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OPTIMAL QUANTIZATION FOR SOME TRIADIC UNIFORM CANTOR DISTRIBUTIONS WITH EXACT BOUNDS

MRINAL KANTI ROYCHOWDHURY

Abstract. Let \( \{S_j : 1 \leq j \leq 3\} \) be a set of three contractive similarity mappings such that \( S_j(x) = rx + \frac{1}{2}(1-r) \) for all \( x \in \mathbb{R} \), and \( 1 \leq j \leq 3 \), where \( 0 < r < \frac{1}{3} \). Let \( P = \sum_{j=1}^{3} \frac{1}{3} P \circ S_j^{-1} \). Then, \( P \) is a unique Borel probability measure on \( \mathbb{R} \) such that \( P \) has support the Cantor set generated by the similarity mappings \( S_j \) for \( 1 \leq j \leq 3 \). Let \( r_0 = 0.1622776602 \), and \( r_1 = 0.2317626315 \) (which are ten digit rational approximations of two real numbers). In this paper, for \( 0 < r \leq r_0 \), we give a general formula to determine the optimal sets of \( n \)-means and the \( n \)th quantization errors for the triadic uniform Cantor distribution \( P \) for all positive integers \( n \geq 2 \). Previously, Roychowdhury gave an exact formula to determine the optimal sets of \( n \)-means and the \( n \)th quantization errors for the standard triadic Cantor distribution, i.e., when \( r = \frac{1}{3} \). In this paper, we further show that \( r = r_0 \) is the greatest lower bound, and \( r = r_1 \) is the least upper bound of the range of \( r \)-values to which Roychowdhury formula extends. In addition, we show that for \( 0 < r \leq r_1 \) the quantization coefficient does not exist though the quantization dimension exists.

1. Introduction

Let \( P \) be a Borel probability measure on \( \mathbb{R}^d \), where \( d \geq 1 \). For a finite set \( \alpha \subset \mathbb{R}^d \), write

\[
V(P; \alpha) = \int \min_{a \in \alpha} \|x - a\|^2 dP(x), \quad V_n := V_n(P) = \inf \left\{ V(P; \alpha) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},
\]

where \( \| \cdot \| \) represents the Euclidean norm on \( \mathbb{R}^d \). Then, \( V(P; \alpha) \) is called the cost or distortion error for \( P \) with respect to the set \( \alpha \), and \( V_n \) is called the \( n \)th quantization error for \( P \) with respect to the squared Euclidean distance. A set \( \alpha \subset \mathbb{R}^d \) is called an optimal set of \( n \)-means for \( P \) if \( V_n(P) = V(P; \alpha) \). It is well-known that for a continuous Borel probability measure an optimal set of \( n \)-means contains exactly \( n \)-elements (see [4]). To see some work in the direction of optimal sets of \( n \)-means, one is referred to [2][5][10]. For theoretical results in quantization we refer to [1][6][8][11], and for its promising application see [12][13]. For a finite set \( \alpha \subset \mathbb{R}^d \) and \( a \in \alpha \), by \( M(a|\alpha) \) we denote the set of all elements in \( \mathbb{R}^d \) which are nearest to \( a \) among all the elements in \( \alpha \), i.e.,

\[
M(a|\alpha) = \{ x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\| \}.
\]

\( M(a|\alpha) \) is called the Voronoi region generated by \( a \in \alpha \). On the other hand, the set \( \{ M(a|\alpha) : a \in \alpha \} \) is called the Voronoi diagram or Voronoi tessellation of \( \mathbb{R}^d \) with respect to the set \( \alpha \).

Definition 1.1. A set \( \alpha \subset \mathbb{R}^d \) is called a centroidal Voronoi tessellation (CVT) with respect to a probability distribution \( P \) on \( \mathbb{R}^d \), if it satisfies the following two conditions:

(i) \( P(M(a|\alpha) \cap M(b|\alpha)) = 0 \) for \( a, b \in \alpha \), and \( a \neq b \);

(ii) \( E(X : X \in M(a|\alpha)) = a \) for all \( a \in \alpha \),

where \( X \) is a random variable with distribution \( P \), and \( E(X : X \in M(a|\alpha)) \) represents the conditional expectation of the random variable \( X \) given that \( X \) takes values in \( M(a|\alpha) \).

A Borel measurable partition \( \{ A_a : a \in \alpha \} \) is called a Voronoi partition of \( \mathbb{R}^d \) with respect to the probability distribution \( P \), if \( P \)-almost surely \( A_a \subset M(a|\alpha) \) for all \( a \in \alpha \). Let us now state the following proposition (see [3][4]).
Proposition 1.2. Let $\alpha$ be an optimal set of $n$-means, $a \in \alpha$, and $M(a|\alpha)$ be the Voronoi region generated by $a \in \alpha$, i.e., $M(a|\alpha) = \{x \in \mathbb{R}^d : \|x-a\| = \min_{b \in \alpha} \|x-b\|\}$. Then, for every $a \in \alpha$, (i) $P(M(a|\alpha)) > 0$, (ii) $P(\partial M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$.

The number $D(P) := \lim_{n \to \infty} \frac{2\log n}{-\log V_n(P)}$, if it exists, is called the quantization dimension of the probability measure $P$. On the other hand, for $s \in (0, +\infty)$, the number $\lim_{n \to \infty} n^{\frac{s}{2}} V_n(P)$, if it exists, is called the $s$-dimensional quantization coefficient for $P$. To know details about the quantization dimension and the quantization coefficient one is referred to [4].

Let $\{S_j : 1 \leq j \leq 3\}$ be a set of three contractive similarity mappings such that $S_j(x) = rx + \frac{r}{2}(1-r)$ for all $x \in \mathbb{R}$, where $0 < r < \frac{1}{2}$ and $1 \leq j \leq 3$. For an integer $n$, if $\sigma := \sigma_1 \sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, then we say that $\sigma$ is a word of length $n$. By $\{1, 2, 3\}^*$, we denote the set of all words including the empty word $\emptyset$. The empty word $\emptyset$ has length zero. For $\sigma := \sigma_1 \sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, by $S_{\sigma}$ it is meant that $S_{\sigma} := S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}$, and by $a(\sigma)$, we mean $a(\sigma) := S_{\sigma} \left( \frac{1}{2} \right)$. For the empty word $\emptyset$, by $S_{\emptyset}$ it is meant the identity mapping on $\mathbb{R}$. For $\sigma := \sigma_1 \sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, set $J_{\sigma} := S_{\sigma}(0, 1)$. For the empty word $\emptyset$, write $J := J_{\emptyset} = S_{\emptyset}([0, 1]) = [0, 1]$. Then, the set $C := \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2, 3\}^n} J_{\sigma}$ is known as the Cantor set generated by the mappings $S_{\sigma}$, and equals the support of the probability measure $P$ given by $P = \sum_{j=1}^{3} \frac{1}{3} P \circ S_{j}^{-1}$. Notice that $C$ satisfies the invariance equality $C = \bigcup_{j=1}^{3} S_{j}(C)$ (see [10]). In this paper a Cantor set $C$, which is generated by a set of three contractive similarity mappings, is called a triadic Cantor set, and a probability measure $P$ which has support the triadic Cantor set, is called a triadic Cantor distribution. For words $\beta, \gamma, \cdots, \delta \in \{1, 2, 3\}^*$, we write

$$a(\beta, \gamma, \cdots, \delta) := E(X|X \in J_{\beta} \cup J_{\gamma} \cup \cdots \cup J_{\delta}) = \frac{1}{P(J_{\beta} \cup \cdots \cup J_{\delta})} \int_{J_{\beta} \cup \cdots \cup J_{\delta}} xdP(x),$$

where $X$ is a random variable with probability distribution $P$, and $E(X)$ and $V := V(X)$ represent the expectation and the variance of the random variable $X$. Notice that for any $\omega \in \{1, 2, 3\}^*$, the similarity mapping $S_{\omega}$ is an injective mapping on $\mathbb{R}$; on the other hand, for any discrete subset $A$ of $\mathbb{R}$, the set $S_{\omega}(A)$ represents the set of values obtained by applying $S_{\omega}$ to each of the elements in $A$. Let us now give the following two definitions.

