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Non-Archimedean quantum mechanics via quantum groups

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Abstract

We present a new non-Archimedean realization of the Fock representation of the q-oscillator algebras where the creation and annihilation operators act on complex-valued functions, which are defined on a non-Archimedean local field of arbitrary characteristic, for instance, the field of p-adic numbers. This new realization implies that many quantum models constructed using q-oscillator algebras are non-Archimedean models, in particular, p-adic quantum models. In this framework, we select a q-deformation of the Heisenberg uncertainty relation and construct the corresponding q-deformed Schrödinger equations. In this way we construct a p-adic quantum mechanics which is a p-deformed quantum mechanics. We also solve the time-independent Schrödinger equations for the free particle, and a particle in a non-Archimedean box. In the last case, we show the existence of a discrete sequence of energy levels. We determine the eigenvalues of Schrödinger operator for a general radial potential. By choosing the potential in a suitable form we recover the energy levels of the q-hydrogen atom.

1. Introduction

In the last thirty-five years the connection between q-oscillator algebras and quantum physics have been studied intensively, see, e.g., [3], [5], [6], [15], [19], [20]-[21], [29], [37]-[38], [42],

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[43], [51], [57–60], and the references therein. From the seminal work of Biedenharn [6] and Macfarlane [42], it was clear that the q-analysis, [18], [27], [29], plays a central role in the representation of q-oscillator algebras which in turn has a deep physical meaning. In particular, the q-deformation of the Heisenberg algebra drives naturally to several types of q-deformed Schrödinger equations, see, e.g., [3], [38], [40], [57–60].

Also, in the last thirty years the p-adic quantum mechanics and the p-adic Schrödinger equations have been studied extensively, see, e.g., [1], [9–17], [30], [33–35], [44], [47], [52], [55]-[50], [61], among many available references. Here, we present a new perspective: a non-Archimedean quantum mechanics, which includes p-adic quantum mechanics as a particular case, is a q-deformation of the classical quantum mechanics. The construction is based on a new non-Archimedean realization of the Fock representation of the q-oscillator algebras, where the creation and annihilation operators act on complex-valued functions defined on a non-Archimedean local field $\mathbb{K}$ of arbitrary characteristic.

On the other hand, the emergence of ultrametricity in physics, which is the occurrence of ultrametric spaces in physical models, has driven to the development of deep connections between p-adic analysis and physics, see, e.g., [17,53,54,45,50,31,32,49,62] and the references therein. The existence of a Planck length implies that the spacetime considered as a topological space is completely disconnected. The points (which are the connected components) play the role of spacetime quanta. This is precisely the Volovich conjecture on the non-Archimedean nature of the spacetime below the Planck scale, [53,54], [49, Chapter 6]. In the last forty years, the above mentioned ideas have motivated many developments in quantum field theory and string theory, see, e.g., [13], [17], [23], [22], [26], [50], [53]-[54], and more recently, [4], [10], [11], [10], [24]-[25], among others.

In [3] Aref'eva and Volovich pointed out the existence of deep analogies between p-adic and q-analysis, and between q-deformed quantum mechanics and p-adic quantum mechanics. In this work we start the investigation of these matters. We present a new non-Archimedean realization of the Fock representation of q-oscillator algebras, where the creation and annihilation operators act on functions $f : \mathbb{K} \to \mathbb{C}$, where $\mathbb{K}$ is a non-Archimedean local field, for instance, the field of formal Laurent series:

$$\mathbb{F}_q((T)) = \left\{ \sum_{k=k_0}^{\infty} a_k T^k; a_k \in \mathbb{F}_q, a_{k_0} \neq 0 \text{ with } k_0 \in \mathbb{Z} \right\},$$

where $\mathbb{F}_q$ is the finite field with q elements. Our results imply, for instance, that the results on the q-deformed harmonic oscillator of Biedenharn [6] and Macfarlane [42] are valid in $\mathbb{F}_q((T))$.

We also study some analogues of the Schrödinger equation coming from a q-deformation of the Heisenberg algebra. There are several different ways of choosing a q-deformation of the classical uncertainty relation which in turn produces several different q-deformations of the Schrödinger equation. In the case of the free particle, these q-deformed equations admit plane waves which are constructed using the classical exponential functions from q-analysis. There are two basic exponential functions, one admitting a meromorphic continuation to the complex plane, and the other admitting an entire analytic continuation to the complex plane. We pick a q-deformation of the uncertainty relation so that the corresponding q-deformed Schrödinger equation admits plane waves which are entire functions.

In this framework, we study, in a rigorous mathematical way, the time-independent q-deformed Schrödinger equations for a free particle, a particle confined in a non-Archimedean box, and a particle subject to an arbitrary radial potential.

We now discuss some applications of our results and do some comparisons. To fix ideas we discuss all the results using the field $\mathbb{F}_q((T))$. The standard norm on $\mathbb{F}_q((T))$ is defined as

$$|x| = \begin{cases} 0 & \text{if} \quad x = 0 \\ q^{-k_0} & \text{if} \quad x = \sum_{k=k_0}^{\infty} a_k T^k,\, a_{k_0} \neq 0. \end{cases}$$

Notice that $|T| = q^{-1}$. We denote by $S = \left\{x \in \mathbb{F}_q((T));\, |x| = 1\right\}$ the unit sphere. Then

$$\mathbb{F}_q((T)) \smallsetminus \{0\} = \bigsqcup_{k=-\infty}^{\infty} T^k S.$$

Which implies that $\mathbb{F}_q((T)) \smallsetminus \{0\}$ is a self-similar set, and that $(\mathbb{Z}, +)$ is a scale group acting on $\mathbb{F}_q((T)) \smallsetminus \{0\}$ as

$$\begin{array}{lll} \mathbb{Z} \times \left(\mathbb{F}_q((T)) \smallsetminus \{0\}\right) & \rightarrow & \mathbb{F}_q((T)) \smallsetminus \{0\} \\ (k, x) & \rightarrow & T^k x. \end{array}$$

In the classical applications of the q-analysis to mathematical physics the background space is $\mathbb{R}$ or $\mathbb{C}$, these fields do not admit $(\mathbb{Z}, +)$ as a scale group. On these fields, by using the Jackson derivative, it is possible to construct certain fractals, see [19]. In the non-Archimedean setting, the background space is a non-Archimedean local field, which has a fractal nature.

We introduce a non-Archimedean version of the Jackson derivative. Let n, m be non-negative integers, and $f\colon \mathbb{F}_q((T)) \rightarrow \mathbb{C}$, we set

$$(\partial(n,m) f)(x) := \frac{f(T^{-n}x) - f(T^m x)}{\left|T^{-n}x\right| - \left|T^m x\right|} = \frac{f(T^{-n}x) - f(T^m x)}{\left(q^n - q^{-m}\right)|x|} \text{ for } x \neq 0.$$

This derivative measures the speed of deformation of function f under the scale group of $\mathbb{F}_q((T))$. Notice that this derivative is not defined at the origin. In this article we use mainly the case $\partial(1,1) =: \partial$. We also use the operators $\left(q^{\pm N} f\right)(x) := f(T^{\mp 1}x)$. The q-oscillator algebras $\mathcal{A}_q$, $\mathcal{A}_q^c$ are generated by the symbols $a, a^{\dagger}, q^{\pm N}$. By interpreting a as ∂, $a^{\dagger}$ as the multiplication $|x|$ and using $q^{\pm N}$, we show the existence of a non-Archimedean realization of the Fock representation of the algebras $\mathcal{A}_q$, $\mathcal{A}_q^c$. The underlying Hilbert space of the representation is isometric to $L^2(\mathbb{K}, dx)$, where dx is the normalized Haar measure of $(\mathbb{K}, +)$. This new realization implies that many models constructed using q-oscillator algebras are indeed non-Archimedean models, in particular, p-adic models.

In q-analysis, the parameter q is a complex number, meanwhile in $\mathbb{K}$-analysis, $q = p^n$, where p is a prime number and n is a positive integer. By specializing the q parameter to a power of p, we pass from q-analysis to $\mathbb{K}$-analysis.

