexity). At points p of A, (9) holds for every $L(p) \in T_p^{\mathbb{C}}(b\Omega)$. By continuity and homogeneity, this estimate holds for p in a neighborhood U_η of A (by, say, doubling the constant,which stays independent of η). The assumption of comparable q-sums of eigenvalues implies that the complement of A in the boundary is the set of points where the sum of *any* q eigenvalues is strictly positive. This implies (again by continuity and homogeneity) that if $A \subset V_\eta \subset\subset U_\eta$, with V_η open, then on $b\Omega \setminus V_\eta$

$$(10) \quad \sum_J{}' |\alpha_\eta(L_u^J)|^2 \lesssim (1-\eta)\left(\sum_J{}' \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial \overline{z_k}} u_{jJ}\overline{u_{kJ}} + O(|u|^2)\right) + M_\eta \sum_J{}' \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}} u_{jJ}\overline{u_{kJ}}$$

for a big enough constant M_η (because there is a strictly positive lower bound on the form in the second line of (10) on $b\Omega \setminus V_\eta$). Since the estimate in the first line of (10) holds on U_η, and Ω is pseudoconvex (i.e. the form on the second line of (10) is nonnegative on the whole boundary), (10) holds on U_η as well, and so holds on all of $b\Omega$, and therefore in a neighborhood W_η of $b\Omega$. Note that upon inserting (5) into the left hand side of (10), the estimate becomes, roughly speaking, a self bounded gradient condition for $(-h_\eta)$ with constant $(1-\eta)$, modulo terms that turn out to be under control. This observation is the key to proving estimates despite the absence of bounds on the functions h_η.

We note in passing that the use of the comparable eigenvalues condition is merely convenient, rather than necessary, to obtain (10). One can always estimate components of a form in strictly pseudoconvex directions by the Levi form; an argument analogous to the one above, with a little additional work, then gives (10).

In the proof of Theorem 1, we will 'only' need to estimate $\int_\Omega |\alpha_\eta(L_u^J)|^2$, rather than work with the point wise estimate (10). This is essential: once we integrate the right hand side of (10), we can use L^2 methods (and in particular the approximate self bounded gradient condition for $(-h_\eta)$ mentioned above) to obtain estimate (11) below. This estimate is the crux of the matter.

Proposition 1. *Assumptions as in Theorem 1, $q_0 \leq q \leq (n-1)$. There are a constant C and, for $(1-\eta)$ small enough, a constant C_η and a relatively compact subdomain Ω_η such that*

$$\sum_J{}' \int_\Omega |\alpha_\eta(L_u^J)|^2 \leq C(1-\eta)(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2) + C_\eta(\|\overline{\partial}u\|^2_{\Omega_\eta} + \|\overline{\partial}^*u\|^2_{\Omega_\eta} + \|u\|^2_{-1})\ ; \quad u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)\ ; \tag{11}$$

recall that $L_u^J = \sum_j u_{jJ}(\partial/\partial z_j)$.

Proof. Integrate the right hand side of (10) over Ω. For the term coming from the Hessian of ρ, we use that this Hessian (the Levi from) acts like a subelliptic multiplier (combine [26], Proposition 4.7, part (C), with the Cauchy–Schwarz inequality). The contribution from this term can therefore be estimated by $(1-\eta)(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2) + C_\eta\|u\|^2_{-1}$. The resulting estimate is

$$\sum_J{}' \int_\Omega |\alpha_\eta(L_u^J)|^2 \lesssim (1-\eta)\left(\sum_J{}' \int_\Omega \sum_{j,k=1}^n \frac{\partial^2(-h_\eta)}{\partial z_j \partial\overline{z_k}} u_{jJ}\overline{u_{kJ}} + \|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2\right) + C_\eta\|u\|^2_{-1}\ ; \quad u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)\ . \tag{12}$$

The Kohn–Morrey–Hörmander formula gives for a $(0,q)$–form $w \in C^\infty_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)$

$$\sum_{|J|=(q-1)}{}' \int_\Omega \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial\overline{z_k}} w_{jJ}\overline{w_{kJ}} e^{h_\eta} \leq \|\overline{\partial}w\|^2_{-h_\eta} + \|\overline{\partial}^*_{-h_\eta}w\|^2_{-h_\eta}\ . \tag{13}$$

The idea is now to set $w = e^{-(h_\eta/2)}u$ (in view of what is needed in (12)). The $\overline{\partial}^*$–term on the right in (13) then simplifies to $\overline{\partial}^*u + (1/2)\sum'_J \left(\sum_j(\partial h_\eta/\partial z_j)u_{jJ}\right) d\overline{z_J}$ (see the computation on page 116 in [34], where this observation is also used). The term $\sum'_J \left(\sum_j(\partial h_\eta/\partial z_j)u_{jJ}\right) d\overline{z_J}$ can be handled via $dh_\eta = (\alpha_\eta - \alpha)$ on $T^{1,0}(b\Omega)$. However, handling the resulting $\overline{\partial}$–term requires a modification of this idea. Set $w := e^{-\widetilde{(h_\eta/2)}}u$ instead, where $\widetilde{h_\eta}$ agrees with h_η on the boundary, and is extended in such a way that $\overline{L_n}\widetilde{h_\eta} = 0$ on the boundary (this amounts to prescribing the real normal derivative at points of the boundary; $\widetilde{h_\eta}$ need no longer be real away from the boundary). This definition will have the effect that the normal component of $\overline{\partial}\widetilde{h_\eta}$ vanishes on the boundary. Estimate (13) becomes

$$\sum_J{}' \int_\Omega \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial\overline{z_k}} u_{jJ}\overline{u_{kJ}} e^{(h_\eta - \widetilde{h_\eta})} \leq \|\overline{\partial}(e^{-\widetilde{(h_\eta/2)}}u)\|^2_{-h_\eta} + \|\overline{\partial}^*_{-h_\eta}(e^{-\widetilde{(h_\eta/2)}}u)\|^2_{-h_\eta}\ . \tag{14}$$