Definition 1.3. For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\beta_2 := \{a(1), a(2), 3\}$ and $\beta_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\beta_n := \beta_n(I)$ as follows:

$$\beta_n(I) = \left\{ \begin{array}{ll}
\{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \bigcup \bigcup_{\omega \in I} S_{\omega}(\beta_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\
\{S_{\omega}(\beta_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \bigcup \bigcup_{\omega \in I} S_{\omega}(\beta_3) & \text{if } 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1},
\end{array} \right.$$
Roychowdhury showed that if \( r = \frac{1}{3} \), then the sets \( \gamma_n \) given by Definition 1.3 determine the optimal sets of \( n \)-means for all positive integers \( n \geq 2 \) (see [13]). Proposition 2.3 implies that \( \gamma_n \) forms a CVT if \( \frac{1}{79} (21 - 2\sqrt{51}) \leq r \leq \frac{1}{41} (2\sqrt{31} - 1) \), i.e., if \( 0.08502712839 \leq r \leq 0.2472080177 \). Thus, we see that the range of \( r \) values for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means is bounded below by \( \frac{1}{79} (21 - 2\sqrt{51}) \), and bounded above by \( \frac{1}{41} (2\sqrt{31} - 1) \). But, the lower bound and the least upper bound of the range of \( r \) values for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means were not known. In this paper, in Theorem 5.1 we give an answer of it.

**Remark 1.6.** Notice that if \( r = 0 \), then \( S_1(x) = 0 \), \( S_2(x) = \frac{1}{2} \), and \( S_3(x) = 1 \) for all \( x \in \mathbb{R} \), and then the probability measure \( P \) becomes a discrete uniform distribution with support \( \{0, \frac{1}{2}, 1\} \). Because of that in our study we are assuming that the contractive ratios \( r \) are positive.

The arrangement of the paper is as follows: In Section 2, we give the basic preliminaries. In Section 3, we show that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( r = \frac{1}{25} \). In Section 4, we prove the following theorem:

**Theorem 1.7.** Let \( \gamma_n := \gamma_n(I) \) be the set for arbitrary \( I \) as defined by Definition 1.4. Let \( r_0, r_1 \in (0, \frac{1}{3}) \) be the unique real numbers satisfying, respectively, the equations

\[
- \frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)} = - \frac{3r^3 + 3r^2 + r - 1}{24(r + 1)},
\]

\[
- \frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)} = - \frac{3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121}{2184(r + 1)}.
\]

Then, \( r_0 = 0.1622776602 \), and \( r_1 = 0.2317626315 \). Then, for all \( n \geq 3 \), the sets \( \gamma_n \) form the optimal sets of \( n \)-means for \( r = r_0 \) and \( r = r_1 \).

In Theorem 5.1, we show that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( 0 < r \leq r_0 \), and the sets \( \gamma_n \) form the optimal sets of \( n \)-means if \( r_0 \leq r \leq r_1 \). Thus, Theorem 5.1 implies the fact that the greatest lower bound, and the least upper bound of \( r \) for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means are, respectively, given by \( r = r_0 \) and \( r = r_1 \). Notice that for \( r = r_0 \) both the sets \( \beta_n \) and \( \gamma_n \) form the optimal sets of \( n \)-means for \( P \). In addition, in Theorem 5.2 we show that the quantization coefficient for \( 0 < r \leq r_1 \) does not exist though the quantization dimension exists.

## 2. Preliminaries

As defined in the previous section, let \( S_j \) for \( 1 \leq j \leq 3 \) be the contractive similarity mappings on \( \mathbb{R} \) given by \( S_j(x) = rx + \frac{1}{2} (1 - r) \) for all \( x \in \mathbb{R} \), and \( 1 \leq j \leq 3 \), where \( 0 < r < \frac{1}{3} \). For \( \sigma := \sigma_1 \sigma_2 \cdots \sigma_k \in \{1, 2, 3\}^k \) and \( \tau := \tau_1 \tau_2 \cdots \tau_l \in \{1, 2, 3\}^l \), by \( \sigma \tau := \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_l \) we mean the word obtained from the concatenation of the words \( \sigma \) and \( \tau \). For \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n \), \( n \geq 0 \), write \( p_\sigma := \frac{1}{3^n} \) and \( s_\sigma := \frac{1}{r^n} \). Recall that if \( C \) is the Cantor set, then \( C := \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2, 3\}^n} J_\sigma \). For \( n \geq 1 \), the intervals \( J_\sigma \), where \( \sigma \in \{1, 2, 3\}^n \), are called the \( n \)th level basic intervals of the Cantor set \( C \).

The following two lemmas are well-known and easy to prove (see [3,15]).

**Lemma 2.1.** Let \( f : \mathbb{R} \to \mathbb{R}^+ \) be Borel measurable and \( k \in \mathbb{N} \), and \( P \) be the probability measure on \( \mathbb{R} \) given by \( P = \sum_{j=1}^3 \frac{1}{3} P \circ S_j^{-1} \). Then,

\[
\int f(x) dP(x) = \sum_{\sigma \in \{1, 2, 3\}^k} \frac{1}{3^k} \int f \circ S_\sigma(x) dP(x).
\]

**Lemma 2.2.** Let \( X \) be a random variable with the probability distribution \( P \). Then,

\[
E(X) = \frac{1}{2} \quad \text{and} \quad V := V(X) = \frac{1 - r}{6(r + 1)}, \quad \text{and} \quad \int (x - x_0)^2 dP(x) = V(X) + (x_0 - \frac{1}{2})^2,
\]

where \( x_0 \in \mathbb{R} \).
The following corollary is useful to obtain the distortion errors.

**Corollary 2.3.** Let \( \sigma \in \{1, 2, 3\}^k \) for \( k \geq 1 \), and \( x_0 \in \mathbb{R} \). Then,
\[
\int_{J_0} (x - x_0)^2 dP(x) = \frac{1}{3k} \left( r^{2k}V + (S_\sigma(\frac{1}{2}) - x_0)^2 \right).
\]

**Proof.** By induction, \( P = \frac{1}{3} \sum_{j=1}^{3} P \circ S_j^{-1} \) implies \( P = \sum_{\sigma \in \{1, 2, 3\}^k} p_\sigma P \circ S_\sigma^{-1} \). Using this fact, Lemma 2.1 and Lemma 2.2, the proof of the corollary follows. \( \square \)

**Proposition 2.4.** Let \( \beta_n(I) \) be the set given by Definition 1.3. Then, \( \beta_n(I) \) forms a CVT if \( 0 < r \leq 2 - \sqrt{3} \), i.e., if \( 0 < r \leq 0.2679491924 \). Moreover, if \( 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)} \), then
\[
V(P, \beta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)} \left( (2 \cdot 3^{\ell(n)} - n)V(P; \beta_2) + (n - 3^{\ell(n)})V(P; \beta_2) \right),
\]
and if \( 2 \cdot 3^{\ell(n)} \leq n < 3^{\ell(n)+1} \), then
\[
V(P, \beta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)} \left( (3^{\ell(n)+1} - n)V(P; \beta_2) + (n - 2 \cdot 3^{\ell(n)})V(P; \beta_3) \right).
\]