The q-deformed harmonic oscillators have been studied intensively, see, e.g., [6], [21], [37–41], [42], [51], among others. These models can be formulated on $\mathbb{K} = \mathbb{F}_q((T))$. The energy levels of these harmonic oscillators have the form

$$E_n = \frac{1}{2}\hbar\omega \frac{\sinh\left(\frac{2n+1}{2}\ln q\right)}{\sinh\left(\frac{1}{2}\ln q\right)} \sim \frac{1}{2}\hbar\omega \frac{\exp\left(\frac{2n+1}{2}\ln q\right)}{\exp\left(\frac{1}{2}\ln q\right)} \text{ as } n \rightarrow \infty.$$

These energy levels are no longer uniformly spaced since q is a power of a prime number. The interpretation of the non-uniform distribution of the energy levels of the q-harmonic oscillator is a challenging problem. In the non-Archimedean framework, they obey a scale law. Consider another background space $\mathbb{K}_m = \mathbb{F}_{q^m}((T))$, which is a $\mathbb{K}$-vector space of dimension $m \geq 2$. In this new background space, we have a copy of the q-deformed harmonic oscillator, with energy levels:

$$E_n^{(m)} = \frac{1}{2}\hbar\omega \frac{\sinh\left(\frac{2n+1}{2}\ln q^m\right)}{\sinh\left(\frac{1}{2}\ln q^m\right)} \sim \frac{1}{2}\hbar\omega \left(\frac{\exp\left(\frac{2n+1}{2}\ln q\right)}{\exp\left(\frac{1}{2}\ln q\right)}\right)^m.$$

Then

$$\frac{E_n^{(m)}}{\frac{1}{2}\hbar\omega} \sim \left(\frac{E_n}{\frac{1}{2}\hbar\omega}\right)^m \text{ as } n \to \infty.$$

This scale law is a reinterpretation of a well-known number-theoretic result, which is available only in the non-Archimedean framework.

Let $V(|x|) : \mathbb{F}_q((T)) \to \mathbb{R}$ be an arbitrary radial potential with a unique singularity at the origin. In this article we study the following time-independent Schrödinger equation:

$$\begin{cases} \Psi_n : \mathbb{F}_q[[T]] \to \mathbb{R} \\ \Psi_n \mid_S = 0 \\ \frac{-\hbar^2}{2m}\left\{\left(q^{-N}\partial\right)^2 + V(|x|)\right\}\Psi_n(x) = E_n\Psi_n(x), \end{cases}$$

here $\mathbb{F}_q[[T]]$ is the unit ball centered at the origin. We show that the energy levels have the form

$$E_n = \frac{-\hbar^2}{2m}\left(1 - q^{-2}\right)^2 q^{4n-4} + V(q^{-n}), \text{ for } n = 1, 2, \ldots. \tag{1.1}$$

Here we determine only the point spectrum of $\frac{-\hbar^2}{2m}\left\{\left(q^{-N}\partial\right)^2 + V(|x|)\right\}$. The determination of the whole spectrum is an open problem.

By a suitable selection of the potential $V(|x|)$, the energy levels of several q-models can be obtained from (1.1). For instance by taking,

$$V_{HA}(|x|) = \frac{\hbar^2\left(1-q^{-2}\right)^2}{2mq^2|x|^4} - \frac{1}{2}mc^2\left(\frac{e^2}{\hbar c}\right)^2 q^{4\mu}\frac{\left(q-q^{-1}\right)^2}{\left(|x|-|x|^{-1}\right)^2},\ x \in \mathbb{F}_q[[T]],$$

formula (1.1) gives the energy levels of the Finkelstein q-hydrogen atom [21]:

$$E_n(\mu) = -\frac{1}{2}mc^2\left(\frac{e^2}{\hbar c}\right)^2\frac{q^{4\mu}}{[2n+1]^2},$$

where μ is a real parameter, and $[j] = \frac{q^j - q^{-j}}{q - q^{-1}}$. In the limit q tends to one, (1.1) gives the Balmer energy formula [21].

The limits $\lim T \to 1$, $\lim q \to 1$ are completely different. To the best of our knowledge, the understanding of the first one requires motivic integration, while the second requires ordinary calculus, of course, after extending the parameter q as a real variable. Notice that this difference is not very clear in the p-adic case, where T is replaced by p and $q = p^{-1}$, for this reason, we prefer $\mathbb{F}_q((T))$ over $\mathbb{Q}_p$. It is interesting to mention that in the limit p tends to one, the p-adic strings relate to ordinary strings see, e.g., [12] and the references therein.

The non-Archimedean difference equations introduced here are new mathematical objects. There are several open problems and intriguing connections between these π-difference equations with several mathematical theories.

2. The q-oscillator algebras and Fock representations

In this section we review some basic aspects of the q-oscillator algebras (also called q-boson algebras) and their Fock representations. For further details the reader may consult [29, Chapter 5].

2.1. The q-oscillator algebras

Let q be a fixed complex number such that $q \neq \pm 1$. We recall that the one-dimensional harmonic oscillator algebra is generated by two elements a, $a^{\dagger}$ satisfying the commutation relation

$$\left[a, a^{\dagger}\right] = aa^{\dagger} - a^{\dagger}a = 1. \tag{2.1}$$

In the Fock representation the generators a, $a^{\dagger}$ correspond to the annihilation and creation operators, respectively, and $N = a^{\dagger}a$ corresponds to the particle number operator. Furthermore,

$$\left[N, a^{\dagger}\right] = a^{\dagger},\ [N, a] = -a. \tag{2.2}$$

Definition 1. The centrally extended q-oscillator algebra $\mathcal{A}_q^c$ is the associative, unital, $\mathbb{C}$-algebra generated by four elements a, $a^{\dagger}$, q^N, q^{-N} subject to the relations

1A. $q^{-N}q^N = q^N q^{-N} = 1$,
2A. $q^N a^{\dagger} = qa^{\dagger}q^N$,
3A. $q^N a = q^{-1}aq^N$,
4A. $\left[a, a^{\dagger}\right]_q := aa^{\dagger} - qa^{\dagger}a = q^{-N}$.

It is relevant to note that in the above definition q^N, q^{-N}, are symbols for two elements and that N is not an element of the algebra $\mathcal{A}_q^c$.

For $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}$, we use the notation

$$q^{kN+\alpha} := q^{\alpha}\left(q^N\right)^k \text{ and } [N+\alpha] := \frac{q^{N+\alpha} - q^{-N-\alpha}}{q - q^{-1}}.$$

Definition 2. The symmetric q-oscillator algebra $\mathcal{A}_q$ is the associative, unital, $\mathbb{C}$-algebra generated by four elements a, $a^{\dagger}$, q^N, q^{-N} subject to the relations

1B. $q^{-N}q^N = q^N q^{-N} = 1$,
2B. $q^N a^{\dagger} = qa^{\dagger}q^N$,
3B. $q^N a = q^{-1}aq^N$,
4B. $\left[a, a^{\dagger}\right]_q := aa^{\dagger} - qa^{\dagger}a = q^{-N}$,
5B. $\left[a, a^{\dagger}\right]_{q^{-1}} := aa^{\dagger} - q^{-1}a^{\dagger}a = q^N$.

Note that relations (4B)-(5B) imply that

$$a^{\dagger}a = [N]_q\,,\quad aa^{\dagger} = [N+1]_q\,,$$

which in turn imply that

$$[N]_q\, a^{\dagger} = a^{\dagger}aa^{\dagger} = a^{\dagger}\,[N+1]_q\,, \text{ and } a\,[N]_q = aa^{\dagger}a = [N+1]_q\, a.$$

In the limit $q \to 1$ the relations (1B)-(5B) of the algebra $\mathcal{A}_q$ reduce to (2.1) and $N = a^{\dagger}a$.

2.2. *The Fock representation of* $\mathcal{A}_q$

In this section we assume that $q > 0$, $q \neq 1$. We do not use the bra-ket notation, we directly identify the generators of $\mathcal{A}_q$ with operators acting on a certain Hilbert space. This direct approach is more convenient for our purposes, see [51], [29, Section 5.3]

The Fock space $\mathcal{F}_q$ is an $\mathcal{A}_q$-module with basis vectors

$$v_n, \ \ n \in \mathbb{N} := \{0, 1, \dots, l, l+1, \dots\},$$

and the action of the generators of $\mathcal{A}_q$ is given by

$$q^{\pm N} v_n = q^{\pm n} v_n, \ \ a^{\dagger} v_n = \sqrt{[n+1]} v_{n+1}, \ \ a v_n = \sqrt{[n]} v_{n-1},$$

where

$$[r] := (q^r - q^{-r})/(q - q^{-1}) = \frac{\sinh(r \ln q)}{\sinh(\ln q)}.$$

Notice that

$$a^{\dagger} a v_n = [n] v_n, \ \ a a^{\dagger} v_n = [n+1] v_n.$$

The Fock space becomes a Hilbert space with respect to the inner product $\langle v_m, v_n \rangle := \delta_{m,n}$. Furthermore, a and $a^{\dagger}$ are adjoint to each other, whereas those of q^N and q^{-N} are self-adjoint operators.