We have

$$\begin{aligned} \|\overline{\partial}(e^{-\widetilde{(h_\eta/2)}}u)\|^2_{-h_\eta} &= \|e^{(h_\eta/2)}\overline{\partial}(e^{-\widetilde{(h_\eta/2)}}u)\|^2 \\ &= \|e^{(h_\eta-\widetilde{h_\eta})/2}(-\frac{1}{2}\overline{\partial}\widetilde{h_\eta}\wedge u + \overline{\partial}u)\|^2 \lesssim \|\overline{\partial}\widetilde{h_\eta}\wedge u\|^2 + \|\overline{\partial}u\|^2 + C_\eta\left(\|\rho u\|^2 + \|\rho\overline{\partial}u\|^2\right)\ ; \end{aligned} \tag{15}$$

the error term on the right again results because $\widetilde{h_\eta}$ and h_η agree on the boundary. It is in controlling the wedge term that we will need the normal component of $\overline{\partial}\widetilde{h_\eta}$ to vanish on the boundary. The modification also introduces a (benign) error in the $\overline{\partial}^*$–term in (14):

$$(16)\quad \|\overline{\partial}^*_{-h_\eta}(e^{-(\widetilde{h_\eta}/2)}u)\|^2_{-h_\eta} = \|e^{(h_\eta/2)}\overline{\partial}^*_{-h_\eta}(e^{-(\widetilde{h_\eta}/2)}u)\|^2$$
$$= \|e^{(h_\eta-\widetilde{h_\eta})/2}(\overline{\partial}^*u - \sum_J{}' (\sum_{j=1}^n \frac{\partial}{\partial z_j}(h_\eta - \frac{\widetilde{h_\eta}}{2})u_{jJ})d\overline{z_J})\|^2$$
$$\lesssim \|\overline{\partial}^*u\|^2 + \|\sum_J{}'(\sum_{j=1}^n \frac{\partial h_\eta}{\partial z_j}u_{jJ})d\overline{z_J}\|^2 + C_\eta(\|\rho u\|^2 + \|\rho\overline{\partial}^*u\|^2 + \|u_N\|^2)\ ,$$

where u_N denotes the normal component of u. We have used that the tangential components of $\partial(h_\eta - \widetilde{h_\eta}/2)$ agree with those of $\partial(h_\eta/2)$ at the boundary. Note that $\sum_{j=1}^n \frac{\partial h_\eta}{\partial z_j}u_{jJ} = \partial h_\eta(L_u^J) = dh_\eta(L_u^J) = (\alpha_\eta - \alpha)(L_u^J)$ at the boundary. Therefore, $\|\sum_{j=1}^n \frac{\partial h_\eta}{\partial z_j}u_{jJ}\|^2 \lesssim \|\alpha_\eta(L_u^J)\|^2 + \|\alpha(L_u^J)\|^2 + C_\eta\|\rho u\|^2$. Combining (14)–(16) and collecting the error terms, we arrive at

$$(17)\quad \sum_J{}' \int_\Omega \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial\overline{z_k}}u_{jJ}\overline{u_{kJ}}$$
$$\lesssim \sum_J{}' \int_\Omega \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial\overline{z_k}}u_{jJ}\overline{u_{kJ}}e^{(h_\eta-\widetilde{h_\eta})} + C_\eta\|\rho u\|^2$$
$$\lesssim \|\overline{\partial}\widetilde{h_\eta}\wedge u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2$$
$$+ \sum_J{}' \|\alpha_\eta(L_u^J)\|^2 + C_\eta\left(\|\rho u\|^2 + \|\rho\overline{\partial}u\|^2 + \|\rho\overline{\partial}^*u\|^2 + \|u_N\|^2\right)\ .$$

In the first inequality, we have used that h_η and $\widetilde{h_\eta}$ agree on the boundary; in addition, we have estimated $\|\alpha(L_u^J)\|^2$ by $\|u\|^2 \lesssim \|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2$. Finally, estimating $\sum{}'_J \|\alpha_\eta(L_u^J)\|^2$ by (12) and choosing $(1-\eta)$ so small that the Hessian term can be absorbed into the left hand side of (17) gives

$$(18)\quad \sum_J{}' \int_\Omega \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial\overline{z_k}}u_{jJ}\overline{u_{kJ}} \lesssim \|\overline{\partial}\widetilde{h_\eta}\wedge u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2$$
$$+ C_\eta\left(\|\rho u\|^2 + \|\rho\overline{\partial}u\|^2 + \|\rho\overline{\partial}^*u\|^2 + \|u_N\|^2 + \|u\|^2_{-1}\right)$$
$$\lesssim \|\overline{\partial}\widetilde{h_\eta}\wedge u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^*u\|^2 + C_\eta(\|\rho\overline{\partial}u\|^2 + \|\rho\overline{\partial}^*u\|^2 + \|u\|^2_{-1})\ .$$

In the last inequality, we have used that ρ is a subelliptic multiplier, and the subelliptic estimate for the normal component of a form ([34], Lemma 2.12).