**Proof.** By the definition, we have \( \beta_2 = \{a(1), a(2, 3)\} \) and \( \beta_3 = \{a(1), a(2), a(3)\} \). Recall that \( \beta_n := \beta_n(I) \) is defined for \( n \geq 3 \), where \( I \subset \{1, 2, 3\}^{\ell(n)} \) with \( \text{card}(I) = n - 3^{\ell(n)} \) if \( 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)} \); and \( \text{card}(I) = n - 2 \cdot 3^{\ell(n)} \) if \( 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1} \). Notice that for \( n \geq 3 \), if \( n \neq 3^{\ell(n)} \) or \( n \neq 2 \cdot 3^{\ell(n)} \), the subset \( I \) can be chosen more than one way. This leads to the fact that if \( n \neq 3^{\ell(n)} \) or \( n \neq 2 \cdot 3^{\ell(n)} \), the sets \( \beta_n \) can be chosen multiple ways. Let us take
\[
\beta_4 = \{a(1), a(2), a(31), a(32, 33)\} \text{ (by choosing } I = \{3\}\text{),}
\beta_5 = \{a(1), a(21), a(22, 23), a(31), a(32, 33)\} \text{ (by choosing } I = \{2, 3\}\text{),}
\beta_6 = \{a(11), a(12, 13), a(21), a(22, 23), a(31), a(32, 33)\} \text{ (where } I = \{1, 2, 3\}\text{),}
\beta_7 = \{a(11), a(12), a(13), a(21), a(22, 23), a(31), a(32, 33)\} \text{ (by choosing } I = \{1\}\text{).}
\]
Since similarity mappings preserve the ratio of the distances of a point from any other two points, \( \beta_n(I) \) will form a CVT if we can show that \( \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7 \) form a CVT. Recall that \( a(1) = E(X : X \in J_1) \) and \( a(2, 3) = E(X : X \in J_2 \cup J_3) \), and also recall the Definition 1.1. Thus, \( \beta_2 \) will form a CVT if
\[
(1) \quad P(M(a(1) | \beta_2) \cap M(a(2, 3) | \beta_2)) = 0.
\]
Since the basic intervals in the first level are \( J_1 := [S_1(0), S_1(1)] \), \( J_2 := [S_2(0), S_2(1)] \), and \( J_3 := [S_3(0), S_3(1)] \), the relation (1) will be true if
\[
S_1(1) \leq \frac{1}{2} (a(1) + a(2, 3)) \leq S_2(0).
\]
Similarly, \( \beta_3 \) will form a CVT if \( S_1(1) < \frac{1}{2} (a(i) + a(i + 1)) < S_{i+1}(0) \) for \( i = 1, 2 \); \( \beta_4 \) will form a CVT if
\[
S_1(1) < \frac{1}{2} (a(1) + a(2)) < S_2(0) < S_1(1) < \frac{1}{2} (a(2) + a(31)) < S_{31}(0) < S_{31}(1)
\]
\[
< \frac{1}{2} (a(31) + a(32, 33)) < S_{32}(0).
\]
Similarly, we can obtain the inequalities for which \( \beta_5, \beta_6, \) and \( \beta_7 \) will form a CVT. Due to similarity, combining all the inequalities, we see that they will be true if the following inequalities
are true:

\[ S_1(1) \leq \frac{1}{2} (a(1) + a(2, 3)) \leq S_2(0), \]
\[ S_1(1) \leq \frac{1}{2} (a(1) + a(21)) \leq S_{21}(0), \]
\[ S_{13}(1) \leq \frac{1}{2} (a(12, 13) + a(21)) \leq S_{21}(0), \]
\[ S_{13}(1) \leq \frac{1}{2} (a(13) + a(21)) \leq S_{21}(0). \]

Upon some simplification, we see that the above inequalities are true if \( 0 < r \leq 2 - \sqrt{3} \), i.e., if \( 0 < r \leq 0.2679491924 \). If \( 3^\ell(n) \leq n \leq 2 \cdot 3^\ell(n) \), then

\[
V(P; \beta_n(I)) = \sum_{\sigma \in \{1, 2, 3\}^\ell(n) \setminus I} \int_{J_\sigma} (x - a(\sigma))^2 dP + \sum_{\sigma \in I} \int_{J_\sigma} \min_{a \in S_\sigma(\beta_2)} (x - a)^2 dP
\]
\[
= \frac{1}{3^\ell(n)} r^{2\ell(n)} \left( \sum_{\sigma \in \{1, 2, 3\}^\ell(n) \setminus I} V + \sum_{\sigma \in I} V(P; \beta_2) \right)
\]
\[
= \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (2 \cdot 3^\ell(n) - n)V + (n - 3^\ell(n))V(P; \beta_2) \right).
\]

Similarly, if \( 2 \cdot 3^\ell(n) \leq n < 3^\ell(n)+1 \), then

\[
V(P; \beta_n(I)) = \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (3^\ell(n)+1 - n)V(P; \beta_2) + (n - 2 \cdot 3^\ell(n))V(P; \beta_3) \right).
\]

Thus, the proof of the proposition is complete. \( \square \)

**Proposition 2.5.** Let \( \gamma_n(I) \) be the set given by Definition 1.4. Then, \( \gamma_n(I) \) forms a CVT if \( \frac{1}{40} (21 - 2\sqrt{31}) \leq r \leq \frac{1}{11} (2\sqrt{31} - 1) \), i.e., if \( 0.08502712839 \leq r \leq 0.2472080177 \). Moreover, if \( 3^\ell(n) \leq n \leq 2 \cdot 3^\ell(n) \), then

\[
V(P; \gamma_n(I)) = \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (2 \cdot 3^\ell(n) - n)V + (n - 3^\ell(n))V(P; \gamma_2) \right),
\]

and if \( 2 \cdot 3^\ell(n) \leq n < 3^\ell(n)+1 \), then

\[
V(P; \gamma_n(I)) = \frac{1}{3^\ell(n)} \cdot r^{2\ell(n)} \left( (3^\ell(n)+1 - n)V(P; \gamma_2) + (n - 2 \cdot 3^\ell(n))V(P; \gamma_3) \right).
\]

**Proof.** By the definition, we have \( \gamma_2 = \{a(1, 21), a(22, 2, 3)\} \) and \( \gamma_3 = \{a(1), a(2), a(3)\} \). For \( n \geq 3 \), if \( n \neq 3^\ell(n) \) or \( n \neq 2 \cdot 3^\ell(n) \), the subset \( I \) can be chosen more than one way. This leads to the fact that if \( n \neq 3^\ell(n) \) or \( n \neq 2 \cdot 3^\ell(n) \), the sets \( \gamma_n \) can be chosen multiple ways. Proceeding in the similar way, as Proposition 2.4, let us choose

\[
\gamma_4 = \{a(1), a(2), a(31, 321), a(322, 323, 33)\}
\]
\[
\gamma_5 = \{a(1), a(21, 221), a(222, 223, 23), a(31, 321), a(322, 323, 33)\}
\]
\[
\gamma_6 = \{a(11, 121), a(122, 123, 13), a(21, 221), a(222, 223, 23), a(31, 321), a(322, 323, 33)\}
\]
\[
\gamma_7 = \{a(11), a(12), a(13), a(21, 221), a(222, 223, 23), a(31, 321), a(322, 323, 33)\}.
\]
Due to the same reasoning as described in the proof of Proposition 2.4 to show $\gamma_n(I)$ forms a CVT, it is enough to prove that the following inequalities are true:

\[
S_{21}(1) \leq \frac{1}{2}((a(1, 21) + a(22, 23, 3)) \leq S_{22}(0),
\]

\[
S_1(1) \leq \frac{1}{2}(a(1) + a(21, 221)) \leq S_2(0),
\]

\[
S_{13}(1) \leq \frac{1}{2}(a(122, 123, 13) + a(21, 221)) \leq S_{21}(0),
\]

\[
S_{13}(1) \leq \frac{1}{2}(a(13) + a(21, 221)) \leq S_{21}(0).
\]

Upon some simplification, we see that the above inequalities are true if

\[
\frac{1}{79}(21 - 2\sqrt{51}) \leq r \leq \frac{1}{47}(2\sqrt{31} - 1),
\] i.e., if $0.08502712839 \leq r \leq 0.2472080177$. The rest of the proof follows in the similar way as it is given for $V(P; \beta_n)$ in Proposition 2.4. Thus, the proof of the proposition is complete. 

**Definition 2.6.** For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\delta_2 := \{a(1, 21, 221), a(222, 223, 23, 3)\}$ and $\delta_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\delta_n := \delta_n(I)$ as follows:

\[
\delta_n(I) = \begin{cases} 
\{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_{\omega}(\delta_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\
\{S_{\omega}(\delta_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_{\omega}(\delta_3) & \text{if } 2 \cdot 3^{\ell(n)} < n \leq 3^{\ell(n)+1},
\end{cases}
\]

where $I \subset \{1, 2, 3\}^{\ell(n)}$ with $\text{card}(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$; and $\text{card}(I) = n - 2 \cdot 3^{\ell(n)}$ if $2 \cdot 3^{\ell(n)} < n \leq 3^{\ell(n)+1}$.