3. Non-Archimedean local fields

We recall that the field of rational numbers $\mathbb{Q}$ admits two types of norms: the Archimedean norm (the usual absolute value), and the non-Archimedean norms (the p-adic norms) which are parameterized by the prime numbers. The field of real numbers $\mathbb{R}$ arises as the completion of $\mathbb{Q}$ with respect to the Archimedean norm. Fix a prime number p, the p-adic norm is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma} \frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $ord(x) := \gamma$, with $ord(0) := \infty$, is called the *p-adic order of x*. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$.

A non-Archimedean local field $\mathbb{K}$ is a locally compact topological field with respect to a non-discrete topology, which comes from a norm $|\cdot|_{\mathbb{K}}$ satisfying

$$|x + y|_{\mathbb{K}} \leq \max\left\{ |x|_{\mathbb{K}}, |y|_{\mathbb{K}} \right\},$$

for $x, y \in \mathbb{K}$. Such a norm is called an *ultranorm or non-Archimedean*. Any non-Archimedean local field $\mathbb{K}$ of characteristic zero is isomorphic (as a topological field) to a finite extension of $\mathbb{Q}_p$. The field $\mathbb{Q}_p$ is the basic example of non-Archimedean local field of characteristic zero. In the case of positive characteristic, $\mathbb{K}$ is isomorphic to the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field $\mathbb{F}_q$, where q is a power of a prime number p.

Notation 1. From now on, we fix the parameter q to be a power of p. In addition, we use q to denote only the cardinality of $\mathbb{F}_q$.

The *ring of integers* of $\mathbb{K}$ is defined as

$$R_{\mathbb{K}} = \{x \in \mathbb{K}; |x|_{\mathbb{K}} \le 1\}.$$

Geometrically $R_{\mathbb{K}}$ is the unit ball of the normed space $(\mathbb{K}, |\cdot|_{\mathbb{K}})$. This ring is a domain of principal ideals having a unique maximal ideal, which is given by

$$P_{\mathbb{K}} = \{x \in \mathbb{K}; |x|_{\mathbb{K}} < 1\}.$$

We fix a generator π of $P_{\mathbb{K}}$, i.e., $P_{\mathbb{K}} = \pi R_{\mathbb{K}}$. Such a generator is also called a *local uniformizing parameter of* $\mathbb{K}$, and it plays the same role as p in $\mathbb{Q}_p$.

The *group of units* of $R_{\mathbb{K}}$ is defined as

$$R_{\mathbb{K}}^{\times} = \{x \in R_{\mathbb{K}}; |x|_{\mathbb{K}} = 1\}.$$

The natural map $R_{\mathbb{K}} \to R_{\mathbb{K}}/P_{\mathbb{K}} \cong \mathbb{F}_q$ is called the *reduction mod* $P_{\mathbb{K}}$. The quotient $\overline{\mathbb{K}} := R_{\mathbb{K}}/P_{\mathbb{K}} \cong \mathbb{F}_q$, $q = p^l$, is called the *residue field* of $\mathbb{K}$. Every non-zero element x of $\mathbb{K}$ can be written uniquely as $x = \pi^{ord(x)}u$, $u \in R_{\mathbb{K}}^{\times}$. We call $u := ac(x)$ the *angular component* of x We set $ord(0) = \infty$. The normalized valuation of $\mathbb{K}$ is the mapping

$$\begin{array}{ccc} \mathbb{K} & \to & \mathbb{Z} \cup \{\infty\} \\ x & \to & ord(x). \end{array}$$

Then $|x|_{\mathbb{K}} = q^{-ord(x)}$ and $|\pi|_{\mathbb{K}} = q^{-1}$.

We fix $\mathfrak{S} \subset R_{\mathbb{K}}$ a set of representatives of $\mathbb{F}_q$ in $R_{\mathbb{K}}$, i.e., the reduction mod $P_{\mathbb{K}}$ is a bijection from $\mathfrak{S}$ onto $\mathbb{F}_q$. We assume that $0 \in \mathfrak{S}$. Any non-zero element x of $\mathbb{K}$ can be written as

$$x = \pi^{ord(x)} \sum_{i=0}^{\infty} x_i \pi^i, \tag{3.1}$$

where $x_i \in \mathfrak{S}$ and $x_0 \neq 0$. This series converges in the norm $|\cdot|_{\mathbb{K}}$. Notice that $ac(x) = \sum_{i=0}^{\infty} x_i \pi^i$.

A multiplicative character (or quasi-character) of the group $(\mathbb{K}^{\times}, \cdot)$ is a continuous homomorphism $\omega : \mathbb{K}^{\times} \to \mathbb{C}^{\times}$ satisfying $\omega(xy) = \omega(x)\omega(y)$. Every multiplicative character has the form

$$\omega(x) = |x|_{\mathbb{K}}^{s} \, \omega_0(ac(x)), \text{ for some } s \in \mathbb{C},$$

where ω_0 is the restriction of ω to $R_{\mathbb{K}}^{\times}$; ω_0 is a continuous multiplicative character of $(R_{\mathbb{K}}^{\times}, \cdot)$ into the complex unit circle.

For an in-depth exposition of non-Archimedean local fields, the reader may consult [56,48], see also [2,50].

4. Non-Archimedean analogues of the Jackson derivative

In this article we introduce several non-Archimedean analogues of the Jackson derivative. For an in-depth presentation of the classical q-analysis the reader may consult [18], [27]. Given $f : \mathbb{K} \to \mathbb{C}$ we define

$$\partial f(x) = \frac{f(\pi^{-1}x) - f(\pi x)}{(q - q^{-1})\,|x|_{\mathbb{K}}}, \text{ for } x \neq 0.$$

The existence of $\partial f(0)$ depends on f. In the classical case, the value at the origin of the Jackson derivative is given by the standard derivative, this approach cannot be used here.

If $g:\mathbb{K}\rightarrow\mathbb{C}$, then the following Leibniz-type rule holds true:

$$\begin{aligned}\partial\left(f(x)\,g(x)\right)&=g(\pi x)\partial f(x)+f(\pi^{-1}x)\partial g(x)\\&=g(\pi^{-1}x)\partial f(x)+f(\pi x)\partial g(x).\end{aligned}\tag{4.1}$$

Notice that for any function $f(ac(x))$, it holds that $\partial f(ac(x))=0$, for $x\neq 0$. In particular, $\partial\omega_0(ac(x))=0$, $x\neq 0$, for any multiplicative character ω_0 of $\left(R_{\mathbb{K}}^{\times},\cdot\right)$. We also have

$$\partial\left|x\right|_{\mathbb{K}}^{m}=[m]\left|x\right|_{\mathbb{K}}^{m-1}\ \text{ for } m\in\mathbb{N}\smallsetminus\{0\}.\tag{4.2}$$

We set $[m]!:=\prod_{i=1}^{m}[i]$, with $[0]!=1$.

Another non-Archimedean Jackson-type derivative is defined as

$$\widetilde{\partial}f(x)=\frac{f\left(\pi^{-1}x\right)-f(x)}{(q-1)\left|x\right|_{\mathbb{K}}},\ x\neq 0,$$

where $f:\mathbb{K}\rightarrow\mathbb{C}$. Now, for $g:\mathbb{K}\rightarrow\mathbb{C}$, we have

$$\widetilde{\partial}\left(f(x)\,g(x)\right)=g\left(\pi^{-1}x\right)\widetilde{\partial}f(x)+f(x)\widetilde{\partial}g(x).$$

Notice that

$$\widetilde{\partial}\left|x\right|_{\mathbb{K}}^{m}=[[m]]\left|x\right|_{\mathbb{K}}^{m-1},\ m\in\mathbb{N}\smallsetminus\{0\},\tag{4.3}$$

where $[[m]]:=\frac{q^{m}-1}{q-1}$. We also set $[[m]]!=\prod_{i=1}^{m}[[i]]$, with $[[0]]!:=1$. In case of functions depending on several variables, say $f(x,t)$, we use the notation $\partial_x f(x,t)$, $\widetilde{\partial}_x f(x,t)$ to mean a derivative with respect to x.

