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Abstract In this paper, we study the Cauchy problem of the following complex cubic Camassa-Holm (ccCH) equation

$$m_t = bu_x + \frac{1}{2}\left[m\left(|u|^2 - |u_x|^2\right)\right]_x - \frac{1}{2}m\left(u\bar{u}_x - u_x\bar{u}\right), \quad m = u - u_{xx},$$

where b is an arbitrary real constant. Long-time asymptotics of the equation is obtained through the $\bar{\partial}$-steepest descent method. Firstly, based on the spectral analysis of the Lax pair and scattering matrix, the solution of the equation is able to be constructed via solving the corresponding Riemann-Hilbert problem (RHP). Then, we present different long time asymptotic expansions of the solution $u(y,t)$ in different space-time solitonic regions of $\xi = y/t$. The half-plane $(y,t) : -\infty < y < \infty, t > 0$ is divided into four asymptotic regions: $\xi \in (-\infty, -1)$, $\xi \in (-1, 0)$, $\xi \in (0, \frac{1}{8})$ and $\xi \in (\frac{1}{8}, +\infty)$. When ξ falls in $(-\infty, -1) \cup (\frac{1}{8}, +\infty)$, no stationary phase point of the phase function $\theta(z)$ exists on the jump profile in the space-time region. In this case, corresponding asymptotic approximations can be characterized with an $N(\Lambda)$-solitons with diverse residual error order $O(t^{-1+2\varepsilon})$. There are four stationary phase points and eight stationary phase points on the jump curve as $\xi \in (-1, 0)$ and $\xi \in (0, \frac{1}{8})$, respectively. The corresponding asymptotic form is accompanied by a residual error order $O(t^{-\frac{3}{4}})$.

Keywords Complex cubic Camassa-Holm (ccCH) equation; Riemann-Hilbert problem; $\bar{\partial}$-steepest descent method; Long-time asymptotics.

Contents

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1. Introduction

The study of solitons and their interactions not only helps clarifying the laws of motion of matter under nonlinear effects but also promotes the development of methods and techniques for solving nonlinear evolution equations. As a result, a large number of methods have been proposed, developed and promoted [1–11, 13]. One of the most significant approaches is the well-known inverse scattering transform (IST) method proposed by Gardner, Green, Kruskal and Miura 1967 [14], and since then the IST has widely been used in the study of integrable systems and soliton solutions with analyzing long-time asymptotic behavior.

In 1973, Manakov first employed the IST method to discuss the long-time asymptotic behavior of the nonlinear wave equations [15]. Thereafter, the IST method has attracted much attention and been developed broadly in nonlinear sciences [12, 16, 17]. Particularly, in 1993 Deift and Zhou proposed the remarkable nonlinear steepest descent method to investigate the long-time asymptotic behavior of integrable equations via the Riemann-Hilbert problem [3]. They successfully cast their nonlinear steepest descent approach to the nonlinear Schrödinger equation [18]. In 2006, McLaughlin and Miller obtained the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights and via $\bar{\partial}$ steepest descent method [19]; and in 2008, they got Plancherel-Rotach type asymptotics valid in all regions of the complex plane for orthogonal polynomials with varying weights of the form $e^{-NV(x)}$ on the real line through $\bar{\partial}$ steepest descent method [20]. In 2009, Krüger and Teschl get the long-time asymptotics of the Toda lattice for decaying initial data in the soliton and in the similarity region by the method of nonlinear steepest descent [21]. In 2009, Grunert and Teschl gained long-time asymptotics for the Korteweg de Vries equation via nonlinear steepest descent [22].

Later in 2016, the Deift-Zhou method is generalized to the so-called $\bar{\partial}$-steepest descent approach [23–29]. Those important models for the study of asymptotic stability of N-soliton solutions by using the $\bar{\partial}$-steepest descent approach include: 1) The defocusing nonlinear schrödinger equation by Cuccagna and Jenkins [23]; 2) The derivative nonlinear schrödinger equation by Jenkins, Liu, Perry and Sulem [24]; 3) The focusing nonlinear schrödinger equation by Borghese, Jenkins, McLaughlin, and Miller [25], and other integrable PDE models by Dieng and McLaughlin [26] 4) The cubic Camassa-Holm equation in space-time solitonic regions by Yang and Fan [27];and 5) The focusing Fokas-Lenells equation in the solitonic region of space-time by Cheng and Fan [28].