**Proposition 2.7.** Let $\delta_n(I)$ be the set given by Definition 2.6. Then, $\delta_n(I)$ forms a CVT if $0.1845020699 \leq r \leq 0.2705731187$. Moreover, if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$, then

\[
V(P, \delta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)} \left((2 \cdot 3^{\ell(n)} - n)V + (n - 3^{\ell(n)})V(P; \delta_2)\right),
\]

and if $2 \cdot 3^{\ell(n)} \leq n < 3^{\ell(n)+1}$, then

\[
V(P, \delta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)} \left((3^{\ell(n)+1} - n)V(P; \delta_2) + (n - 2 \cdot 3^{\ell(n)})V(P; \delta_3)\right).
\]

**Proof.** By the definition, we have $\delta_2 = \{a(1, 21, 221), a(222, 223, 23, 3)\}$ and $\delta_3 = \{a(1), a(2), a(3)\}$. For $n \geq 3$, if $n \neq 3^{\ell(n)}$ or $n \neq 2 \cdot 3^{\ell(n)}$, the subset $I$ can be chosen more than one way. This leads to the fact that if $n \neq 3^{\ell(n)}$ or $n \neq 2 \cdot 3^{\ell(n)}$, the sets $\delta_n$ can be chosen multiple ways. Proceeding in the similar way, as Proposition 2.4 let us choose

\[
\delta_4 = \{a(1), a(2), a(31, 321, 3221), a(3222, 3223, 323, 33)\}
\]

\[
\delta_5 = \{a(1), a(21, 221, 2221), a(2222, 2223, 223, 23), a(31, 321, 3221), a(3222, 3223, 323, 33)\}
\]

\[
\delta_6 = \{a(11, 121, 1221), a(1222, 1223, 123, 13), a(21, 221, 2221), a(2222, 2223, 223, 23), a(31, 321, 3221), a(3222, 3223, 323, 33)\}
\]

\[
\delta_7 = \{a(11), a(12), a(13), a(21, 221, 2221), a(2222, 2223, 223, 23), a(31, 321, 3221), a(3222, 3223, 323, 33)\}.
\]
Due to the same reasoning as described in the proof of Proposition 2.4 to show $\delta_n(I)$ forms a CVT, it is enough to prove that the following inequalities are true:

$$S_{221}(1) \leq \frac{1}{2}(a(1, 21, 221) + a(222, 223, 23, 3)) \leq S_{222}(0),$$

$$S_1(1) \leq \frac{1}{2}(a(1) + a(21, 221, 2221)) \leq S_{21}(0),$$

$$S_{13}(1) \leq \frac{1}{2}(a(1222, 1223, 123, 13) + a(21, 221, 2221)) \leq S_{21}(0),$$

$$S_{13}(1) \leq \frac{1}{2}(a(13) + a(21, 221, 2221)) \leq S_{21}(0).$$

The above inequalities are true if $0 < r \leq 0.2705731187$. The rest of the proof follows in the similar way as it is given for $V(P; \beta_n(I))$ in Proposition 2.4. Thus, the proof of the proposition is complete.

The following proposition is useful to establish Lemma 3.1 and Lemma 4.1.

**Proposition 2.8.** Let $\kappa := \{a_1, a_2\}$, where $a_1 := E(X : X \in [0, \frac{1}{2}])$, and $a_2 := E(X : X \in [\frac{1}{2}, 1])$. Then, $a_1 = \frac{+1}{6 - 2r}$, and $a_2 = \frac{5 - 3r}{6 - 2r}$, and the corresponding distortion error is given by

$$V(P; \kappa) = \frac{-7r^3 + 13r^2 - 9r + 3}{6(r - 3)^2(r + 1)}.$$

**Proof.** By the hypothesis, we have

$$a_1 = E(X : X \in [0, \frac{1}{2}]) = E\left(X : X \in J_1 \cup J_{21} \cup J_{221} \cup \cdots\right),$$

$$a_2 = E(X : X \in [\frac{1}{2}, 1]) = E\left(X : X \in J_3 \cup J_{23} \cup J_{223} \cup \cdots\right),$$

yielding

$$a_1 = 2 \sum_{n=1}^{\infty} \frac{1}{3^n} \frac{1}{2}(-r^{n-1} + r^n + 1) = \frac{r + 1}{6 - 2r},$$

$$a_2 = 2 \sum_{n=1}^{\infty} \frac{1}{3^n} \frac{1}{2}(r^{n-1} - r^n + 1) = \frac{5 - 3r}{6 - 2r},$$

and the corresponding distortion error is given by

$$V(P; \kappa) = 2 \int_{J_1 \cup J_{21} \cup J_{221} \cup J_{2221} \cdots} \left(x - \frac{r + 1}{6 - 2r}\right)^2 dP$$

implying

$$V(P; \kappa) = 2\left(\sum_{n=1}^{\infty} \frac{r^{2n}}{3^n} V + \sum_{n=1}^{\infty} \frac{1}{3^n} \left(\frac{1}{2}(-r^{n-1} + r^n + 1) - \frac{r + 1}{6 - 2r}\right)^2\right) = \frac{-7r^3 + 13r^2 - 9r + 3}{6(r - 3)^2(r + 1)}.$$

Thus, the proposition is proved.

**3. Optimal sets of n-means and the n-th quantization errors for $r = \frac{1}{25}$**

Let $\beta_n$ be the set given by Definition 1.3. In this section, we show that for all $n \geq 2$, the sets $\beta_n$ form the optimal sets of $n$-means for $r = \frac{1}{25}$. To calculate the distortion errors we will frequently use the formula given by Corollary 2.3. Notice that by Lemma 2.2 in this case, we have $E(X) = \frac{1}{2}$ and $V := V(X) = \frac{1 - r}{6(r + 1)} = \frac{2}{13}$.

**Lemma 3.1.** The set $\beta := \{a(1), a(2, 3)\}$ forms the optimal set of two-means, and the corresponding quantization error is given by $V_2 = \frac{314}{8125} = 0.0386462$. 

Proof. Let \( \beta := \{a_1, a_2\} \) be an optimal set of two-means. Since the points in an optimal set are the conditional expectations in their own Voronoi regions, without any loss of generality, we can assume that \( 0 < a_1 < a_2 < 1 \). Let us consider the set \( \kappa := \{a(1), a(2, 3)\} \). The distortion error due to the set \( \kappa \) is given by

\[
V(P; \kappa) = \int_{J_1} (x - a(1))^2 dP + \int_{J_2 \cup J_3} (x - a(2, 3))^2 dP = 0.0386462.
\]

Since \( V_2 \) is the quantization for two-means, we have \( V_2 \leq 0.0386462 \). Assume that \( 0.38 < a_1 \).

Then,

\[
V_2 \geq \int_{J_1} (x - 0.38)^2 dP = 0.0432821 > V_2,
\]

which is a contradiction. Hence, \( a_1 \leq 0.38 \). Similarly, \( 0.62 \leq a_2 \). Since \( \frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}(0.38 + 1) = 0.69 < S_3(0) = 0.96 \), the Voronoi region of \( a_1 \) does not contain any point from \( J_3 \). Similarly, the Voronoi region of \( a_2 \) does not contain any point from \( J_1 \). Since the union of the Voronoi regions of \( a_1 \) and \( a_2 \) covers \( J_1 \cup J_2 \cup J_3 \), without any loss of generality, we can assume that the Voronoi region of \( a_2 \) contains points from \( J_2 \), and \( \frac{1}{2}(a_1 + a_2) = \frac{1}{2} \). If \( \frac{1}{2}(a_1 + a_2) = \frac{1}{2} \), then substituting \( r = \frac{1}{25} \), by Proposition 2.8, we have

\[
V_2 = \frac{866}{17797} = 0.0486599 > V_2,
\]

which leads to a contradiction. Hence, we can conclude that \( \frac{1}{2}(a_1 + a_2) < \frac{1}{2} \). Using the similar technique as it is given in the proof of Lemma 3.1 in [15], we can show that \( S_1(1) \leq \frac{1}{2}(a_1 + a_2) \leq S_2(0) \) yielding the fact that \( a_1 = a(1), a_2 = a(2, 3) \), and \( V_2 = \frac{314}{8125} = 0.0386462 \). Hence, the proof of the lemma is complete.