5. A non-Archimedean Fock representation of $\mathcal{A}_q$

5.1. Some operators

We introduce the operators:

$$a^{\dagger}f(x)=\left|x\right|_{\mathbb{K}}f(x),\ af(x)=\partial f(x),\ q^{N}f(x)=f(\pi^{-1}x),\ q^{-N}f(x)=f(\pi x),$$

which act on functions $f:\mathbb{K}\rightarrow\mathbb{C}$. We now fix a multiplicative character ω_{vac} of $\left(R_{\mathbb{K}}^{\times},\cdot\right)$. By simplicity we take $\omega_{vac}=1$. We call such a function the *vacuum eigenstate*. We define

$$u_n(x)=\frac{\left|x\right|_{\mathbb{K}}^{n}}{\sqrt{[n]!}},\ x\in\mathbb{K},\ \text{for } n\in\mathbb{N}.$$

Then

$$a^{\dagger}u_n(x)=\sqrt{[n+1]}u_{n+1}(x)\ \text{ for } n\in\mathbb{N},\tag{5.1}$$

$$au_n(x)=\sqrt{[n]}u_{n-1}(x)\ \text{ for } n\in\mathbb{N}\smallsetminus\{0\},\tag{5.2}$$

$$au_0(x)=0,\tag{5.3}$$

$$q^{\pm N}u_n(x)=q^{\pm n}u_n(x)\ \text{for } n\in\mathbb{N}.\tag{5.4}$$

5.2. A non-Archimedean Bargmann-Fock type realization

Let $\mathcal{F}_q^{\blacklozenge}$ be the $\mathbb{C}$-vector space of formal series of the form

$$f(x)=\sum_{n=0}^{\infty}c_n\,|x|_{\mathbb{K}}^{n}\,,\,x\in\mathbb{K},\,c_n\in\mathbb{C}\text{ for every }n.$$

We introduce a sesquilinear form on $\mathcal{F}_q^{\blacklozenge}$ by taking

$$(f,g):=\overline{f(\partial)}g(x)\mid_{x=0},$$

where $f(\partial):=\sum_{n=0}^{\infty}c_n\partial^n$. If $g(x)=\sum_{n=0}^{\infty}d_n\,|x|_{\mathbb{K}}^{n}$, by using (4.2), we have

$$(f,g)=\sum_{n=0}^{\infty}\overline{c_n}d_n\,[n]!.$$

We set $\|f\|^2:=\sum\limits_{n=0}^{\infty}|c_n|^2\,[n]!$. Notice that

$$(u_n(x)\,,u_m(x))=\delta_{n,m},$$

i.e., $\{u_n(x)\}_{n\in\mathbb{N}}$ is an orthonormal basis. We now set

$$\mathcal{F}_q:=\left\{f=\sum_{n=0}^{\infty}c_nu_n(x)\in\mathcal{F}_q^{\blacklozenge};\,\|f\|^2=\sum_{n=0}^{\infty}|c_n|^2<\infty\right\}.$$

Then, the space $\mathcal{F}_q$ endowed with the inner product $(\cdot,\cdot)$ becomes a Hilbert space having $\{u_n(x)\}_{n\in\mathbb{N}}$ as an orthonormal basis. The space $\mathcal{F}_q$ is isomorphic to the classical $l^2(\mathbb{C})$ Hilbert space consisting of the square-summable complex sequences.

By formulae (5.1)-(5.4), the operators $a^{\dagger}$, a, q^N, q^{-N} are well-defined in the $\mathbb{C}$-vector space

$$\mathcal{F}_q^{\text{fin}}:=\left\{f(x)=\sum_{n=0}^{M}c_nu_n(x)\in\mathcal{F}_q;\text{ for some }M=M(f)\right\},$$

which is a dense subspace of $\mathcal{F}_q$. These operators extend to linear unbounded operators on $\mathcal{F}_q$. One easily verifies that $a^{\dagger}$ is the adjoint of a and that q^N, q^{-N} are self-adjoint operators on $\mathcal{F}_q^{\text{fin}}$.

Finally, by using formulae (5.1)-(5.4), one verifies that the operators $a^{\dagger}$, a, q^N, q^{-N} satisfy the relations (1)-(5) given in Definition 2. A similar realization for the algebra $\mathcal{A}_q^c$ exists.

5.3. L^2 is isometric to $\mathcal{F}_q$

We fix a Haar measure dx on the additive group $(\mathbb{K},+)$ satisfying that $\int_{R_{\mathbb{K}}}dx=1$. The space $L^2(\mathbb{K})$ consists of all the functions $f:\mathbb{K}\to\mathbb{C}$ satisfying

$$\|f\|_2^2=\int_{\mathbb{K}}|f(x)|^2\,dx<\infty.$$

In the case $\mathbb{K}=\mathbb{Q}_p$ it is well-known that $L^2(\mathbb{Q}_p)$ admits a countable orthonormal basis, see, e.g., [32], [35], [50]. Consequently $L^2(\mathbb{Q}_p)$ is isometric to $l^2(\mathbb{C})$. This fact is indeed valid for any non-Archimedean local field. We set

$$\omega_{rbk}(x):=q^{\frac{-r}{2}}\chi_{\mathbb{K}}\left(\pi^{-1}k\left(\pi^{r}x-b\right)\right)\Omega\left(\left|\pi^{r}x-b\right|_{\mathbb{K}}\right),$$

where $r \in \mathbb{Z}$, $k \in \mathfrak{S} \smallsetminus \{0\}$, $b \in \mathbb{K}/R_{\mathbb{K}}$, $b = \sum_{i=\beta}^{-1} n_i \pi^i$, with $n_i \in \mathfrak{S}$, $\beta \in \mathbb{Z}$, $\beta < 0$ ($\mathfrak{S}$ is a set of representatives of $\mathbb{F}_q$ in $R_{\mathbb{K}}$), $\chi_{\mathbb{K}}$ is the standard additive character of the additive group $(\mathbb{K}, +)$, i.e., $\chi_{\mathbb{K}} : \mathbb{K} \to \mathbb{C}$ is a continuous mapping satisfying: $\left|\chi_{\mathbb{K}}(x)\right| = 1$, $\chi_{\mathbb{K}}(x+y) = \chi_{\mathbb{K}}(x)\chi_{\mathbb{K}}(y)$, $\chi_{\mathbb{K}} \mid_{R_{\mathbb{K}}} = 1$ but $\chi_{\mathbb{K}} \mid_{\mathbb{K} \smallsetminus R_{\mathbb{K}}} \neq 1$. Finally, $\Omega(|\pi^r x - b|_{\mathbb{K}})$ denotes the characteristic function of the ball $\pi^{-r} b + \pi^{-r} R_{\mathbb{K}}$. The family $\{\omega_{rbk}(x)\}_{rbk}$ forms a complete orthonormal basis of $L^2(\mathbb{K})$. The proof of this result follows using the argument of the case $\mathbb{K} = \mathbb{Q}_p$, see [36, Theorem 2]. Therefore $L^2(\mathbb{K})$ is isometric to $l^2(\mathbb{C})$, the Fock space.

Remark 1. (i) Given $f, g, \partial f, \partial g \in L^2(\mathbb{K})$, by using changes of variables, one verifies that

$$\langle g, \partial f \rangle := \int_{\mathbb{K}} \overline{g(x)}\, \partial f(x)\, dx = -\int_{\mathbb{K}} \overline{\partial g(x)}\, f(x)\, dx.$$

(ii) The operators $|x|_{\mathbb{K}}$, ∂ are well-defined on the space of test functions which is dense in $L^2(\mathbb{K})$. However these operators cannot be directly interpreted as creation and annihilation operators in $L^2(\mathbb{K})$. Let $\mathfrak{i} : L^2(\mathbb{K}) \to l^2(\mathbb{C})$ be the above mentioned isometry. Then the creation operator, respectively annihilation operator, in $L^2(\mathbb{K})$ are $\mathfrak{i}^{-1} \circ |x|_{\mathbb{K}} \circ \mathfrak{i}$, respectively $\mathfrak{i}^{-1} \circ \partial \circ \mathfrak{i}$.