In this paper, we consider the following complex cubic Camassa-Holm equation

$$m_t = bu_x + \frac{1}{2}\left[m\left(|u|^2 - |u_x|^2\right)\right]_x - \frac{1}{2}m\left(u\bar{u}_x - u_x\bar{u}\right), \quad m = u - u_{xx}, \tag{1.1}$$

which was proposed by Xia and Qiao 2015 [30]. In [30], the authors mainly studied the peakon interplaying and integrability of the following two-component system

$$
\begin{aligned}
&m_t = bu_x + \tfrac{1}{2}\left[m\left(uv - u_x v_x\right)\right]_x - \tfrac{1}{2}m\left(uv_x - u_x v\right),\\
&n_t = bv_x + \tfrac{1}{2}\left[n\left(uv - u_x v_x\right)\right]_x + \tfrac{1}{2}n\left(uv_x - u_x v\right),\\
&m = u - u_{xx}, \quad n = v - v_{xx},
\end{aligned} \tag{1.2}
$$

where b is an arbitrary real constant. By imposing the complex conjugate reduction on $v = \bar{u}$, equation (1.2) is reduced to (1.1). The above reduction of the two component system (1.2) looks very like the AKNS system, which can include the KdV equation, the MKdV equation, the Gardner equation, and the nonlinear Schrödinger equation as its special cases. Moreover, the complex cubic Camassa-Holm equation (ccCH) (1.1) has the complex valued N-peakon solutions and kink wave solutions [30]. Within the best of our knowledge, equation (1.1) is the first integrable model admitting both complex peakon and kink solutions. When u is a real, equation (1.1) becomes the following cubic CH equation (sometimes called the FQXL model [31])

$$
m_t = bu_x + \frac{1}{2}\left[m\left(u^2 - u_x^2\right)\right]_x, \quad m = u - u_{xx}, \tag{1.3}
$$

which is the notable FORQ/MCH equation $m_t = bu_x + \frac{1}{2}\left[m\left(u^2 - u_x^2\right)\right]_x$ plus a linear dispersion term bu_x. Clearly, equation 1.3 is not equivalent to the FORQ/MCH equation, instead equivalent to the FQXL model [31] via a twisted Galileo transformation.

The long time asymptotic behaviors for different initial value problems of (1.3) have been studied in [27]Yang and Fan considered the initial value problem of (1.3) with the weighted Sobolev initial data $u(x,0) = u_0(x) \in H^{4,2}(\mathbb{R})$ under the assumption of the presence of discrete spectrum and discussed the asymptotic analysis for the cubic CH/FQXL equation (1.3) in different space-time solitonic regions in the form of following solutions [27]:

$$
u(x,t) = u^r(x,t;\tilde{\mathcal{D}}) + \mathcal{O}\left(t^{-1+2\rho}\right), \text{ for } \xi < -\frac{1}{4} \text{ or } \xi > 2, \tag{1.4}
$$

and

$$
u(x,t) = u^r(x,t;\tilde{\mathcal{D}}) + f_{11}t^{-1/2} + \mathcal{O}\left(t^{-3/4}\right), \text{ for } -\frac{1}{4} < \xi < 0 \text{ or } 0 \leq \xi < 2. \tag{1.5}
$$

In this paper, we focus on the following initial value problem for the ccCH equation

$$
m_t = bu_x + \frac{1}{2}\left[m\left(|u|^2 - |u_x|^2\right)\right]_x - \frac{1}{2}m\left(u\bar{u}_x - u_x\bar{u}\right), \quad m = u - u_{xx}, \tag{1.6}
$$

$$
u(x,0) = u_0(x) \in H^{4,2}(\mathbb{R}). \tag{1.7}
$$

In comparison with the case [27] where the symmetry in $S(z) = \overline{S(1/\bar{z})}$ requires m, u be real, but the discrete spectrum of (1.1) lacks such a symmetry. Moreover, the asymptotic behaviors of the solution from the Riemann-Hilbert problem proposed here in our paper are also different from the procedure in [27] with respect to the spectral parameter z. To overcome these issues, we have added a new transformation on the eigenfunction $\psi(x,t;z)$ and given a new reconstruction formula for the complex case. More detailed differences are described in the later sections. In general, we successfully obtain the long time asymptotic behavior for the initial value problem (1.6)-(1.7) for the ccCH equation (1.1) in different space-time solitonic regions in the form of following solutions:

$$u(x,t)e^{2g_-} = u^r(x,t;\tilde{\mathcal{D}})e^{2\tilde{g}_-} + O(t^{-1+2\varepsilon}), \text{ for } \xi < -1 \text{ or } \xi > \frac{1}{8}, \tag{1.8}$$

and

$$u(x,t)e^{2g_-} = u^r(x,t;\tilde{\mathcal{D}})e^{2\tilde{g}_-} + k_{11}t^{-\frac{1}{2}} + O(t^{-\frac{3}{4}}), \text{ for } -1 < \xi < \frac{1}{8}, \tag{1.9}$$

where g_- is a purely imaginary function (2.10), $\tilde{g}_-$ is yielded through replacing u in equation (2.10) with $u^r(x,t;\tilde{\mathcal{D}})$ that is the $\mathcal{N}(\Lambda)$-soliton solution of (1.6)-(1.7) with the scattering data $\tilde{\mathcal{D}}_\Lambda = \{0, \{\varrho_n, c_n T^2(\varrho_n)\}_{n\in\Lambda}\}$ (also see Corollary 6.2), and k_{11} is given by (9.11). When u and u^r are real (namely, $g_- = \tilde{g}_- = 0$), we note that (1.8) and (1.9) are degenerated into equations (1.4) and (1.5) but with different regions. Moreover, since $M(z)$ and $M(i)$ in (3.14) and (3.15) are the same as the ones in [27] (due to real u and u^r), we can prove that $k_{11} = f_{11}$ in this case. Thus, the two formulations given here by (1.8) and (1.9) for the complex case are more generalized than the results in [27] for the real case.

The rest of this paper is organized as follows. In Section 2, on the basis of Lax pair of the complex cubic Camassa-Holm equation (1.1), we introduce two kinds of eigenfunctions $\mu_\pm^0$ and $\mu_\pm$ depending on the initial data u_0 to deal with the spectral singularity. In Section 3, Through providing the relation between the solution of equation (1.1) and the solution of the corresponding Riemann-Hilbert problem, we prove the existence and uniqueness of the solution to the Riemann-Hilbert problem of the ccCH model. Then in Section 4, we show that the scatter data $r(z)$ is in $H^{1,1}(\mathbb{R})$ for the given initial value $u_0(x) \in H^{4,2}(\mathbb{R})$. In Section 5, we present two triangular decompositions of the jump matrix by employing a matrix-valued function $T(z)$ so that the original RH problem is transformed into a new RH problem $M^{(1)}(z)$. Also, a mixed $\bar{\partial}$-RH problem $M^{(2)}(z)$ is cast via another matrix-valued function $R(z)$ and a continuous extension of $M^{(1)}(z)$. The mixed $\bar{\partial}$-RH problem $M^{(2)}(z)$ is needed to break into the practical model of RH problem $M^{(R)}(z)$ and the pure $\bar{\partial}$-problem $M^{(3)}(z)$. In Section 6, we obtain $M^{R}(z)$ from the reflectionless RH problem $M^{(r)}(z)$ for the soliton components. In Section 7, we apply the inner model $M^{lo}(z)$ to match the parabolic cylindrical functions near each stationary state point ξ_j and figure out the error function $E(z,\xi)$ by using a small-norm RH problem. In Section 8, the $\bar{\partial}$-problem for $M^{(3)}(z)$ is analyzed. In Section 9, we finally find a decomposition formula which yields the long-time asymptotic behavior of the solution to the initial value problem (1.6)-(1.7).

2. Spectral analysis of complex cubic Camassa-Holm equation

Without loss of generality, let us choose $b = 1$ in equation (1.1) by the transformation $u(x,t) = \sqrt{b}\tilde{u}(\tilde{x},\tilde{t})$, $x = \tilde{x}$, $t = \frac{1}{b}\tilde{t}$. Equation (1.1) becomes

$$\tilde{m}_{\tilde{t}} = \tilde{u}_{\tilde{x}} + \frac{1}{2}\left[\tilde{m}\left(|\tilde{u}|^2 - |\tilde{u}_{\tilde{x}}|^2\right)\right]_{\tilde{x}} - \frac{1}{2}\tilde{m}\left(\tilde{u}\bar{\tilde{u}}_{\tilde{x}\tilde{x}} - \tilde{u}_{\tilde{x}}\bar{\tilde{u}}\right), \quad \tilde{m} = \tilde{u} - \tilde{u}_{\tilde{x}\tilde{x}}. \tag{2.1}$$