To estimate $\|\overline{\partial}\widetilde{h_\eta}\wedge u\|^2$, we temporarily switch notation, and also express forms in special boundary charts; let $v = \sum{}'_{|K|=q} v_K\ \overline{\omega_K}$ (supported in a special boundary chart). Then

$$(19)\qquad |\overline{\partial}\widetilde{h_\eta}\wedge v|^2 \leq |\overline{\partial}\widetilde{h_\eta}|^2|v|^2 \lesssim \sum_{j<n,K}{}' |(\overline{L_j}\widetilde{h_\eta})v_K|^2 + \sum_K{}' |(\overline{L_n}\widetilde{h_\eta})v_K|^2\ .$$

Because $\overline{L_n}\widetilde{h_\eta}=0$ on the boundary, the last term is $O_\eta(\rho|v|^2)$, and so will be under control. For the others, note that $\overline{L_j}\widetilde{h_\eta}=\overline{L_j}h_\eta=\overline{L_j(h_\eta)}$ on the boundary when $j<n$. Therefore

$$
\begin{aligned}
(20)\quad |(\overline{L_j}\widetilde{h_\eta})v_K|^2 &\lesssim |L_jh_\eta)v_K|^2+C_\eta|\rho v|^2\\
&= |\overline{\partial}h_\eta(v_KL_j)|^2+C_\eta|\rho v|^2=|(\alpha_\eta-\alpha)(v_KL_j)|^2+C_\eta|\rho v|^2\\
&\lesssim |\alpha_\eta(v_KL_j)|^2+|v|^2+C_\eta|\rho v|^2\ .
\end{aligned}
$$

The error that results from the fact that $dh_\eta=\alpha_\eta-\alpha$ only on $T^{1,0}(b\Omega)$ (i.e. at the boundary) is also covered by $C_\eta|\rho v|^2$. Integrating over Ω, invoking the case $q=1$ of (10) and (12) with $u=v_K\overline{\omega_j}$ (so $L_u=v_KL_j$), and then (18), we get

$$
\begin{aligned}
(21)\quad \int_\Omega|\overline{L_j}\widetilde{h_\eta}v_K|^2 &\lesssim \int_\Omega|\alpha_\eta(v_KL_j)|^2+\|v\|^2+C_\eta\|\rho v\|^2\\
&\lesssim (1-\eta)\left(\|\overline{\partial}\widetilde{h_\eta}\wedge(v_K\overline{\omega_j})\|^2+\|\overline{\partial}(v_K\overline{\omega_j})\|^2+\|\overline{\partial}^*(v_K\overline{\omega_j})\|^2\right)\\
&\quad +\|v\|^2+C_\eta\left(\|\rho\overline{\partial}(v_K\overline{\omega_j})\|^2+\|\rho\overline{\partial}^*(v_K\overline{\omega_j})\|^2+\|\rho v\|^2+\|v\|^2_{-1}\right)\ .
\end{aligned}
$$

Because $q_0\le q$, and in view of Lemma 2, the comparable q_0–sums assumption on the Levi eigenvalues in Theorem 1 implies the maximal estimate (3) for $(0,q)$–forms. Accordingly

$$
(22)\quad \|\overline{\partial}(v_K\overline{\omega_j})\|^2+\|\overline{\partial}^*(v_K\overline{\omega_j})\|^2\lesssim\sum_s\|\overline{L_s}v_K\|^2+\|L_jv_K\|^2+\|v_K\|^2\lesssim\|\overline{\partial}v\|^2+\|\overline{\partial}^*v\|^2\ ,
$$

where the second inequality results from the maximal estimates applied to the $(0,q)$–form v (note that $j<n$). Also, by Lemma 1, with $e^{-h}=\rho^2$,

$$
\begin{aligned}
(23)\quad \|\rho\overline{\partial}(v_K\overline{\omega_j})\|^2+\|\rho\overline{\partial}^*(v_K\overline{\omega_j})\|^2 &\lesssim \|\rho\overline{\partial}v\|^2+\|\rho\overline{\partial}^*v\|^2+\|\rho v\|\|\rho v\|_1\\
&\lesssim \|\rho\overline{\partial}v\|^2+\|\rho\overline{\partial}^*v\|^2+\|\rho v\|(\|\overline{\partial}v\|+\|\overline{\partial}^*v\|),
\end{aligned}
$$

again because ρ is a subelliptic multiplier. On the last term, we use the $l.c.-s.c.$ estimate and then once more that ρ is a subelliptic multiplier to obtain $\|\rho v\|(\|\overline{\partial}v\|+\|\overline{\partial}^*v\|)\le(1-\eta)(\|\overline{\partial}v\|^2+\|\overline{\partial}^*v\|^2)+C_\eta\|v\|^2_{-1}$. Starting from (19), inserting (22) into (21), and using $|\overline{\partial}\widetilde{h_\eta}\wedge(v_K\overline{\omega_j}|^2\lesssim|\overline{\partial}\widetilde{h_\eta}|^2|v|^2$, we have

$$
\begin{aligned}
(24)\quad \int_\Omega|\overline{\partial}\widetilde{h_\eta}|^2|v|^2 &\le \sum_{j<n,K}{}'\int_\Omega|(\overline{L_j}\widetilde{h_\eta})v_K|^2+C_\eta\|\rho v\|^2\\
&\lesssim (1-\eta)\left(\int_\Omega|\overline{\partial}\widetilde{h_\eta}|^2|v|^2+\|\overline{\partial}v\|^2+\|\overline{\partial}^*v\|^2\right)\\
&\quad +\|v\|^2+C_\eta\left(\|\rho\overline{\partial}v\|^2+\|\rho\overline{\partial}^*v\|^2+\|v\|^2_{-1}\right)\ .
\end{aligned}
$$