6. The non-Archimedean harmonic oscillator

Motivated by Biedenharn's work [6], see also [29] and the references therein, we introduce the π-momentum operator $\mathbf{\Pi}$ and the π-position operator $\boldsymbol{Q}$, in terms of $a^\dagger = |x|_K$, $a = \partial$, as

$$\mathbf{\Pi} := i\sqrt{\frac{m\hbar\omega}{2}}\left(a^\dagger - a\right), \quad \boldsymbol{Q} := \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right).$$

The Hamiltonian of the π-harmonic oscillator (or non-Archimedean oscillator) is defined as

$$\boldsymbol{H} = \frac{\mathbf{\Pi}^2}{2m} + \frac{m\omega^2}{2}\boldsymbol{Q}^2 = \frac{1}{2}\hbar\omega\left(aa^\dagger + a^\dagger a\right).$$

Then $\boldsymbol{H} u_n(x) = E_n u_n(x)$, i.e., $\boldsymbol{H}$ is diagonal on the eigenstates $u_n(x)$ with eigenvalues

$$E_n := \frac{1}{2}\hbar\omega\left([n+1] + [n]\right) = \frac{1}{2}\hbar\omega\frac{\sinh\left(\frac{2n+1}{2}\ln q\right)}{\sinh\left(\frac{1}{2}\ln q\right)}.$$

Since q is a power of a prime number, the energy levels are no longer uniformly spaced. In the limit $q \to 1$ these numbers give the eigenvalues of the usual quantum harmonic oscillator, see, e.g., [6]. The interpretation of the non-uniform distribution of the energy levels of the q-harmonic oscillator is a challenging problem.

In the non-Archimedean framework, the nonuniform spacing of the energy levels of the π-harmonic oscillator obeys to a scale law. We set $S_r = \{x \in \mathbb{K}; |x|_{\mathbb{K}} = q^r\}$ for the sphere with center at the origin and radius $r \in \mathbb{Z}$. Notice that $S_0 = \bigsqcup_j (j + \pi R_{\mathbb{K}})$, where $j \in \mathfrak{S} \smallsetminus \{0\}$, see (3.1), and each ball $j + \pi R_{\mathbb{K}}$ can be identified with an infinite regular rooted tree. The regularity means that each vertex has exactly q children. The set $\mathbb{K} \smallsetminus \{0\}$ is the disjoint union of a countable number of copies of scaled trees. More precisely,

$$\mathbb{K} \smallsetminus \{0\} = \bigsqcup_{r=-\infty}^{\infty} S_r = \bigsqcup_{r=-\infty}^{\infty} \pi^{-r} S_0.$$

The group $(\mathbb{Z}, +)$ is a scale group acting on $\mathbb{K} \smallsetminus \{0\}$ as

$$\begin{array}{lll}\mathbb{Z}\times(\mathbb{K}\smallsetminus\{0\}) & \rightarrow & \mathbb{K}\smallsetminus\{0\}\\ (r,x) & \rightarrow & \pi^{-r}x.\end{array}$$

Then $\mathbb{K}\smallsetminus\{0\}$ is a self-similar set obtained from S_0 by the action of the scale group $(\mathbb{Z},+)$.

Let us fix $\mathbb{K}$ a non-Archimedean local field, which plays the role of a one-dimensional background space. Now, let $\mathbb{K}_m$ be an extension of $\mathbb{K}$ of degree m, this means that $\mathbb{K}_m$ is a local field, containing $\mathbb{K}$, which is a $\mathbb{K}$-vector space of dimension $m\geq 2$. The ring of integers $R_{\mathbb{K}}$ of $\mathbb{K}$ is a subring of the ring of integers $R_{\mathbb{K}_m}$ of $\mathbb{K}_m$. The local uniformizing parameter π of $\mathbb{K}$ generates an ideal $\pi R_{\mathbb{K}_m}$ in $R_{\mathbb{K}_m}$. Since any ideal of $R_{\mathbb{K}_m}$ has the form $\pi_m^l R_{\mathbb{K}_m}$, where π_m denotes a local uniformizing parameter of $\mathbb{K}_m$, we have $\pi R_{\mathbb{K}_m}=\pi_m^e R_{\mathbb{K}_m}$, for some positive integer e (called *the ramification index* of the extension $\mathbb{K}/\mathbb{K}_m$). By a well-known result, we have $m=ef$, where the positive integer f (called *the inertia index* of the extension $\mathbb{K}/\mathbb{K}_m$) is the dimension of $\overline{\mathbb{K}_m}$ considered as a $\overline{\mathbb{K}}$-vector space, i.e., $\overline{\mathbb{K}_m}=\mathbb{F}_{q^f}$, see, e.g., [56].

Since any function $f:\mathbb{K}_m\rightarrow\mathbb{C}$ has a restriction to $\mathbb{K}$, the operators $a_m^{\dagger}=|x|_{\mathbb{K}_m}$, $a_m=\partial$ have natural restrictions which act on functions defined on $\mathbb{K}$. We denote these restrictions as $a^{\dagger}$, a. Then, we may assume the existence of 'two identical copies' of a non-Archimedean harmonic oscillator, one in $\mathbb{K}$ and the other in $\mathbb{K}_m$. The energy levels of these oscillators are

$$E_n(\mathbb{K})=\frac{1}{2}\hbar\omega\frac{\sinh\left(\frac{2n+1}{2}\ln q\right)}{\sinh\left(\frac{1}{2}\ln q\right)},\quad E_n(\mathbb{K}_m)=\frac{1}{2}\hbar\omega\frac{\sinh\left(\frac{2n+1}{2}\ln q^f\right)}{\sinh\left(\frac{1}{2}\ln q^f\right)},\tag{6.1}$$

respectively.

By using (6.1),

$$E_n(\mathbb{K})\sim\frac{1}{2}\hbar\omega\frac{\exp\left(\frac{2n+1}{2}\ln q\right)}{\exp\left(\frac{1}{2}\ln q\right)},\quad E_n(\mathbb{K}_m)\sim\frac{1}{2}\hbar\omega\left[\frac{\exp\left(\frac{2n+1}{2}\ln q\right)}{\exp\left(\frac{1}{2}\ln q\right)}\right]^f$$

for $n\rightarrow\infty$, then

$$\left(\frac{E_n(\mathbb{K}_m)}{\frac{1}{2}\hbar\omega}\right)\sim\left(\frac{E_n(\mathbb{K})}{\frac{1}{2}\hbar\omega}\right)^f\text{ for }n\rightarrow\infty.$$

7. Non-Archimedean quantum mechanics

In this section we introduce a new class of q-deformed Schrödinger equations and study the cases of the free particle and a particle in a non-Archimedean box.

7.1. A non-Archimedean Heisenberg uncertainty relations

In the algebra $\mathcal{A}_q$, it verifies that $\partial\,|x|_{\mathbb{K}}-q^{-1}\,|x|_{\mathbb{K}}\,\partial=q^N$, and by using $q^{-1}q^{-N}\,|x|_{\mathbb{K}}=|x|_{\mathbb{K}}\,q^{-N}$, we have

$$\begin{aligned}1&=q^{-N}\partial\,|x|_{\mathbb{K}}-q^{-1}q^{-N}\,|x|_{\mathbb{K}}\,\partial=q^{-N}\partial\,|x|_{\mathbb{K}}-q^{-2}\,|x|_{\mathbb{K}}\,q^{-N}\partial\\&=:\left[q^{-N}\partial,|x|_{\mathbb{K}}\right]_{q^{-2}}.\end{aligned}\tag{7.1}$$

We propose using operator $-i\hbar q^{-N}\partial$ as a non-Archimedean analogue of the momentum operator, and propose operator $|x|_{\mathbb{K}}$ as an analogue of the position operator. By using (7.1), the Heisenberg uncertainty formula becomes

$$\left[-i\hbar q^{-N}\partial, |x|_{\mathbb{K}}\right]_{q^{-2}} = -i\hbar. \tag{7.2}$$

The relation (7.2) is a q-deformation of the classical Heisenberg uncertainty relation. In the limit $q \to 1$ the relation (7.2) becomes the standard Heisenberg uncertainty relation.

7.2. Some mathematical results

We review some results on q-analysis following [18], [29, Chapter 2], [27]. In this framework q is a complex parameter, since here q represents the cardinality of a finite field, we use a complex parameter ρ in our review of the ρ-analysis, later on we specialize ρ to q.