The complex cubic Camassa-Holm equation (2.1) admits the Lax pair [30]

$$\phi_x(x,t,z) = U(x,t,z)\phi(x,t,z), \tag{2.2}$$

$$\phi_t(x,t,z) = V(x,t,z)\phi(x,t,z), \tag{2.3}$$

where $U = \frac{1}{2}\begin{pmatrix} -\alpha & \lambda m \\ -\lambda n & \alpha \end{pmatrix}$, $V = -\frac{1}{2}\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}$, z is a spectral parameter, and $m = u - u_{xx}$, $n = \bar{u} - \bar{u}_{xx}$, $\alpha = \sqrt{1-\lambda^2}$,

$$\begin{aligned} V_{11} &= \lambda^{-2}\alpha + \tfrac{\alpha}{2}\left(|u|^2 - u_x\bar{u}_x\right) + \tfrac{1}{2}\left(u\bar{u}_x - u_x\bar{u}\right), \\ V_{12} &= -\lambda^{-1}\left(u - \alpha u_x\right) - \tfrac{1}{2}\lambda m\left(|u|^2 - u_x\bar{u}_x\right), \\ V_{21} &= \lambda^{-1}\left(\bar{u} + \alpha\bar{u}_x\right) + \tfrac{1}{2}\lambda n\left(|u|^2 - u_x\bar{u}_x\right) \end{aligned} \tag{2.4}$$

with

$$\alpha = \alpha(z) = \tfrac{i}{2}\left(z - \tfrac{1}{z}\right), \qquad \lambda = \lambda(z) = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right).$$

It is worth noting that Lax pair (2.2) has singularities at $z = 0$, $z = \infty$ and branch cut points $z = i$ in the extended complex z-plane. we need to use different transformations to analyze these singularities respectively.

Case I : $z = \infty$.

Considering the singularity at $z = \infty$, we introduce a transformation to control asymptotic behavior of the Lax pair (2.2)-(2.3) as $z \to \infty$, define

$$A(x,t) = \sqrt{\frac{d+1}{2d}} \begin{pmatrix} 1 & \frac{-im}{d+1} \\ \frac{-i\bar{m}}{d+1} & 1 \end{pmatrix} \tag{2.5}$$

and

$$h(x,t,z) = x - \textstyle\int_x^\infty (d-1)ds + \frac{t}{\lambda^2}, \quad d = \sqrt{|m|^2+1}.$$

We make the transformation

$$\phi = A(x,t)\psi e^{-\frac{i}{4}\left(z-\frac{1}{z}\right)h\sigma_3}, \tag{2.6}$$

ψ satisfies the following Lax pair

$$\psi_x = -\frac{i}{4}\left(z-\frac{1}{z}\right) h_x \left[\sigma_3, \psi\right] + U_2\psi, \tag{2.7}$$

$$\psi_t = -\frac{i}{4}\left(z-\frac{1}{z}\right) h_t \left[\sigma_3, \psi\right] + V_2\psi, \tag{2.8}$$

where

$$U_2 = \begin{pmatrix} \frac{m_x\bar{m}-m\bar{m}_x}{4d(d+1)} & \frac{i(d+1)m_x}{4d^2} - \frac{im^2\bar{m}_x}{4d^2(d+1)} \\ \frac{i(d+1)\bar{m}_x}{4d^2} - \frac{i\bar{m}^2 m_x}{4d^2(d+1)} & \frac{-\bar{m}m_x+m\bar{m}_x}{4d(d+1)} \end{pmatrix} + \frac{1}{2zd}\begin{pmatrix} -i|m|^2 & m \\ -\bar{m} & i|m|^2 \end{pmatrix},$$