Lemma 3.2. The set \( \beta := \{a(1), a(2), a(3)\} \) forms an optimal set of three-means, and the corresponding quantization error is given by \( V_3 = \frac{2}{8125} = 0.000246154 \).

**Proof.** Consider the set of three points \( \kappa := \{a(1), a(2), a(3)\} \). The distortion error due to the set \( \kappa \) is given by

\[
V(P; \kappa) = \sum_{j=1}^{3} \int_{J_j} (x - a(j))^2 dP = \frac{2}{8125} = 0.000246154.
\]

Since \( V_3 \) is the quantization error for three-means, we have \( V_3 \leq 0.000246154 \). Let \( \beta := \{a_1, a_2, a_3\} \), where \( 0 < a_1 < a_2 < a_3 < 1 \), be an optimal set of three-means. If \( S_1(1) = \frac{1}{25} < \frac{1}{23} < a_1 \), then

\[
V_3 \geq \int_{J_1} (x - \frac{1}{23})^2 dP = \frac{13709}{51577500} = 0.000265794 > V_3,
\]

which gives a contradiction. Thus, we can assume that \( a_1 \leq \frac{1}{23} \). Similarly, \( \frac{22}{23} \leq a_3 \). Suppose that \( \beta \cap J_1 = \emptyset \). Then, due to symmetry, we can assume that \( \beta \cap J_3 = \emptyset \), and then

\[
V_3 \geq 2 \int_{J_1} (x - a_1)^2 dP = 2 \int_{J_1} (x - S_1(1))^2 dP = \frac{7}{16250} = 0.000430769 > V_3,
\]

which leads to a contradiction. So, we can assume that \( \beta \cap J_1 \neq \emptyset \), i.e., \( a_1 < S_1(1) \). Similarly, \( \beta \cap J_3 \neq \emptyset \), i.e., \( S_3(0) < a_3 \). Now, we show that \( \beta \cap J_2 \neq \emptyset \). Suppose that \( \beta \cap J_2 = \emptyset \). Then, either \( a_2 < \frac{12}{25} = S_2(0) \), or \( \frac{12}{25} = S_2(1) < a_2 \). First, assume that \( a_2 < S_2(0) \). Then, notice that \( S_2(1) = \frac{13}{25} < \frac{1}{2}(S_2(0) + S_3(0)) < S_3(0) \) yielding the fact that the Voronoi region of \( S_2(0) \) contains \( J_2 \). Hence,

\[
V_3 \geq \int_{J_2} (x - S_2(0))^2 dP + \int_{J_3} (x - a(3))^2 dP = \frac{29}{97500} = 0.000297436 > V_3,
\]
which is a contradiction. Similarly, we can show that a contradiction arises if \( \frac{13}{25} = S_2(1) < a_2 \). Thus, we can assume that \( \beta \cap J_2 \neq \emptyset \). Now, if the Voronoi region of \( a_1 \) contains points from \( J_2 \), we have \( \frac{1}{2}(a_1 + a_2) > \frac{12}{25} = S_2(0) \) implying \( a_2 > \frac{24}{25} - a_1 \geq \frac{24}{25} - \frac{1}{25} = \frac{23}{25} > S_2(1) \), which is a contradiction as \( \beta \cap J_2 \neq \emptyset \). Hence, we can assume that the Voronoi region of \( a_1 \) does not contain any point from \( J_2 \), and so from \( J_3 \). Similarly, we can show that the Voronoi region of \( a_2 \) does not contain any point from \( J_1 \) and \( J_3 \), and the Voronoi region of \( a_3 \) does not contain any point from \( J_2 \), and so from \( J_1 \). Thus, by Proposition 1.2 we conclude that \( a_1 = a(1), a_2 = a(2), \) and \( a_3 = a(3) \), and the corresponding quantization error is given by \( V_3 = \frac{2}{2^{5125}} = 0.000246154 \), which is the lemma.

**Proposition 3.3.** Let \( \beta_n \) be an optimal set of \( n \)-means for any \( n \geq 3 \). Then, \( \beta_n \cap J_i \neq \emptyset \) for all \( 1 \leq j \leq 3 \), and \( \beta_n \) does not contain any point from the open intervals \( (S_1(1), S_2(0)) \) and \( (S_2(1), S_3(0)) \). Moreover, the Voronoi region of any point in \( \beta_n \cap J_i \) does not contain any point from \( J_i \), where \( 1 \leq i \neq j \leq 3 \).

**Proof.** By Lemma 3.2, the proposition is true for \( n = 3 \). Let us prove the lemma for \( n \geq 4 \). Let \( \beta_n := \{a_1, a_2, \ldots, a_n\} \) be an optimal set of \( n \)-means for \( n \geq 4 \). Since the points in an optimal set are the conditional expectations in their own Voronoi regions, without any loss of generality, we can assume that \( 0 < a_1 < a_2 < \cdots < a_n < 1 \). Consider the set of four elements \( \kappa := S_1(\beta_2) \cup \{a(2), a(3)\} \). Then,

\[
V(P; \kappa) = \int_{J_1} \min_{a \in S_1(\beta_2)} (x-a)^2 dP + \int_{J_2} (x-a(2))^2 dP + \int_{J_3} (x-a(3))^2 dP = \frac{938}{5078125} = 0.000184714.
\]

Since \( V_n \) is the quantization error for \( n \)-means for \( n \geq 4 \), we have \( V_n \leq V_4 \leq 0.000184714 \).

Suppose that \( S_1(1) \leq a_1 \). Then,

\[
V_n \geq \int_{J_1} (x - S_1(1))^2 dP = \frac{7}{32500} = 0.000215385 > V_n,
\]

which is a contradiction. So, we can assume that \( a_1 < S_1(1) \), i.e., \( \beta_n \cap J_1 \neq \emptyset \). Similarly, \( \beta_n \cap J_2 \neq \emptyset \). We now show that \( \beta_n \cap J_2 \neq \emptyset \). For the sake of contradiction, assume that \( \beta_n \cap J_2 = \emptyset \). Let \( a_j := \max \{a_i : a_i < S_2(0) \text{ for } 1 \leq i \leq n-1\} \). Then, \( a_j < S_2(0) \). As \( \beta_n \cap J_2 = \emptyset \), we have \( S_2(1) < a_{j+1} \). If \( a_j < \frac{1}{2}(S_1(1) + S_2(0)) = \frac{13}{50} \), then as \( \frac{1}{2}(a_j + a_{j+1}) < \frac{1}{2}(\frac{13}{50} + S_2(1)) = \frac{39}{100} < \frac{12}{25} = S_2(0) \), we have

\[
V_n \geq \int_{J_2} (x - S_2(0))^2 dP = \frac{7}{32500} = 0.000215385 > V_n,
\]

which leads to a contradiction. So, we can assume that \( \frac{13}{50} \leq a_j < S_2(0) \). Then, by Proposition 1.2 we have \( \frac{1}{2}(a_{j-1} + a_j) < \frac{12}{25} \) implying \( a_{j-1} < \frac{24}{25} - a_j \leq \frac{24}{25} - \frac{13}{50} = -\frac{9}{50} < 0 \), which gives a contradiction as \( \beta_n \cap J_1 \neq \emptyset \). Hence, we can conclude that \( \beta_n \cap J_2 \neq \emptyset \). Notice that \( (S_1(1), S_2(0)) = (\frac{1}{25}, \frac{12}{25}) \). Suppose that \( \beta_n \) contains a point from the open interval \( (\frac{1}{25}, \frac{12}{25}) \). Let \( a_j := \max \{a_i : a_i < \frac{1}{25} \text{ for } 1 \leq i \leq n-2\} \). Then, due to Proposition 1.2, \( a_{j+1} \in (\frac{1}{25}, \frac{12}{25}) \), and \( a_{j+2} \in J_2 \). The following cases can arise:

**Case 1.** \( \frac{1}{25} < a_{j+1} \leq \frac{13}{50} \).

Then, \( \frac{1}{2}(a_{j+1} + a_{j+2}) > \frac{12}{25} \) implying \( a_{j+2} > \frac{24}{25} - a_{j+1} \geq \frac{24}{25} - \frac{13}{50} = \frac{35}{50} > S_2(1) \), which leads to a contradiction because \( a_{j+2} \in J_2 \).