For any nonzero complex number ρ, the ρ-number $[a]_\rho$, $a \in \mathbb{C}$, is defined as

$$[a]_\rho = \frac{\rho^a - \rho^{-a}}{\rho - \rho^{-1}} = \frac{\sinh(a \ln \rho)}{\sinh(\ln \rho)}.$$

We also define

$$[[a]]_\rho = \frac{\rho^a - 1}{\rho - 1} = \rho^{\frac{a(a-1)}{2}} [a]_{\sqrt{\rho}}.$$

In the case $\rho = q$, we use the simplified notation $[a]_q = [a]$, $[[a]]_q = [[a]]$. We use this convention for any function depending on ρ.

For $m \in \mathbb{N}$, we set ρ-factorial $[m]_\rho! := \prod_{j=1}^m [m]_\rho$ with $[0]_\rho! := 1$, and $[[m]]_\rho! := \prod_{j=1}^m [[m]]_\rho$ with $[[0]]_\rho! := 1$. By convention, $[m]_q! = [m]!$, $[[m]]_q! = [[m]]!$.

We also set for $m \in \mathbb{N}$,

$$(a; \rho)_m := \begin{cases} (1-a)(1-a\rho)\cdots\left(1-a\rho^{m-1}\right) & \text{for} \quad m \geq 1 \\ 1 & \text{for} \quad m = 0. \end{cases}$$

Then

$$[m]_\rho! = \frac{\rho^{-\frac{m(m-1)}{2}}}{\left(1-\rho^2\right)^m}\left(\rho^2; \rho^2\right)_m. \tag{7.3}$$

For $|\rho| < 1$, we set

$$(a; \rho)_\infty := \prod_{j=1}^{\infty}\left(1 - a\rho^{j-1}\right).$$

This product converges for all $a \in \mathbb{C}$ and defines an analytic function.

7.2.1. The ρ-exponential functions

There are two ρ-analogues of the exponential function:

$$e_\rho(z) := \sum_{n=0}^{\infty} \frac{z^n}{(\rho; \rho)_n} = \frac{1}{(z; \rho)_\infty} \text{ for } z, \rho \in \mathbb{C}, \text{ with } |\rho| < 1,$$

$$\mathcal{E}_\rho(z) := \sum_{n=0}^{\infty} \frac{\rho^{\frac{n(n-1)}{2}} z^n}{(\rho; \rho)_n} = (-z; \rho)_\infty \text{ for } z, \rho \in \mathbb{C}, \text{ with } |\rho| < 1. \tag{7.4}$$

Furthermore, $e_\rho(z)\,\mathcal{E}_\rho(z) = 1$,

$$e_{\rho}(z) := 1 + \sum_{n=1}^{\infty} \frac{\left(\frac{z}{1-\rho}\right)^{n}}{(1-\rho)\left(1-\rho^{2}\right)\cdots(1-\rho^{n})} = \sum_{n=1}^{\infty} \frac{\left(\frac{z}{1-\rho}\right)^{n}}{[[n]]_{\rho}!}$$
$$= \frac{1}{\prod_{j=1}^{\infty}\left(1 - \frac{z}{1-\rho}\rho^{j-1}\right)},$$

and also

$$\mathcal{E}_{\rho}(z) = \sum_{n=0}^{\infty} \frac{\rho^{\frac{n(n-1)}{2}}\left(\frac{z}{1-\rho}\right)^{n}}{[[n]]_{\rho}!} = \prod_{j=1}^{\infty}\left(1 - \frac{z}{1-\rho}\rho^{j-1}\right),$$
$$\mathcal{E}_{\rho^{2}}(z) = \sum_{n=0}^{\infty} \frac{\rho^{\frac{n(n-1)}{2}}\left(\frac{z}{1-\rho^{2}}\right)^{n}}{[n]_{\rho}!} = \prod_{j=1}^{\infty}\left(1 + \frac{z}{1-\rho^{2}}\rho^{j-1}\right). \tag{7.5}$$

7.2.2. The π-exponential functions

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{K}$, by using $\rho = q^{-1}$ in (7.5) and $[n]_{\rho} = [n]_{\rho^{-1}}$, we set

$$\mathcal{E}(x,\lambda) := \sum_{l=0}^{\infty} \frac{q^{\frac{-(l-1)l}{2}}\lambda^{l}|x|_{\mathbb{K}}^{l}}{[l]!} = \mathcal{E}_{q^{-2}}\left(\frac{\lambda|x|_{\mathbb{K}}}{1-q^{-2}}\right) = \prod_{j=1}^{\infty}\left(1 + \frac{\lambda|x|_{\mathbb{K}}}{1-q^{-2}}q^{-2j+2}\right), \tag{7.6}$$

and

$$e(x,\lambda) := \sum_{l=0}^{\infty} \frac{\lambda^{l}|x|_{\mathbb{K}}^{l}}{[[l]]_{q^{-1}}!} = e_{\rho}\left(\left(1-q^{-1}\right)\lambda|x|_{\mathbb{K}}\right) = \frac{1}{\prod_{j=1}^{\infty}\left(1 - \left(1-q^{-1}\right)\lambda|x|_{\mathbb{K}}\, q^{-j+1}\right)}. \tag{7.7}$$

Notice that the series $e(x,\lambda)$ converges for $|\lambda|\,|x|_{\mathbb{K}} < \frac{1}{1-q^{-1}}$, and that $\mathcal{E}(x,\lambda)$ and $e(x,\lambda)$ are radial functions of x. We call $e(x;\lambda)$, $\mathcal{E}(x,\lambda)$ the *π-exponential functions*.

By using (4.3), we have

$$\widetilde{\partial}_{x}^{m} e(x;\lambda) = \lambda^{m} e(x,\lambda)\,, \text{ for } m \in \mathbb{N}\smallsetminus\{0\}\,. \tag{7.8}$$

On the other hand, $\partial_{x}\mathcal{E}(x,\lambda) = \lambda\mathcal{E}\left(\pi^{-1}x,\lambda\right) = \lambda q^{N}\mathcal{E}(x,\lambda)$, i.e.,

$$\left(q^{-N}\partial_{x}\right)\mathcal{E}(x,\lambda) = \lambda\mathcal{E}(x,\lambda)\,.$$

By induction on m,

$$\left(q^{-N}\partial_{x}\right)^{m}\mathcal{E}(x;\lambda) = \lambda^{m}\mathcal{E}(x,\lambda)\,, \text{ for } m \in \mathbb{N}\smallsetminus\{0\}\,. \tag{7.9}$$

7.2.3. ρ-Trigonometric functions

The ρ-trigonometric functions attached to $\mathcal{E}_{\rho}(z)$ are defined as

$$\sin_{\rho}(z) = \frac{1}{2i}\left(\mathcal{E}_{\rho}(iz) - \mathcal{E}_{\rho}(-iz)\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n}\rho^{(2n+1)n}\left(\frac{z}{1-\rho}\right)^{2n+1}}{[[2n+1]]_{\rho}!},$$

$$\cos_\rho(z) = \frac{1}{2}\left(\mathcal{E}_\rho(iz) + \mathcal{E}_\rho(-iz)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{(2n-1)n}\left(\frac{z}{1-\rho}\right)^{2n}}{[[2n]]_\rho!},$$

where $i = \sqrt{-1}$.