Now we choose $(1-\eta)$ small enough so that the term containing $\overline{\partial}\widetilde{h_\eta}$ in (24)can be absorbed into the left hand side. Then

$$
\begin{aligned}
(25)\quad \|\overline{\partial}\widetilde{h_\eta}\wedge v\|^2\le\int_\Omega|\overline{\partial}\widetilde{h_\eta}|^2|v|^2 &\lesssim (1-\eta)\left(\|\overline{\partial}v\|^2+\|\overline{\partial}^*v\|^2\right)\\
&\quad +\|v\|^2+C_\eta\left(\|\rho\overline{\partial}v\|^2+\|\rho\overline{\partial}^*v\|^2+\|v\|^2_{-1}\right)\ .
\end{aligned}
$$

Via a partition of unity, (25) carries over as usual to when v is not supported in a special boundary chart; the compactly supported term that arises is also dominated by the right hand side of (25) (by interior elliptic regularity). Insert this estimate into (18). The result is the first inequality below

$$\sum_J{}' \int_\Omega \sum_{j,k} \frac{\partial^2(-h_\eta)}{\partial z_j \partial \overline{z_k}} u_{jJ}\overline{u_{kJ}} \lesssim \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 + C_\eta \left(\|\rho\overline{\partial}u\|^2 + \|\rho\overline{\partial}^* u\|^2 + \|u\|^2_{-1}\right)$$
$$\lesssim \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 + C_\eta \left(\|\overline{\partial}u\|^2_{\Omega_\eta} + \|\overline{\partial}^* u\|^2_{\Omega_\eta} + \|u\|^2_{-1}\right) \; ; \tag{26}$$

for the second inequality we choose $\Omega_\eta \subset\subset \Omega$ such that on $\Omega \setminus \Omega_\eta$, $C_\eta \rho^2 \leq 1$. Inserting (26) into (12) gives (11). The proof of Proposition 1 is now complete. □

4. Proof of Theorem 1

Proof of Theorem 1. It suffices to prove the statement for the $\overline{\partial}$–Neumann operator ([7]; [34], Theorem 5.5). The proof follows the first part of the proof of Theorem 1 in section 4 of [33], but then uses the commutator formulas from [23] (i.e.Lemmas 3 and 4 above) instead of Lemmas 4 and 5 in [33].

We use a downward induction on the degree q. In the top degree $q = n$, the $\overline{\partial}$–Neumann boundary conditions reduce to Dirichlet boundary conditions, and N_n gains two derivatives in Sobolev norms. So to show that N_{q_0} satisfies Sobolev estimates, it suffices to show: if N_{q+1} satisfies Sobolev estimates, and $q \geq q_0$, then so does N_q. So fix such a q. The induction assumption will be invoked to conclude that the Bergman projection P_q on $(0, q)$–forms satisfies Sobolev estimates ([34], Theorem 5.5).

As usual, the arguments will involve absorbing terms, and so one has to know that these terms are finite. In order to insure that, we work with the regularized operators $N_{\delta,q}$ that result form elliptic regularization. That is, $N_{\delta,q}$ is the inverse of the selfadjoint operator $\square_{\delta,q}$ associated with the quadratic form $Q_{\delta,q}(u, u) = \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 + \delta\|\nabla u\|^2$, with form domain $W^1_{(0,q)}(\Omega) \cap dom(\overline{\partial}^*)$, where ∇ denotes the vector of all first order derivatives of all coefficients of u. $N_{\delta,q}$ maps $L^2_{(0,q)}(\Omega)$ continuously into the form domain, endowed with the norm $Q_{\delta,q}(u, u)^{1/2}$. We will prove estimates with constants that are uniform in δ; letting $\delta \to 0$ then gives the desired estimates for N_q ([12], pages 102–103 and [34], section 3.3).

So q is now fixed, $q_0 \leq q$, and N_{q+1}, hence P_q, satisfy Sobolev estimates. We want to prove Sobolev estimates for $N_{\delta,q}$, with constants uniform in δ (δ small). Because P_q is regular in Sobolev norms, Lemma 7 in [33] says that for all $s > 0$, there is a constant C_s independent of δ such that we have the estimate

$$\|N_{\delta,q}u\|_s \leq C_s(\|\overline{\partial}N_{\delta,q}u\|_s + \|\overline{\partial}^* N_{\delta,q}u\|_s) \; . \tag{27}$$

Therefore, it suffices to prove estimates for the right hand side of (27) (with constants independent of δ). In order to bring $Q_{\delta,q}$ into play, we prove estimates for $\overline{\partial}N_{\delta,q}$, $\overline{\partial}^* N_{\delta,q}$, and $\delta^{1/2}\nabla N_{\delta,q}$ as in [33]. We do this by induction on the Sobolev index k.

Because $N_{\delta,q}$ maps $L^2_{(0,q)}(\Omega)$ continuously into the form domain of $Q_{\delta,q}$, with norm less than or equal to 1, the following inductive assumption holds for $l = 0$: there are $\eta \in (0, 1)$, a constant C and $\delta_0 > 0$ such that

$$\|\overline{\partial}N_{\delta,q}u\|^2_{l,2h_\eta} + \|\overline{\partial}^* N_{\delta,q}u\|^2_{l,2h_\eta} + \delta\|\nabla N_{\delta,q}u\|^2_{l,2h_\eta} \leq C\|u\|^2_l \; , \; 0 < \delta \leq \delta_0 \; . \tag{28}$$

Indeed, take any $\eta \in (0, 1)$, then choose C big enough and $\delta > 0$ arbitrary. We now assume this induction assumption holds for $0 \leq l \leq (k - 1)$ and show that it then holds for $l = k$. The induction assumption (28) corresponds to (31) in [33], with the modification that the norms on the left hand side are weighted, and that there is no term $\delta^2\|u\|^2_{l+1}$. It turns out that with Lemmas 3 and 4 for the commutators with $\overline{\partial}$ and $\overline{\partial}^*$ (rather than Lemmas 4 and 5 from [33]), this term is not needed for the induction to run.