**Case 2.** \( \frac{13}{50} \leq a_{j+1} \leq \frac{12}{25} \).

Then, \( \frac{1}{2}(a_j + a_{j+1}) < \frac{12}{25} \) implying \( a_j \leq \frac{2}{25} - a_{j+1} \leq \frac{2}{25} - \frac{13}{50} = -\frac{9}{50} \), which is a contradiction because \( a_j > 0 \).

Thus, by Case 1 and Case 2, we can conclude that \( \beta_n \) does not contain any point from the open interval \( (S_1(1), S_2(0)) \). Reflecting the situation with respect to the point \( \frac{1}{2} \), we can conclude that \( \beta_n \) does not contain any point from the open interval \( (S_2(1), S_3(0)) \) as well. To prove the last part of the proposition, we proceed as follows: Let \( a_j := \max \{a_i : a_i < \frac{1}{25} \text{ for } 1 \leq i \leq n-2\} \). Then, \( a_j \) is the rightmost element in \( \beta_n \cap J_1 \), and \( a_{j+1} \in \beta_n \cap J_2 \). Suppose that the Voronoi region of \( a_j \)
contains points from $J_2$. Then, $\frac{1}{2}(a_j + a_{j+1}) > \frac{12}{25}$ implying $a_{j+1} > \frac{24}{25} - a_j \geq \frac{24}{25} - \frac{1}{25} = \frac{23}{25} > S_2(1)$, which yields a contradiction as $a_{j+1} \in J_2$. Thus, the Voronoi region of any point in $\beta_n \cap J_1$ does not contain any point from $J_2$, and $J_3$ as well. Similarly, we can prove that the Voronoi region of any point in $\beta_n \cap J_2$ does not contain any point from $J_1$ and $J_3$, and the Voronoi region of any point in $\beta_n \cap J_3$ does not contain any point from $J_1$ and $J_2$. Thus, the proof of the proposition is complete.

The following lemma is a modified version of Lemma 4.5 in [5], and the proof follows similarly. One can also see Lemma 3.5 in [15].

**Lemma 3.4.** Let $n \geq 3$, and let $\beta_n$ be an optimal set of $n$-means such that $\beta_n \cap J_j \neq 0$ for all $1 \leq j \leq 3$, and $\beta_n$ does not contain any point from the open intervals $(S_1(1), S_2(0))$ and $(S_2(1), S_3(0))$. Further assume that the Voronoi region of any point in $\beta_n \cap J_j$ does not contain any point from $J_i$, where $1 \leq i \neq j \leq 3$. Set $\kappa_j := \beta_n \cap J_j$, and $n_j := \text{card} (\kappa_j)$ for $1 \leq j \leq 3$. Then, $S_j^{-1}(\kappa_j)$ is an optimal set of $n_j$-means, and $V_n = \frac{1}{1875} (V_{n_1} + V_{n_2} + V_{n_3})$.

Let us now state and prove the following theorem which gives the optimal sets of $n$-means.

**Theorem 3.5.** Let $P$ be the probability measure on $\mathbb{R}$ with support the Cantor set $C$ generated by the three contractive similarity mappings $S_j$ for $j = 1, 2, 3$. Let $n \in \mathbb{N}$ with $n \geq 3$. Take $r = \frac{1}{25}$. Then, the sets $\beta_n := \beta_n(I)$ given by Definition 1.3 form the optimal sets of $n$-means for $P$ with the corresponding quantization error $V_n := V(P; \beta_n(I))$, where $V(P; \beta_n(I))$ is given by Proposition 2.4.

**Proof.** We will proceed by induction on $\ell(n)$. If $n = 3$, then by Lemma 3.2 the theorem is true. Now, we show that the theorem is true if $n = 4$. Let $\kappa_j := \beta_n \cap J_j$, and $n_j := \text{card} (\kappa_j)$ for $1 \leq j \leq 3$. Since $S_j^{-1}(\kappa_j)$ is an optimal set of $n_j$-means for $1 \leq j \leq 3$, and for $n = 4$ the possible choices for the triplet $(n_1, n_2, n_3)$ are $(2, 1, 1)$, $(1, 1, 2)$, and $(1, 1, 2)$, by Proposition 2.4 and Lemma 3.4, the set $\beta_4$ forms an optimal set of four-means with quantization error $V(P; \beta_4)$ given by Proposition 2.4. Remember that for a given $n$, among all the possible choices of the triplets $(n_1, n_2, n_3)$, the triplets $(n_1, n_2, n_3)$ which give the smallest distortion error will give the optimal sets of $n$-means. Notice that for $n = 5$, the possible choices of the triplets are $(3, 1, 1)$, $(1, 1, 3)$, $(1, 1, 2)$, $(2, 1, 2)$, $(2, 1, 2)$, $(2, 1, 2)$, $(2, 1, 2)$, and $(2, 1, 2)$, which gives the smallest distortion error. Hence, the optimal sets of five-means are $\{a(1)\} \cup S_2(\beta_2) \cup S_3(\beta_2)$, $S_1(\beta_2) \cup \{a(2)\} \cup S_3(\beta_2)$, and $S_1(\beta_2) \cup S_2(\beta_2) \cup \{a(3)\}$ which are the sets $\beta_5$ given by Definition 1.3. Similarly, we can calculate the optimal sets of six- and seven-means. Thus, the theorem is true for $\ell(n) = 1$. Let us assume that the theorem is true for all $\ell(n) < m$, where $m \in \mathbb{N}$ and $m \geq 2$. We now show that the theorem is true if $\ell(n) = m$. Let us first assume that $3^m \leq n \leq 2 \cdot 3^m$. Let $n_j$ be an optimal set of $n$-means for $P$ such that $3^m \leq n \leq 2 \cdot 3^m$. Let $\text{card} (\beta_n \cap J_j) = n_j$ for $j = 1, 2, 3$, and then by Lemma 3.4, we have

$$V_n = \frac{1}{1875} (V_{n_1} + V_{n_2} + V_{n_3}).$$

Without any loss of generality, we can assume that $n_1 \geq n_2 \geq n_3$. Let $u, v, w \in \mathbb{N}$ be such that

$$3^w \leq n_1 \leq 2 \cdot 3^w, \quad 3^v \leq n_2 \leq 2 \cdot 3^v, \quad \text{and} \quad 3^w \leq n_3 \leq 2 \cdot 3^w.
$$

Proceeding in the similar lines as the proof of Theorem 3.6 in [15], we can show that $u = v = w = m - 1$. Since by Lemma 3.4 for $S_j^{-1}(\beta_n \cap J_j)$ is an optimal set of $n_j$ means where $3^{m-1} \leq n_j \leq 2 \cdot 3^{m-1}$, we have

$$S_j^{-1}(\beta_n \cap J_j) = \{a(\omega) : \omega \in \{1, 2, 3\}^{m-1} \setminus I_j\} \cup (\cup_{\omega \in I_j} S_\omega(\beta_2)),$$

where $I_j \subseteq \{1, 2, 3\}^{m-1}$ with card $(I_j) = n_j - 3^{m-1}$ for $1 \leq j \leq 3$. Hence,

$$\beta_n := \beta_n(I) = \bigcup_{j=1}^{3} S_j^{-1}(\beta_n \cap J_j) = \{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup (\cup_{\omega \in I} S_\omega(\beta_2)).$$
where \( I \subseteq \{1, 2, 3\}^m \) with \( \text{card}(I) = n - 3^m \), is an optimal set of \( n \)-means. The corresponding quantization error is

\[
V_n = \frac{1}{3^m} r^{2m} ((2 \cdot 3^m - n)V + (n - 3^m)V_2) = V(P; \beta_n(I)),
\]

where \( V(P; \beta_n(I)) \) is given by Proposition 2.3. Thus, the theorem is true if \( 3^m \leq n \leq 2 \cdot 3^m \). Similarly, we can prove that the theorem is true if \( 2 \cdot 3^m < n < 3^{m+1} \). Hence, by the induction principle, the proof of the theorem is complete. \( \square \)

4. Optimal sets of \( n \)-means and the \( n \)th quantization errors for \( r = r_0 \) and \( r = r_1 \)

In this section, we give the proof of Theorem 1.7. First, we prove the following two lemmas.