$$\begin{aligned} V_2 = &\begin{pmatrix} \frac{m_t\bar{m}-m\bar{m}_t}{4d(d+1)} & \frac{i(d+1)m_t}{4d} - \frac{im^2\bar{m}_t}{4d^2(d+1)} \\ \frac{i(d+1)\bar{m}_t}{4d^2} - \frac{i\bar{m}^2 m_t}{4d^2(d+1)} & \frac{m\bar{m}_t-\bar{m}m_t}{4d(d+1)} \end{pmatrix} - \frac{1}{4zd}(|u|^2 - u_x\bar{u}_x)\begin{pmatrix} i|m|^2 & -m \\ \bar{m} & -i|m|^2 \end{pmatrix} \\ &- \frac{1}{4\lambda d}(u-\alpha u_x)\begin{pmatrix} i\bar{m} & -(d+1) \\ \frac{-\bar{m}^2}{d+1} & -i\bar{m} \end{pmatrix} - \frac{1}{4\lambda d}(\bar{u}+\alpha\bar{u}_x)\begin{pmatrix} im & \frac{m^2}{d+1} \\ d+1 & -im \end{pmatrix} \\ &- \frac{1}{4d}(u\bar{u}_x - u_x\bar{u})\begin{pmatrix} 1 & -im \\ i\bar{m} & -1 \end{pmatrix} - \frac{\alpha}{2\lambda^2}\begin{pmatrix} \frac{1}{d}-1 & \frac{-im}{d} \\ \frac{i\bar{m}}{d} & 1-\frac{1}{d} \end{pmatrix}. \end{aligned}$$

Based on the analysis of the Lax pair (2.7)-(2.8), we know that the solutions of spectral problem do not approximate the identity matrix as $z \to \infty$. To solve this problem, we make the transformation

$$\psi(x,t;z) = e^{g_-\hat{\sigma}_3}\mu(x,t;z)e^{-g_+\sigma_3}, \tag{2.9}$$

where

$$g_- = \int_{-\infty}^{x} \frac{m_x\bar{m} - m\bar{m}_x}{4d(d+1)}(s,t)ds, \tag{2.10}$$

$$g_+ = \int_x^{+\infty} \frac{m_x\bar{m} - m\bar{m}_x}{4d(d+1)}(s,t)ds, \tag{2.11}$$

$$g = g_- + g_+ = \int_{-\infty}^{+\infty} \frac{m_x\bar{m} - m\bar{m}_x}{4d(d+1)}(s,t)ds. \tag{2.12}$$

We obtain a new Lax pair

$$\mu_x = -\frac{i}{4}\left(z - \frac{1}{z}\right) h_x\left[\sigma_3, \mu\right] + e^{-g_-\hat{\sigma}_3}U_3\mu, \tag{2.13}$$

$$\mu_t = -\frac{i}{4}\left(z - \frac{1}{z}\right) h_t\left[\sigma_3, \mu_\pm\right] + e^{-g_-\hat{\sigma}_3}V_3\mu, \tag{2.14}$$

where

$$U_3 = \begin{pmatrix} 0 & \frac{i(d+1)m_x}{4d^2} - \frac{im^2\bar{m}_x}{4d^2(d+1)} \\ \frac{i(d+1)\bar{m}_x}{4d^2} - \frac{i\bar{m}^2m_x}{4d^2(d+1)} & 0 \end{pmatrix} + \frac{1}{2zd}\begin{pmatrix} -i|m|^2 & m \\ -\bar{m} & i|m|^2 \end{pmatrix},$$

$$\begin{aligned} V_3 = &\begin{pmatrix} 0 & \frac{i(d+1)m_t}{4d^2} - \frac{im^2\bar{m}_t}{4d^2(d+1)} \\ \frac{i(d+1)\bar{m}_t}{4d^2} - \frac{i\bar{m}^2m_t}{4d^2(d+1)} & 0 \end{pmatrix} - \frac{1}{4zd}(|u|^2 - u_x\bar{u}_x)\begin{pmatrix} i|m|^2 & -m \\ \bar{m} & -i|m|^2 \end{pmatrix} \\ &- \frac{1}{2(z+\frac{1}{z})d}\begin{pmatrix} im\bar{u} + i\bar{m}u & \frac{m^2\bar{u}}{d+1} - (d+1)u \\ (d+1)\bar{u} - \frac{\bar{m}^2u}{d+1} & -im\bar{u} - i\bar{m}u \end{pmatrix} + \frac{1}{2(z^2+1)d}\begin{pmatrix} \bar{m}u_x - m\bar{u}_x & 0 \\ 0 & m\bar{u}_x - \bar{m}u_x \end{pmatrix} \\ &- \frac{i(z^2-1)}{4(z^2+1)d}\begin{pmatrix} 0 & \frac{m^2\bar{u}_x}{d+1} + (d+1)u_x \\ \frac{\bar{m}^2u_x}{d+1} + (d+1)\bar{u}_x & 0 \end{pmatrix} - \frac{1}{4d}(u\bar{u}_x - u_x\bar{u})\begin{pmatrix} 0 & -im \\ i\bar{m} & 0 \end{pmatrix} \\ &- \frac{i(z^3-z)}{(z^2+1)^2}\begin{pmatrix} \frac{1}{d} - 1 & \frac{-im}{d} \\ \frac{i\bar{m}}{d} & 1 - \frac{1}{d} \end{pmatrix}. \end{aligned}$$

Remark 2.1. Compared to the cubic Camassa-Holm equation [27], the complex cubic Camassa-Holm equation has an additional transformation (2.9) to remove the diagonal elements of non-zero term of the matrix U_2 and V_2 as $|z| \to \infty$.