For $\eta \in (0,1)$, set $X_\eta = e^{-h_\eta + \rho g_\eta} L_n^\rho$, where g_η is a smooth function that agrees on the boundary with $\alpha_\eta(\overline{L_n})/\overline{L_n}\rho$. The reason for this choice of exponent (rather than just $-h_\eta$) is that the terms where $j = n$ in the commutator (39) below will then vanish on the boundary. This fact is needed in the argument. Being able to make this choice is a manifestation of the principle that 'commutator conditions in the normal (and any strictly pseudoconvex) direction come for free' ([34], Section 5.7, [8], proof of the lemma). Note that because g_η is only prescribed on the boundary, we may assume that $|\rho g_\eta| \le 1/2k$. As usual, we need tangential differential operators to preserve the domain of $\overline{\partial}^*$, and so we let them act in special boundary charts ([34], section 2.2). The error this introduces is one order lower than the operator, and as a result, our estimates are not affected.

Using Lemma 1 for barred derivatives or derivatives that are complex tangential, we have

$$
\begin{aligned}
(29)\quad \|\overline{\partial}^* N_{\delta,q} u\|_{k,2h_\eta}^2 \lesssim{}& \|(e^{-h_\eta}(L_n^\rho - \overline{L_n^\rho}))^k \overline{\partial}^* N_{\delta,q} u\|^2 \\
&+ C_\eta(\|\overline{\partial}\overline{\partial}^* N_{\delta,q} u\|_{k-1}^2 + \|\overline{\partial}^* N_{\delta,q} u\|_{k-1}\|\overline{\partial}^* N_{\delta,q} u\|_k + \|u\|_k^2)\ .
\end{aligned}
$$

The $\|u\|_k^2$ term is needed because of the compactly supported term from the partition of unity used for the special boundary charts; this term is not covered by Lemma 3. The estimate follows from interior elliptic regularity of $\Box_{\delta,q}$, with constants that are uniform in δ. This uniformity holds because for $u \in dom(\Box_{\delta,q})$, $\Box_{\delta,q} u = -(1/4+\delta)\Delta u$, where Δ acts coefficientwise (see [34], formula (3.22)). We have used that when there is a barred derivative, or a complex tangential derivative, we can always commute it so that it acts first. Then there is no issue with with the requirement that a form must be in the domain of $\overline{\partial}^*$ in order for Lemma 1 to apply. The error term this makes is of order at most $(k-1)$. Modulo terms controlled by $\|\overline{\partial}^* N_{\delta,q} u\|_k^2$, $\|(e^{-h_\eta}(L_n^\rho - \overline{L_n^\rho}))^k \overline{\partial}^* N_{\delta,q} u\|^2 \lesssim \|e^{-kh_\eta}(L_n^\rho - \overline{L_n^\rho})^k \overline{\partial}^* N_{\delta,q} u\|^2 \le e\|(X_\eta - \overline{X_\eta})^k \overline{\partial}^* N_{\delta,q} u\|^2$ (since $2k|\rho g_\eta| \le 1$). Upon also estimating the last term in (29) via a s.c.–l.c argument and absorbing the term $s.c\|\overline{\partial}^* N_{\delta,q} u\|_{k,2h_\eta}^2$ into the left hand side of (29), we thus arrive at

$$
(30)\quad \|\overline{\partial}^* N_{\delta,q} u\|_{k,2h_\eta}^2 \lesssim \|(X_\eta - \overline{X_\eta})^k \overline{\partial}^* N_{\delta,q} u\|^2 + C_\eta(\|\overline{\partial}\overline{\partial}^* N_{\delta,q} u\|_{k-1}^2 + \|\overline{\partial}^* N_{\delta,q} u\|_{k-1}^2 + \|u\|_k^2)\ .
$$

This estimate corresponds to estimate (32) in [33], with η in the role of ε. Note that uniform boundedness of h_η, used in [33] for estimate (32), is not needed here because we use the weighted norm on the left hand side.

Applying the same reasoning to $\|\overline{\partial} N_{\delta,q} u\|_{k,2h_\eta}^2$ takes a little more care, because $\overline{\partial} N_{\delta,q} u$ need not be in the domain of $\overline{\partial}^*$ (the free boundary condition imposed by Q_δ is different from that imposed by Q). The necessary modification is explained in detail in [33], to which we refer the reader. The result is the following estimate:

$$
\begin{aligned}
(31)\quad \|\overline{\partial} N_{\delta,q} u\|_{k,2h_\eta}^2 \lesssim{}& \|(X_\eta - \overline{X_\eta})^k \overline{\partial} N_{\delta,q} u\|^2 \\
&+ C_\eta(\|\vartheta\overline{\partial} N_{\delta,q} u\|_{k-1}^2 + \|\overline{\partial} N_{\delta,q} u\|_{k-1}^2 + \|\delta(\partial/\partial\nu) N_{\delta,q} u \wedge \overline{\omega_n}\|_k^2 + \|u\|_k^2)\ .
\end{aligned}
$$

Here, ϑ is the formal adjoint of $\overline{\partial}$, $(\partial/\partial\nu)$ denotes the normal derivative acting coefficientwise on forms, and ω_n is the form dual to the unit normal L_n.