**Lemma 4.1.** Let \( r_0 \) and \( r_1 \) be the real numbers given by Theorem 1.7. Then, the set \( \gamma := \{a(1,21), a(22,23,3)\} \) for \( r = r_0 \) and \( r = r_1 \) form the optimal sets of two-means, and the corresponding quantization errors are, respectively, given by \( V_2 = 0.0324042 \), and \( V_2 = 0.026897 \).

**Proof.** First, we prove that \( \gamma \) forms an optimal set of two-means for \( r = r_0 \). Let \( \gamma := \{a_1, a_2\} \) be an optimal set of two-means. Since, the points in an optimal set are the expected values of their own Voronoi regions, without any loss of generality, we can assume that \( 0 < a_1 < a_2 < 1 \). Let us consider the set \( \kappa := \{a(1,21), a(22,23,3)\} \). The distortion error due to the set \( \kappa \) is given by

\[
V(P; \kappa) = \int_{J_1} (x - a(1,21))^2dP + \int_{J_2 \cup J_3} (x - a(22,23,3))^2dP = 0.0324042.
\]

Since \( V_2 \) is the quantization error for two-means, we have \( V_2 \leq 0.0324042 \). Assume that \( 0.39 < a_1 \). Then,

\[
V_2 \geq \int_{J_1} (x - 0.39)^2dP = 0.0328529 > V_2,
\]

which is a contradiction. Hence, \( a_1 \leq 0.39 \). Similarly, \( 0.61 \leq a_2 \). Since \( \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(0.39+1) = 0.695 < S_3(0) = 0.837722 \), the Voronoi region of \( a_1 \) does not contain any point from \( J_3 \). Similarly, the Voronoi region of \( a_2 \) does not contain any point from \( J_1 \). Since the union of the Voronoi regions of \( a_1 \) and \( a_2 \) covers \( J_1 \cup J_2 \cup J_3 \), without any loss of generality, we can assume that the Voronoi region of \( a_2 \) contains points from \( J_2 \), and \( \frac{1}{2}(a_1 + a_2) \leq \frac{1}{2} \). If \( \frac{1}{2}(a_1 + a_2) = \frac{1}{2} \), then substituting \( r = 0.1622776602 \), by Proposition 2.3, we have

\[
V(P; \kappa) = 0.0329779,
\]

which contradicts \( V_2 \). Hence, we can conclude that \( \frac{1}{2}(a_1 + a_2) < \frac{1}{2} \). Using the similar technique as it is given in the proof of Lemma 3.1 in [15], we can show that either \( \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a(1,21) + a(22,23,3)) = 0.466886 \), or \( \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a(1) + a(2,3)) = 0.395285 \), i.e., either \( S_{21}(1) < \frac{1}{2}(a_1 + a_2) < S_{22}(0) \), or \( S_{1}(1) < \frac{1}{2}(a_1 + a_2) < S_{2}(0) \). Notice that if \( S_{21}(1) < \frac{1}{2}(a_1 + a_2) < S_{22}(0) \), then \( \gamma_2 \), given by Definition 1.4, forms the optimal set of two-means. On the other hand, if \( S_{1}(1) < \frac{1}{2}(a_1 + a_2) < S_{2}(0) \), then \( \beta_2 \), given by Definition 1.3, forms the optimal set of two-means. In fact, later we will see that \( V(P; \gamma_2) = V(P; \beta_2) = 0.0324042 \) for \( r = 0.1622776602 \). Thus, \( \gamma_2 \) forms the optimal set of two-means for \( r = r_0 \) with quantization error \( V_2 = 0.0324042 \). Similarly, we can show that \( \gamma_2 \) forms the optimal set of two-means if \( r = r_1 \) with quantization error \( V_2 = 0.026897 \). Hence, the lemma is yielded. \( \square \)

The following lemma is true analogously as Lemma 3.3 in [15].

**Lemma 4.2.** The set \( \gamma_3 := \{a(1), a(2), a(3)\} \) for \( r = r_0 \), and \( r = r_1 \) form the optimal sets of three-means, and the corresponding quantization errors are, respectively, given by \( V_3 = 0.00316342 \), and \( V_3 = 0.00558347 \).

The following proposition is true analogously as Proposition 3.5 in [15].
Proposition 4.3. Let $n \geq 3$, and let $\gamma_n$ be an optimal set of $n$-means for $r = r_0$, and $r = r_1$. Then, $\gamma_n \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$, and $\gamma_n$ does not contain any point from the open intervals $(S_1(1), S_2(0))$ and $(S_2(1), S_3(0))$. Moreover, the Voronoi region of any point in $\gamma_n \cap J_j$ does not contain any point from $J_i$, where $1 \leq i \neq j \leq 3$.

The following remark is true due to Proposition 4.3.

Remark 4.4. Let $n \geq 3$, and let $\gamma_n$ be an optimal set of $n$-means for $r = r_0$, and $r = r_1$. Set $\gamma_j := \gamma_n \cap J_j$, and $n_j := \text{card} (\gamma_j)$ for $1 \leq j \leq 3$. Then, $S_j^{-1}(\gamma_j)$ is an optimal set of $n_j$-means, and for $r = r_0$ and $r = r_1$, respectively, we have $V_n = \frac{1}{3} r_0^n (V_{n_1} + V_{n_2} + V_{n_3})$ and $V_n = \frac{1}{3} r_1^n (V_{n_1} + V_{n_2} + V_{n_3})$.

Proof of Theorem 1.7. We proceed to prove it by induction on $\ell(n)$. By Lemma 4.2, we see that the theorem is true for $n = 3$. Proceeding in the similar way, as mentioned in the proof of Theorem 3.5, we can show that for $n = 4, 5, 6, 7$, the sets $\gamma_n$ form the optimal sets of $n$-means for $r = r_0$ and $r = r_1$. Thus, the theorem is true if $\ell(n) = 1$. Let us assume that the theorem is true for all $\ell(n) < m$, where $m \in \mathbb{N}$ and $m \geq 2$. We now show that the theorem is true if $\ell(n) = m$. Let us first assume that $3^m \leq n \leq 2 \cdot 3^m$. Let $\gamma_n$ be an optimal set of $n$-means for $P$ such that $3^m \leq n \leq 2 \cdot 3^m$. Let card $(\gamma_n \cap J_j) = n_j$ for $j = 1, 2, 3$, and then by Remark 4.4 we have

$$V_n = \frac{1}{3} r_0^n (V_{n_1} + V_{n_2} + V_{n_3}) \text{ for } r = r_0, \text{ and } V_n = \frac{1}{3} r_1^n (V_{n_1} + V_{n_2} + V_{n_3}) \text{ for } r = r_1.$$ 

The rest of the proof for $r = r_0$ and $r = r_1$ follow in the similar way as the proof of Theorem 3.5. Thus, we complete the proof of the theorem.

5. Main results

The two theorems in this section, state and prove the main results of the paper.

Theorem 5.1. Let $r_0, r_1 \in (0, \frac{1}{3})$ be the unique real numbers satisfying, respectively, the equations

$$\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)} = \frac{-3r^3 - 3r^2 + r - 1}{24(r + 1)},$$ 

$$\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)} = \frac{-3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121}{2184(r + 1)}.$$ 

Then, $r_0 = 0.1622776602$, and $r_1 = 0.2317626315$. Let the sets $\beta_n$ and $\gamma_n$ be, respectively, given by Definition 1.3 and Definition 1.4. Then, $\beta_n$ form the optimal sets of $n$-means for $0 < r \leq r_0$, and $\gamma_n$ forms the optimal sets of $n$-means for $r_0 \leq r \leq r_1$.