Then matrix function $\mu = \mu(x,t;z)$ has the following asymptotics

$$\mu \sim I, \quad x \to \pm\infty. \tag{2.15}$$

The Lax pair (2.13)-(2.14) can be written as the total differential form

$$d\left(e^{\frac{i}{4}\left(z-\frac{1}{z}\right)h\hat{\sigma}_3}\mu\right) = e^{\frac{i}{4}\left(z-\frac{1}{z}\right)h\hat{\sigma}_3}e^{-g_-\hat{\sigma}_3}(U_3dx + V_3dt)\mu. \tag{2.16}$$

By integrating the equation (2.16) in two directions along the parallel real axis, we can get two Volterra type integral equations

$$\begin{aligned} \mu_-(x,t;z) &= \mathbb{I} + \int_{-\infty}^x e^{-\frac{i}{4}\left(z-\frac{1}{z}\right)[h(x,t;z)-h(s,t;z)]\hat{\sigma}_3}e^{-g_-\hat{\sigma}_3}U_3(s,t;z)\mu_-(s,t;z)ds, \\ \mu_+(x,t;z) &= \mathbb{I} - \int_x^{+\infty} e^{-\frac{i}{4}\left(z-\frac{1}{z}\right)[h(x,t;z)-h(s,t;z)]\hat{\sigma}_3}e^{-g_-\hat{\sigma}_3}U_3(s,t;z)\mu_+(s,t;z)ds. \end{aligned} \tag{2.17}$$

Proposition 2.1. $\mu_{+,2}(z)$ and $\mu_{-,1}(z)$ are analytical in $\mathbb{C}_+$; $\mu_{+,1}(z)$ and $\mu_{-,2}(z)$ are analytical in $\mathbb{C}_-$. Here, $\mathbb{C}_+ = \{z \in \mathbb{C} | Imz > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} | Imz < 0\}$, $\mu_{\pm,1}(z)$ and $\mu_{\pm,2}(z)$ are the first and second columns of $\mu_\pm(z)$, respectively.

Proposition 2.2. $\mu_\pm(x, t; z)$ has the following symmetry

$$\mu_\pm(z) = \sigma_2 \overline{\mu_\pm(\bar{z})} \sigma_2 \tag{2.18}$$

and

$$\mu_\pm(z) = e^{-g_- \hat{\sigma}_3} A^{-2} \sigma_3 \mu_\pm(-1/z) \sigma_3 = e^{-g_- \hat{\sigma}_3} A^{-2} \sigma_1 \overline{\mu_\pm(-1/\bar{z})} \sigma_1. \tag{2.19}$$

Due to $\mu_+(x, t; z)$ and $\mu_-(x, t; z)$ are two linear correlation matrix solutions of Lax pair (2.13)-(2.14) , there exists linear relation

$$e^{\frac{i}{4}\left(z - \frac{1}{z}\right) h(x,t;z) \hat{\sigma}_3} \mu_-(x, t; z) = e^{\frac{i}{4}\left(z - \frac{1}{z}\right) h(x,t;z) \hat{\sigma}_3} \mu_+(x, t; z) S(z), \tag{2.20}$$

that is

$$\mu_-(x, t; z) = \mu_+(x, t; z) e^{-\frac{i}{4}\left(z - \frac{1}{z}\right) h(x,t;z) \hat{\sigma}_3} S(z). \tag{2.21}$$

From symmetry relations (2.18), (2.19) and relation (2.21), we can obtain Proposition 2.3.