The argument for $\delta\|\nabla N_{\delta,q} u\|_{k,2h_\eta}^2$ is slightly different from that given in [33] (the argument there uses uniform bounds on the functions h_η). At issue is the $\partial/\partial\nu$ term in ∇, which need not be in the domain of $\overline{\partial}^*$. However, because the boundary is not characteristic for $\overline{\partial} \oplus \vartheta$ ([34], Lemma 2.2), we can write $\nabla N_{\delta,q} u = \nabla_T N_{\delta,q} u$ plus a linear combination of coefficients of $\overline{\partial}N_{\delta,q}u$, $\overline{\partial}^*N_{\delta,q}u$, and $N_{\delta,q}u$ itself. Here, ∇_T stands for tangential derivatives. Applying the same reasoning as for (30) to $\nabla_T N_{\delta,q}u$ gives

$$\delta\|\nabla N_{\delta,q}u\|^2_{k,2h_\eta} \lesssim \delta\|(X_\eta - \overline{X_\eta})^k\nabla N_{\delta,q}u\|^2 + C_\eta\delta(\|\overline{\partial}N_{\delta,q}u\|^2_k + \|\overline{\partial}^*N_{\delta,q}u\|^2_k + \|N_{\delta,q}u\|_k \, \|N_{\delta,q}u\|_{k+1}) \,. \tag{32}$$

We have used that $\|(X_\eta - \overline{X_\eta})^k\nabla_T N_{\delta,q}u\|^2 \le \|(X_\eta - \overline{X_\eta})^k\nabla N_{\delta,q}u\|^2$.

Estimates (30),(31), and (32) correspond to estimates (32), (33), and (34) in [33]. It is shown there in (31)–(39), and the paragraph immediately following (39), that completing the induction step reduces to estimating the following three inner products:

$$\Big((X_\eta - \overline{X_\eta})^k\overline{\partial}N_{\delta,q}u, [\overline{\partial}, X_\eta - \overline{X_\eta}](X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u\Big) \,, \tag{33}$$

$$\Big((X_\eta - \overline{X_\eta})^k\overline{\partial}^*N_{\delta,q}u, [\overline{\partial}^*, X_\eta - \overline{X_\eta}](X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u\Big) \,, \tag{34}$$

and

$$\delta\Big((X_\eta - \overline{X_\eta})^k\nabla N_{\delta,q}u, [\nabla, X_\eta - \overline{X_\eta}](X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u\Big) \,. \tag{35}$$

More precisely, it is shown in [33] that for η given, there is $\delta_0(\eta)$ such that all the error terms in (30)–(32) are acceptable for (28) (for $l = k$), or can be absorbed into the left hand side of (28) when $\delta \le \delta_0(\eta)$. Note that upon adding (33) through (35), using Cauchy–Schwarz on each inner product, followed by a s.c.–l.c. argument, the squares of the norms of the left hand sides in (33)–(35) can be absorbed into the first term on the right hand side of the sum of (30)–(32). Therefore, in order to estimate the sum of the left hand sides of (30)–(32), we only have to estimate

$$\|[\overline{\partial}, X_\eta - \overline{X_\eta}](X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u\|^2 + \|[\overline{\partial}^*, X_\eta - \overline{X_\eta}](X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u\|^2 + \delta\|[\nabla, X_\eta - \overline{X_\eta}](X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u\|^2 \,. \tag{36}$$

This expression differs from

$$\|[\overline{\partial}, (X_\eta - \overline{X_\eta})^k]N_{\delta,q}u\|^2 + \|[\overline{\partial}^*, (X_\eta - \overline{X_\eta})^k]N_{\delta,q}u\|^2 + \delta\|[\nabla, (X_\eta - \overline{X_\eta})^k]N_{\delta,q}u\|^2 \tag{37}$$

by terms that are of order $(k-1)$. This follows from the formula for commutators with powers of an operator in [16], Lemma 2, page 418 (see also [33], formula (3.54)): $[A, T^k] = \sum_{j=1}^{k}\binom{k}{j}[\cdots[[A,T],T]\cdots]T^{k-j}$ (j–fold, note that these iterated commutators are of order one). In view of (27) and the induction assumption, these terms are acceptable. We will now estimate (37).

For the commutators with $\overline{\partial}$ and $\overline{\partial}^*$ we use Lemmas 3 and 4, with $\psi = h_\eta - \rho g_\eta$ and $v = (X_\eta - \overline{X_\eta})^{k-1}N_{\delta,q}u$. Observe that near the boundary

$$\overline{L_j}\psi + \alpha^\rho(\overline{L_j}) = -\overline{L_j}(\rho g_\eta) + \overline{L_j}h_\eta + \alpha^\rho(\overline{L_j}) = -\overline{L_j}(\rho g_\eta) + \alpha_\eta(\overline{L_j}) \,;\; j < n \,; \tag{38}$$

the last equality follows from (5). Inserting this equality into Lemma 3 and estimating the error terms with Lemma 1 gives

$$\begin{aligned} &\|[\overline{\partial}, (X_\eta - \overline{X_\eta})^k]N_{\delta,q}u\|^2 \\ &\quad\lesssim {\sum_J}' \sum_{j\notin J}\int_\Omega |\overline{L_j}(\rho g_\eta) - \alpha_\eta(\overline{L_j})|^2|(X_\eta - \overline{X_\eta})^k(N_{\delta,q}u)_J|^2 \\ &\qquad + C_\eta\,(\|\overline{\partial}N_{\delta,q}u\|^2_{k-1} + \|\overline{\partial}^*N_{\delta,q}u\|^2_{k-1} + \|N_{\delta,q}u\|_{k-1}\|N_{\delta,q}u\|_k + \|u\|^2_{k-2}) \,. \end{aligned} \tag{39}$$