Proof. By Proposition 2.4, Proposition 2.5, and Proposition 2.7, we see that both $\beta_n$ and $\gamma_n$ form CVTs if $0.08502712839 \leq r \leq 0.2472080177$; both $\gamma_n$ and $\delta_n$ form CVTs if $0.1845020699 \leq r \leq 0.2472080177$; both $\beta_n$ and $\delta_n$ form CVTs if $0.1845020699 \leq r \leq 0.2679491924$. Again, $V(P; \beta_0) = V(P; \gamma_0) = V(P; \delta_0)$. Thus, for any $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$, from the aforementioned propositions, in the case of $V(P; \beta_n(I))$ and $V(P; \gamma_n(I))$, we see that $V(P; \beta_n(I)) > V(P; \gamma_n(I))$,

$$V(P; \beta_n(I)) = V(P; \gamma_n(I)),$$

and $V(P; \beta_n) < V(P; \gamma_n)$ will be true if $V(P; \beta_2) > V(P; \gamma_2)$,

$$V(P; \beta_2) = V(P; \gamma_2),$$

and $V(P; \beta_2) < V(P; \gamma_2)$, respectively. Similarly, it hold in the case of $V(P; \beta_n)$ and $V(P; \delta_n)$, and in the case of $V(P; \gamma_n)$ and $V(P; \delta_n)$. Next, we have

$$V(P; \beta_2) = -\frac{3r^3 - 3r^2 + r - 1}{24(r + 1)},$$ 

$$V(P; \gamma_2) = -\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r + 1)},$$ 

$$V(P; \delta_2) = -\frac{3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121}{2184(r + 1)}.$$
After some calculation, we observe that \( V(P; \beta_2) < V(P; \gamma_2) \) is true if 0.08502712839 ≤ \( r < 0.1622776602 \); \( V(P; \beta_2) = V(P; \gamma_2) \) if \( r = 0.1622776602 \), and \( V(P; \beta_2) > V(P; \gamma_2) \) if \( 0.1622776602 < r \leq 0.2472080177 \). Again, \( V(P; \beta_2) > V(P; \delta_2) \) if 0.1701473031 < \( r \leq 0.2679491924 \) and \( V(P; \beta_2) = V(P; \delta_2) \) if \( r = 0.1701473031 \). Recall that the sets \( \beta_n \) form CVTs if 0 < \( r \leq 0.2679491924 \). Hence, we can say that the sets \( \beta_n \) do not form the optimal sets of \( n \)-means if 0.1622776602 < \( r \leq 0.2679491924 \). In Theorem 1.7, we have seen that the sets \( \beta_n \) form the optimal sets of \( n \)-means if \( r = \frac{1}{3} \). Using the similar technique, we can show that the sets \( \beta_n \) form the optimal sets of \( n \)-means if 0 < \( r \leq \frac{1}{3} \). Since \( V(P; \beta_2) = V(P; \gamma_2) \) if \( r = r_0 \); and by Theorem 1.7, the sets \( \gamma_n \) form the optimal sets of \( n \)-means if \( r = r_0 \), we can say that the sets \( \beta_n \) also form the optimal sets of \( n \)-means if \( r = r_0 \). Again, \( V(P; \beta_2) \) is strictly decreasing in the closed interval [0, \( r_0 \]). Hence, the sets \( \beta_n \) form the optimal sets of \( n \)-means for 0 < \( r \leq r_0 \).

To prove the remaining part of the theorem, we see that

(i) \( V(P; \beta_2) < V(P; \gamma_2) \) if 0.08502712839 ≤ \( r < 0.1622776602 \); \( V(P; \beta_2) = V(P; \gamma_2) \) if \( r = 0.1622776602 \), and \( V(P; \beta_2) > V(P; \gamma_2) \) if \( 0.1622776602 < r \leq 0.2472080177 \).

(ii) \( V(P; \delta_2) < V(P; \gamma_2) \) if 0.2317626315 < \( r \leq 0.2472080177 \); \( V(P; \delta_2) = V(P; \gamma_2) \) if \( r = 0.2317626315 \), and \( V(P; \delta_2) > V(P; \gamma_2) \) if \( 0.1845020699 < r < 0.2317626315 \).

Thus, the sets \( \gamma_n \) do not form the optimal sets of \( n \)-means if 0.08502712839 ≤ \( r < 0.1622776602 \), or if 0.2317626315 < \( r \leq 0.2472080177 \); in other words, the range of \( r \) values for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means is bounded below by \( r_0 = 0.1622776602 \) and bounded above by \( r_1 = 0.2317626315 \). By Theorem 1.7, we see that the sets \( \gamma_n \) form the optimal sets of \( n \)-means if \( r = r_0 \), and \( r = r_1 \). Again, \( V(P; \gamma_2) \) is strictly decreasing in the closed interval [\( r_0, r_1 \)]. Hence, the precise range of \( r \) values for which the sets \( \gamma_n \) form the optimal sets of \( n \)-means is given by \( r_0 \leq r \leq r_1 \). Thus, the proof of the theorem is complete.

Since the Cantor set \( C \) under investigation satisfies the strong separation condition, with each \( S_j \) having contracting factor of \( r \), the Hausdorff dimension of the Cantor set is equal to the similarity dimension. Hence, from the equation \( 3(r)^\beta = 1 \), we have \( \dim_H(C) = \beta = -\frac{\log 3}{\log r} \). By Theorem 14.17 in [4], the quantization dimension \( D(P) \) exists and is equal to \( \beta \). In Theorem 5.2, we show that \( \beta \) dimensional quantization coefficient for \( P \) does not exist.

**Theorem 5.2.** The \( \beta \)-dimensional quantization coefficient for 0 < \( r \leq r_1 \) does not exist.

**Proof.** We have \( 3^{\frac{1}{\beta}} = \frac{1}{r} \). Notice that \( \left\{ (3^\ell(n))^\frac{2}{\beta} V_{3^\ell(n)}(P) \right\} \) and \( \left\{ (2 \cdot 3^\ell(n))^\frac{2}{\beta} V_{2 \cdot 3^\ell(n)}(P) \right\} \) are two different subsequences of the sequence \( \left\{ n^{\frac{2}{\beta}} V_n(P) \right\} \). First, assume that 0 < \( r \leq r_0 \). Then, by Theorem 5.1, \( \beta_n \) is an optimal set of \( n \)-means for 0 < \( r \leq r_0 \). Recall Proposition 2.4. Then, we have

\[
\lim_{n \to \infty} (3^\ell(n))^{\frac{2}{\beta}} V_{3^\ell(n)}(P) = \lim_{n \to \infty} \frac{1}{r^{2\ell(n)}} \frac{1}{3^\ell(n)} r^{2\ell(n)} 3^\ell(n) V = V,
\]

and

\[
\lim_{n \to \infty} (2 \cdot 3^\ell(n))^{\frac{2}{\beta}} V_{2 \cdot 3^\ell(n)}(P) = \lim_{n \to \infty} 2 \frac{2}{\beta} \frac{1}{r^{2\ell(n)}} \frac{1}{3^\ell(n)} r^{2\ell(n)} 3^\ell(n) V(P; \beta_2) = 2^{\frac{2}{\beta}} V(P; \beta_2).
\]

By (6) and (7), we see that \( \left\{ n^{\frac{2}{\beta}} V_n(P) \right\} \) has two different subsequences having two different limits, and so \( \lim_{n \to \infty} n^{\frac{2}{\beta}} V_n(P) \) does not exist. Due to Theorem 5.1 and Proposition 2.5, similarly, we can show that if \( r_0 \leq r \leq r_1 \), then \( \lim_{n \to \infty} n^{\frac{2}{\beta}} V_n(P) \) does not exist. Thus, we show that the \( \beta \)-dimensional quantization coefficient for 0 < \( r \leq r_1 \) does not exist, which completes the proof of the theorem.

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References


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