We define the π-trigonometric functions attached to $\mathcal{E}(x, \lambda)$ as

$$\cos(x, \mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n-1)} \mu^{2n} |x|_{\mathbb{K}}^{2n}}{[2n]!},$$
$$\sin(x, \mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)} \mu^{2n+1} |x|_{\mathbb{K}}^{2n+1}}{[2n+1]!},$$

for $x \in \mathbb{K}$, $\mu \in \mathbb{R}$. We call $\cos(x, \mu)$, resp. $\sin(x; \mu)$, the *π-cosine function*, resp., the *π-sine function*. Some useful properties of these trigonometric functions are the following:

$$\cos(0, \mu) = 1,\ \cos(x, -\mu) = \cos(x, \mu),\ q^{-N}\partial \cos(x, \mu) = -\mu \sin(x, \mu),$$
$$\sin(0, \mu) = 0,\ \sin(x, -\mu) = -\sin(x, -\mu),\ q^{-N}\partial \sin(x, \mu) = \mu \cos(x, \mu).$$

7.3. A non-Archimedean analogue of the Schrödinger equation

Given a function $\Psi(t, x) : \mathbb{K} \times \mathbb{K} \to \mathbb{C}$, we set

$$\partial_t u(t, x) := \frac{u(\pi^{-1}t, x) - u(\pi t, x)}{\left(q - q^{-1}\right)|t|_{\mathbb{K}}},\quad q_x^{-N}\partial_x u(t, x) := \frac{u(t, x) - u(t, \pi^2 x)}{\left(1 - q^{-2}\right)|x|_{\mathbb{K}}}.$$

We propose the following non-Archimedean analogue of the Schrödinger equation:

$$i\hbar\partial_t \Psi(t, x) = \left\{\frac{-\hbar^2}{2m}\left(q_x^{-N}\partial_x\right)^2 + V(t, x)\right\}\Psi(t, x), \tag{7.10}$$

where $\Psi(t, x)$ is a wave function, m is the mass of the particle, and $V(t, x)$ is the potential. The time and spatial variables are elements of a non-Archimedean local field of arbitrary characteristic, but the wave functions are complex-valued.

7.4. Free particle

In the case of the free particle, the time-independent Schrödinger equation takes the form

$$\frac{-\hbar^2}{2m}\left(q_x^{-N}\partial_x\right)^2 \Psi(x) = E\Psi(x). \tag{7.11}$$

We study the solutions of (7.11) in spaces of type $\mathcal{F}_q^\blacklozenge$. We first notice that if $\Psi(x)$ is a solution of (7.11), then $h(ac(x))\Psi(x)$ is also a solution of (7.11), where $h(ac(x))$ is an arbitrary function. This means that a regularization at the origin for $\Psi(x)$ is required. By using that $\left(q_x^{-N}\partial_x\right)^2 \mathcal{E}(x; \lambda) = \lambda^2 \mathcal{E}(x, \lambda)$, we obtain that

$$\Psi(x) = c_0 \mathcal{E}\left(x, i\sqrt{\frac{2mE}{\hbar^2}}\right) + c_1 \mathcal{E}\left(x, -i\sqrt{\frac{2mE}{\hbar^2}}\right)$$
$$= a_0 \cos\left(x, \sqrt{\frac{2mE}{\hbar^2}}\right) + a_1 \sin\left(x, \sqrt{\frac{2mE}{\hbar^2}}\right) \tag{7.12}$$

where a_0, a_1 are functions depending only on the angular component of x. Since the angular component is not defined at the origin, function $\Psi(x)$ is not defined at the origin. Since $\cos\left(0, \sqrt{\frac{2mE}{\hbar^2}}\right) = 1$, $\sin\left(0, \sqrt{\frac{2mE}{\hbar^2}}\right) = 0$, the simplest way of regularizing $\Psi(x)$ at the origin is by choosing $a_0 = 0$ and a_1 as a nonzero constant, i.e., by choosing

$$\Psi(x) = a_1 \sin\left(x, \sqrt{\frac{2mE}{\hbar^2}}\right). \tag{7.13}$$

The functions $\mathcal{E}\left(x, \pm i\sqrt{\frac{2mE}{\hbar^2}}\right)$ are the analogs of the classical plane waves. These functions are radial functions defined in $\mathbb{K}$ by a convergent, complex-valued series, see (7.6).

Another possible version of a non-Archimedean, q-deformed, time-independent Schrödinger equation is

$$\frac{-\hbar^2}{2m}\widetilde{\partial}_x^2\Psi(x) = E\Psi(x). \tag{7.14}$$

By (7.8), the planes waves have the form $e\left(x, \pm i\sqrt{\frac{2mE}{\hbar^2}}\right)$ are solutions of (7.14). These series converge when $|x|_{\mathbb{K}}\sqrt{\frac{2mE}{\hbar^2}} < \frac{1}{1-q^{-1}}$, see (7.7). For this reason we focus on equations of type (7.11).

7.5. Particle in a box

We now consider a potential of the form

$$V(x) = \begin{cases} 0 & \text{if} \quad |x|_{\mathbb{K}} \le q^{L-1} \\ \infty & \text{if} \quad |x|_{\mathbb{K}} > q^{L}, \end{cases}$$

where L is a fixed integer. We look for a solution of the Schrödinger equation

$$\left\{\frac{-\hbar^2}{2m}\left(q_x^{-N}\partial_x\right)^2 + V(x)\right\}\Psi(x) = E\Psi(x)$$

subjected to the conditions: $\Psi(0) = 0$, $\Psi(q^L) = 0$. The first is a regularization condition and the second condition guarantees that the particle is confined in the box $\left\{x \in K; |x|_{\mathbb{K}} \le q^{L-1}\right\}$. The first condition implies that the solution has the form (7.13), with $E = \frac{\hbar^2}{2m}\mu^2$, $\mu \in \mathbb{R}$. Notice that $\Psi(x) = \Psi(|x|_{\mathbb{K}})$. To satisfy the second condition, we need

$$\Psi(q^L) = B_1 \sin\left(q^L, \sqrt{\frac{2mE}{\hbar^2}}\right) = 0.$$

There is a sequence of positive real numbers $\omega_1 < \omega_2 < \cdots < \omega_k \cdots$, such that the zeros of $\sin(x, \mu) = 0$ are $|x|_{\mathbb{K}}\,\mu = \omega_k$, $k \ge 1$, then $\mu = q^{-L}\omega_k$, $k \ge 1$. This fact follows from Theorem 5.1 in [14]. Indeed, the exponential function in [14, formula 2.2] is exactly $\mathcal{E}(\left(1-\rho^2\right)z)$, with $q = \rho^2$. Therefore

$$E_k = \frac{\hbar^2\omega_k^2}{2mq^{2L}}, \text{ for } k \ge 1,$$

are the energy levels for a particle confined in a non-Archimedean box.

8. The non-Archimedean Schrödinger equation with a radial potential

The ball B_L with center at the origin and radius $L \in \mathbb{Z}$ is defined as

$$B_L = \left\{ x \in \mathbb{K}; |x|_{\mathbb{K}} \leq q^L \right\}.$$

The sphere S_L with center at the origin and radius $L \in \mathbb{Z}$ is defined as

$$S_L = \left\{ x \in \mathbb{K}; |x|_{\mathbb{K}} = q^L \right\}.$$

We denote by $[-\infty, +\infty]$ the extended numeric line. We fix a function $V : [0, +\infty) \to [-\infty, +\infty]$. We denote by $Sing(V)$ the set of singularities of V. A point $x \in [0, +\infty)$ is a singular point of V, if V is not continuous at x, or if $V(x) = \pm\infty$. The function $V(|x|_K)$ is a radial potential. We now consider the following eigenvalue problem:

$$\begin{cases} \Psi : B_L \to \mathbb{C} \\ \left\{ \frac{-\hbar^2}{2m} \left(q_x^{-N} \partial_x \right)^2 + V(|x|_{\mathbb{K}}) \right\} \Psi(x) = E\Psi(x). \end{cases} \tag{8.1}$$

By using (7.9) with $m = 2$, and taking $\Psi(x) = \mathcal{E}(|x|_{\mathbb{K}}, \lambda)$, one gets

$$\left(q^{-N} \partial_x \right)^2 \mathcal{E}(|x|_{\mathbb{K}}; \lambda) = \lambda^2 \mathcal{E}(|x|_{\mathbb{K}}, \lambda).$$

Now, by replacing $\Psi(x)$ in (8.1), we obtain the condition

$$\left\{ \frac{-\hbar^2}{2m} \lambda^2 + V(|x|_{\mathbb{K}}) - E) \right\} \mathcal{E}(|x|_{\mathbb{K}}, \lambda) = 0. \tag{8.2}$$

Consider the points $|x|_{\mathbb{K}} = q^l \leq q^L$ such that $x \notin Sing(V)$. Then (8.2) becomes

$$\left\{ \frac{-\hbar^2}{2m} \lambda^2 + V(q^l) - E) \right\} \mathcal{E}\left(q^l, \lambda \right) = 0. \tag{8.3}$$

By (7.6) the zeros of $\mathcal{E}\left(q^l, \lambda\right) = 0$ satisfy $1 + \frac{\lambda q^l}{1-q^{-2}} q^{-2j+2} = 0$, for $j = 1, 2, \ldots$, therefore, the energy levels $E = E_{l,\lambda}$ have the form

$$E_{l,\lambda} = \frac{-\hbar^2}{2m} \lambda^2 + V(q^l),$$

where l is an integer satisfying $l \leq L$ such that $S_l \nsubseteq Sing(V)$, and

$$\lambda \in \mathbb{R} \setminus \left\{ -\left(1 - q^{-2}\right) q^{2j-l-2}; \, j \geq 1 \right\}.$$

8.1. Potentials supported in the unit ball

To obtain a more precise description of the energy levels, it is necessary to impose boundary conditions and some additional restrictions to the function V.