We only have to sum over $j \notin J$ because for $j \in J$, the term $\overline{\omega_j} \wedge \overline{\omega_J}$ vanishes. The term $\|u\|^2_{k-2}$ again results from the compactly supported term that arises from letting $(X_\eta - \overline{X_\eta})$

act in special boundary charts; as above, the estimate follows from interior elliptic regularity of $\Box_{\delta,q}$, with constants that are uniform in δ. The order $(k-1)$ error terms are acceptable by the induction hypothesis, and using (27) for the last term shows that it can be split into an absorbable term and a lower order term. From Lemma 4, we similarly obtain

$$\begin{aligned}(40)\quad &\|[\overline{\partial}^*, (X_\eta - \overline{X_\eta})^k]N_{\delta,q}u\|^2 \\ &\lesssim {\sum_J}' \sum_{j\in J} \int_\Omega |L_j(\rho g_\eta - \alpha_\eta(L_j)|^2 |(X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J|^2 \\ &\quad + C_\eta \left(\|\overline{\partial} N_{\delta,q}u\|_{k-1}^2 + \|\overline{\partial}^* N_{\delta,q}u\|_{k-1}^2 + \|N_{\delta,q}u\|_{k-1}\|N_{\delta,q}u\|_k + \|(N_{\delta,q}u)_N\|_k^2 + \|u\|_{k-2}^2 \right) .\end{aligned}$$

The first three error terms are the same as in (39) and so are acceptable, as is the last. For fourth one, we use the subelliptic estimate $\|(N_{\delta,q})_N\|_k^2 \lesssim \|\overline{\partial} N_{\delta,q}u\|_{k-1}^2 + \|\overline{\partial}^* N_{\delta,q}u\|_{k-1}^2 + \|N_{\delta,q}u\|_{k-1}^2$ ([34], estimate (2.96)). Using (27) again shows that this term is also acceptable.

For the commutator with ∇ in (37), we note that it is of order k, so that the term is estimated by $\delta C_\eta \|N_{\delta,q}u\|_k^2 \lesssim \delta C_\eta (\|\overline{\partial} N_{\delta,q}u\|_{k,2h_\eta}^2 + \|\overline{\partial}^* N_{\delta,q}u\|_{k,2h_\eta}^2)$ (again by (27), the weight in the norms just changes the constant C_η). If we choose $\delta_0(\eta)$ small enough, this term can therefore be absorbed when $\delta \le \delta_0(\eta)$. We are therefore left with estimating only the main terms on the right hand sides of (39) and (40), respectively.

For $j < n$, the contributions coming from $\overline{L_j}(\rho g_\eta)$ and $L_j(\rho g_\eta)$, respectively, are $\mathcal{O}_\eta(\|\rho(X_\eta - \overline{X_\eta})^k N_{\delta,q}u\|_k$ and are again acceptable (ρ is a subelliptic multiplier and (27)). Because α_η is real, $|\alpha_\eta(\overline{L_j})|^2 = |\alpha_\eta(L_j)|^2$, and we only have to estimate $\int_\Omega |\alpha_\eta(L_j)|^2 |(X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J|^2 = \int_\Omega |\alpha_\eta((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J L_j)|^2$. We do this via (11) in Proposition 1, applied to the $(0,1)$–form $(X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J \overline{\omega_j}$.Then $L_u = ((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J L_j$, and the resulting estimate is

$$\begin{aligned}(41)\quad &\int_\Omega |\alpha_\eta(L_j)|^2 |(X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J|^2 \\ &\lesssim (1-\eta) \left(\|\overline{\partial}((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J \overline{\omega_j})\|^2 + \|\overline{\partial}^*((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J \overline{\omega_j})\|^2 \right) \\ &\quad + C_\eta \left(\|\overline{\partial}((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J \overline{\omega_j})\|_{\Omega_\eta}^2 + \|\overline{\partial}^*((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J \overline{\omega_j})\|_{\Omega_\eta}^2 \right) \\ &\qquad + C_\eta \|(X_\eta - \overline{X_\eta})^k N_{\delta,q}u\|_{-1}^2 .\end{aligned}$$

The error terms are acceptable; this is clear for the $\|\cdot\|_{-1}$ term (also using (27)). For the others, it follows from interior elliptic regularity of $\Box_{\delta,q}$, with constants that are uniform in δ. The $\overline{\partial}$ and $\overline{\partial}^*$ terms in the second line of (41) are dominated (uniformly in η and δ) by

$$\begin{aligned}(42)\quad &\sum_{s=1}^n \|\overline{L_s}((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J)\|^2 + \|L_j((X_\eta - \overline{X_\eta})^k (N_{\delta,q}u)_J)\|^2 + \|(X_\eta - \overline{X_\eta})^k N_{\delta,q}u\|^2 \\ &\qquad \lesssim \|\overline{\partial}((X_\eta - \overline{X_\eta})^k N_{\delta,q}u)\|^2 + \|\overline{\partial}^*((X_\eta - \overline{X_\eta})^k N_{\delta,q}u)\|^2 ;\end{aligned}$$

we have used the basic estimate for the barred derivatives for the first term, maximal estimates for the second (see (3); note again that $q_0 \le q$, and Lemma 2), and the basic L^2 estimate for the third.

There remain the terms in (39) and (40) where $j = n$. In (39), we have that $|\overline{L_n}(\rho g_\eta) - \alpha_\eta(\overline{L_n})|^2 = 0$ on the boundary, due to the choice of g_η. The contribution from this term is thus $\mathcal{O}_\eta(\|\rho(X_\eta - \overline{X_\eta})^k N_{\delta,q}u\|^2)$. Again because ρ is a subelliptic multiplier and (27), this term is acceptable. In (40), the term with $j = n$ occurs only in a sum over J with $n \in J$.

That is, $(N_{\delta,q}u)_J$ is a component of the normal part $(N_{\delta,q}u)_N$ of $N_{\delta,q}u$ and so is subelliptic; we conclude as above that it is acceptable.

Combining (39)–(42), and the remark above about the commutator with ∇ in (37) gives

$$\begin{aligned}(43)\quad & \|[\overline{\partial},(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2+\|[\overline{\partial}^*,(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2+\delta\|[\nabla,(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2\\ &\lesssim (1-\eta)\left(\|\overline{\partial}((X_\eta-\overline{X_\eta})^kN_{\delta,q}u)\|^2+\|\overline{\partial}^*((X_\eta-\overline{X_\eta})^kN_{\delta,q}u)\|^2\right)\\ &\qquad + \textit{terms acceptable terms}\\ &\lesssim (1-\eta)\left(\|[\overline{\partial},(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2+\|[\overline{\partial}^*,(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2\right)\\ &\quad + (1-\eta)\left(\|(X_\eta-\overline{X_\eta})^k\overline{\partial}N_{\delta,q}u\|^2+\|(X_\eta-\overline{X_\eta})^k\overline{\partial}^*N_{\delta,q}u\|^2\right)\\ &\qquad + \textit{acceptable terms}\ .\end{aligned}$$

For $(1-\eta)$ small enough, the fourth line in (43) can be absorbed into the first line, giving

$$\begin{aligned}(44)\quad & \|[\overline{\partial},(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2+\|[\overline{\partial}^*,(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2+\delta\|[\nabla,(X_\eta-\overline{X_\eta})^k]N_{\delta,q}u\|^2\\ &\lesssim (1-\eta)\left(\|(X_\eta-\overline{X_\eta})^k\overline{\partial}N_{\delta,q}u\|^2+\|(X_\eta-\overline{X_\eta})^k\overline{\partial}^*N_{\delta,q}u\|^2\right)+\textit{acceptable terms}\ .\end{aligned}$$

The left hand side of (44) is what is needed to estimate the sum of the left hand sides of (30), (31), and (32). Thus

$$\begin{aligned}(45)\quad & \|\overline{\partial}N_{\delta,q}u\|^2_{k,2h_\eta}+\|\overline{\partial}^*N_{\delta,q}u\|^2_{k,2h_\eta}+\delta\|\nabla N_{\delta,q}u\|^2_{k,2h_\eta}\\ &\lesssim (1-\eta)\left(\|(X_\eta-\overline{X_\eta})^k\overline{\partial}N_{\delta,q}u\|^2+\|(X_\eta-\overline{X_\eta})^k\overline{\partial}^*N_{\delta,q}u\|^2\right)+\textit{acceptable terms}\ ,\end{aligned}$$

where 'acceptable terms' stands for terms that are of order at most $(k-1)$ in $\overline{\partial}N_{\delta,q}u$, $\overline{\partial}^*N_{\delta,q}u$, or $\delta^{1/2}\nabla N_{\delta,q}u$, or are the same as the terms on the left hand side of (45), but with a small constant in front (for $(1-\eta)$ small enough and then $\delta\leq\delta_0(\eta)$), or are dominated by $\|u\|^2_k$. After combining the estimates $\|(X_\eta-\overline{X_\eta})^k\overline{\partial}N_{\delta,q}u\|^2\lesssim\|\overline{\partial}N_{\delta,q}u\|^2_{k,2h_\eta}+C_\eta\|N_{\delta,q}u\|^2_{k-1}$ and $\|(X_\eta-\overline{X_\eta})^k\overline{\partial}^*N_{\delta,q}u\|^2\lesssim\|\overline{\partial}^*N_{\delta,q}u\|^2_{k,2h_\eta}+C_\eta\|N_{\delta,q}u\|^2_{k-1}$ with the induction hypothesis, and absorbing terms, we obtain that there are a constant C, $\eta\in(0,1)$, and δ_0 such that

$$(46)\qquad \|\overline{\partial}N_{\delta,q}u\|^2_{k,2h_\eta}+\|\overline{\partial}^*N_{\delta,q}u\|^2_{k,2h_\eta}+\delta\|\nabla N_{\delta,q}u\|^2_{k,2h_\eta}\leq C\|u\|^2_k\ ,\ 0<\delta\leq\delta_0\ .$$

This completes the induction on k; (46) holds for all $k\in\mathbb{N}$, with C, η, and δ_0 depending on k.

Fix $k\in\mathbb{N}$. In view of (27), and choosing η so that (46) holds, we see that there is $\delta_0(k)>0$ and a constant C_k such that

$$(47)\qquad \|N_{\delta,q}u\|^2_k\leq C_k\|u\|^2_k\ ,\ 0<\delta\leq\delta_0(k)\ .$$

(The weighted and unweighted norms are equivalent, with constants depending on η; once η is fixed, this dependence is no longer relevant.) As we said at the beginning of this section, letting $\delta\to0$ transfers this estimate to N_q. Since k was arbitrary, so we have now shown that if N_{q+1} satisfies Sobolev estimates for $q\geq q_0$, then so does N_q. This concludes the downward induction on the degree q, and completes the proof of Theorem 1. □

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