We take

$$V(|x|_{\mathbb{K}}) : B_0 \to [-\infty, +\infty],$$

such that $Sing(V)$ is just the origin. We consider the following eigenvalue problem:

$$\begin{cases} \Psi : B_0 \to \mathbb{C} \\ \Psi\mid_{S_0} = 0 \\ \left\{ \frac{-\hbar^2}{2m} \left(q_x^{-N}\partial_x\right)^2 + V(|x|_{\mathbb{K}}) \right\} \Psi(x) = E\Psi(x). \end{cases} \tag{8.4}$$

We take $\Psi(x) = \mathcal{E}(|x|_{\mathbb{K}}, \lambda)$. To satisfy the condition $\Psi\mid_{S_0} = 0$, we require

$$1 + \frac{\lambda}{1-q^{-2}} q^{-2j+2} = 0, \text{ for } j = 1, 2, \ldots,$$

i.e.,

$$\lambda = -\left(1 - q^{-2}\right) q^{2j-2}, \text{ for } j = 1, 2, \ldots. \tag{8.5}$$

Take $|x|_{\mathbb{K}} = q^{-r} < 1$, notice that $x \notin Sing(V)$, then (8.3), with $-r = l$, is satisfied if λ satisfies (8.5). We pick $j = r$, then the energy levels are given by

$$E_r = \frac{-\hbar^2}{2m} \left(1 - q^{-2}\right)^2 q^{4r-4} + V(q^{-r}), \text{ for } r = 1, 2, \ldots, \tag{8.6}$$

and the functions

$$\Psi_r(x) = \mathcal{E}\left(|x|_{\mathbb{K}}, -\left(1-q^{-2}\right) q^{2r-2}\right) \text{ for } r = 1, 2, \ldots,$$

are eigenfunctions. The determination of all the possible eigenfunctions requires solving an equation of the form

$$\frac{-\hbar^2}{2m} \left(\frac{1-q^{-2}}{q^2}\right)^2 y^4 + V(y^{-1}) = E,$$

for $y \in (0, 1)$, where E is known, and then take $r = \frac{-\ln y}{\ln q} \in \mathbb{N}$.

8.2. π-Hydrogen atom

By a suitable selection of the potential $V(|x|_{\mathbb{K}})$, the energy levels of several q-models can be obtained from (8.6). For instance by taking,

$$\begin{aligned} V_{HO}(|x|_{\mathbb{K}}) &= \frac{\hbar^2 \left(1-q^{-2}\right)^2}{2mq^2 |x|_{\mathbb{K}}^4} + \frac{1}{2}\hbar\omega \frac{\sinh\left(\ln q^{\frac{1}{2}} |x|\right)}{\sinh\left(\frac{1}{2}\ln q\right)} \\ &= \frac{\hbar^2 \left(1-q^{-2}\right)^2}{2mq^2 |x|_{\mathbb{K}}^4} + \frac{1}{2}\hbar\omega \frac{q^{\frac{1}{2}} |x| - q^{\frac{-1}{2}} |x|^{-1}}{q^{\frac{1}{2}} - q^{\frac{-1}{2}}}, \end{aligned}$$

formula (8.6) gives the energy levels of the q-harmonic oscillator, see [6].

Many versions of the q-hydrogen atom have been studied. In [21], Finkelstein studied a model of a q-hydrogen atom with energy levels of the form

$$E_n(\mu) = -\frac{1}{2} mc^2 \left(\frac{e^2}{\hbar c}\right)^2 \frac{q^{4\mu}}{[2n+1]^2}, \tag{8.7}$$

where μ is a real parameter. This result was established using classical q-analysis on $\mathbb{C}$. The potential

$$V_{HA}(|x|) = \frac{\hbar^2 \left(1-q^{-2}\right)^2}{2mq^2 |x|_{\mathbb{K}}^4} - \frac{1}{2}mc^2 \left(\frac{e^2}{\hbar c}\right)^2 q^{4\mu} \frac{\sinh^2(\ln q)}{\sinh^2(\ln |x|)}$$
$$= \frac{\hbar^2 \left(1-q^{-2}\right)^2}{2mq^2 |x|_{\mathbb{K}}^4} - \frac{1}{2}mc^2 \left(\frac{e^2}{\hbar c}\right)^2 q^{4\mu} \frac{\left(q-q^{-1}\right)^2}{\left(|x|-|x|^{-1}\right)^2}$$

produces the energy levels (8.7). In the limit q tends to one, (8.7) becomes

$$E = -\frac{1}{2}mc^2 \left(\frac{e^2}{\hbar c}\right)^2 \frac{1}{(2n+1)^2},$$

which is the Balmer energy formula, where the principal quantum number is $2n+1$, see [21].

9. Some open problems

The construction of non-Archimedean quantum mechanics as a q-deformation of the classical quantum mechanics gives rise to several new mathematical problems and intriguing connections.

9.1. Semigroups with non-Archimedean time

A central problem is to determine if there is a semigroup attached to Schrödinger equation (7.10), i.e., if there is a family of operators $\{S_t\}_{t\in\mathbb{K}}$ such that

$$\Psi(t,x) = S_t \Psi_0(x), \text{ with } \Psi(0,x) = \Psi_0(x) : \mathbb{K} \to \mathbb{K},$$

is the solution of the initial valued problem attached to (7.10).

9.2. A non-Archimedean version of the Frobenius method

Set $\boldsymbol{D} = \left(q^{-N}\partial\right)$ and $A_i(x) = \sum_{l=0}^{\infty} \frac{c_{i,l}|x|_{\mathbb{K}}^l}{[l]!}$ for $i = 0, 1, \ldots, M$. To determine if a π-difference equation of the form

$$\sum_{i=1}^{M} A_i(x) \boldsymbol{D}^i \Phi(x) = 0 \tag{9.1}$$

admits a solution of the form $\Phi(x) = \sum_{l=0}^{\infty} \frac{d_l |x|_{\mathbb{K}}^{l+\gamma}}{[l]!} : \mathbb{K} \to \mathbb{C}$. To the best of our knowledge, there is no a theory for equations of type (9.1). It is important to mention here, that nowadays there are at least three different types of theories of p-adic differential equations, see [2], [8], [28], [32], [35], [46], and [62].

9.3. Non-Archimedean representations of q-oscillatory algebras

Suppose that $g : \mathbb{K} \to \mathbb{K}$. We define the operators

$$\Delta g(x) = \frac{g\left(\pi^{-1}x\right) - g(\pi x)}{\left(\pi^{-1} - \pi\right)x}, \text{ for } x \neq 0,$$

and

$$\widetilde{\Delta} g(x)=\frac{g\left(\pi^{-1} x\right)-g(x)}{\left(\pi^{-1}-1\right) x}, \text{ for } x \neq 0.$$

Is it possible to construct a Fock-type representation of $\mathcal{A}_q$, where $\mathfrak{a} g=\Delta f$ and $\mathfrak{a}^{\dagger} g=x g$? A solution of this problem will allow constructing non-Archimedean quantum mechanics with $\mathbb{K}$-valued wave functions via quantum groups.

Another relevant problem is to study π-difference equations of type

$$\sum_{j=1}^{L} a_j(x)\, \Delta^j g(x)=0, \tag{9.2}$$

where $a_j(x)=\sum_{k=0}^{\infty} d_{j,k} x^k$ with the $d_{j,k}$s$\in \mathbb{K}$, and $g: \mathbb{K} \rightarrow \mathbb{K}$. Notice that equations of type (9.1) are radically different to those of type (9.2).

9.4. *Sato-Bernstein-type theorems*

A very relevant problem consists in studying the existence of Sato-Bernstein theorems on algebras of type $\mathbb{C}\left[|x|_{\mathbb{K}}, \partial, q^{-N}, q^N\right]$, see, e.g., [7].

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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