Proposition 2.3. The scattering matrix $S(z)$ has the following symmetry reductions

$$S(z) = \sigma_1 \overline{S(-1/\bar{z})} \sigma_1 = \sigma_3 S(-1/z) \sigma_3 = \sigma_2 \overline{S(\bar{z})} \sigma_2. \tag{2.22}$$

Therefore, the scattering matrix $S(z)$ can be written as

$$S(z) = \begin{pmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{pmatrix} = \begin{pmatrix} a(z) & -\overline{b(\bar{z})} \\ b(z) & \overline{a(\bar{z})} \end{pmatrix}. \tag{2.23}$$

In addition, we define the reflection coefficient as

$$r(z) = \frac{b(z)}{a(z)}. \tag{2.24}$$

From the symmetry relation (2.22), we get

$$r(z) = -r(-1/z). \tag{2.25}$$

According to the relation (2.21), $a(z)$ and $b(z)$ can be expressed as

$$a(z) = \mu_-^{11}(z) \overline{\mu_+^{11}(\bar{z})} + \mu_-^{21}(z) \overline{\mu_+^{21}(\bar{z})}, \tag{2.26}$$

$$\overline{b(\bar{z})}e^{-\frac{i}{2}\left(z-\frac{1}{z}\right)h(x,t;z)}=\overline{\mu_-^{11}(\bar{z})}\mu_+^{12}(z)-\mu_-^{12}(z)\overline{\mu_+^{11}(\bar{z})}. \tag{2.27}$$

From the analyticity of $\mu_\pm(z)$, we know that $a(z)$ is analytic on $\mathbb{C}_+$. Furthermore, $\mu_\pm(z)$ admit the asymptotics

$$\mu_\pm(z)=I+\frac{B_1}{z}+\mathcal{O}\left(z^{-2}\right),\quad z\to\infty, \tag{2.28}$$

where the off-diagonal entries of the matrix $B_1(x,t)$ are

$$B_{1,12}(x,t)=\frac{-2ie^{-2g_-}\left(\frac{i(d+1)m_x}{4d^2}-\frac{im^2\bar{m}_x}{4d^2(d+1)}\right)}{d},\ B_{1,21}(x,t)=\frac{-2ie^{2g_-}\left(\frac{-i(d+1)\bar{m}_x}{4d^2}-\frac{i\bar{m}^2m_x}{4d^2(d+1)}\right)}{d}. \tag{2.29}$$

From Eq.(2.26) and Eq.(2.28), we obtain the asymptotics of $a(z)$

$$a(z)=1+\mathcal{O}\left(z^{-1}\right),\quad z\to\infty. \tag{2.30}$$

Since the zeros of $a(z)$ on $\mathbb{R}$ correspond to spectral singularities, we need to consider the distribution of these zeros. Suppose that $a(z)$ has K_1 simple zeros z_1, ..., z_{K_1} on $\{z\in\mathbb{C}_+:0<argz\le\frac{\pi}{2},|z|>1\}$, and K_2 simple zeros ϑ_1, ..., ϑ_{K_2} on the circle $\{z=\varsigma^{i\varpi}:0<\varpi\le\frac{\pi}{2}\}$, and K_3 simple zeros ς_1, ..., ς_{K_3} on $\{z\in\mathbb{C}_+:0<argz\le\frac{\pi}{2},\mid z\mid<1\}$. Then by symmetries (2.22), we know that

$$a(z_j)=0\quad\Leftrightarrow\quad a(-\tfrac{1}{z_j})=0,\ j=1,...,K_1,$$
$$a(\varsigma_l)=0\quad\Leftrightarrow\quad a(-\tfrac{1}{\varsigma_l})=0,\ l=1,...,K_3$$

and on the circle

$$a(\vartheta_p)=0\quad\Leftrightarrow\quad a(-\tfrac{1}{\vartheta_p})=0,\ p=1,...,K_2.$$

Since the zeros of $a(z)$ occur in pairs, it is convenient to define the zeros of $a(z)$ as $\varrho_j=z_j$, $\varrho_{j+K_1}=-\frac{1}{z_j}$, for $j=1,...,K_1$; $\varrho_{2K_1+p}=\vartheta_p$, $\varrho_{2K_1+K_2+p}=-\frac{1}{\vartheta_p}$ for $p=1,...,K_2$; $\varrho_{2K_1+2K_2+l}=\varsigma_l$ and $\varrho_{2K_1+2K_2+K_3+l}=-\frac{1}{\varsigma_l}$ for $l=1,...,K_3$. Due to ϱ_n is the zeros of $\overline{a(\bar{z})}$, so the discrete spectrum is

$$\mathcal{Z}=\{\varrho_n,\bar{\varrho}_n\}_{n=1}^{2K_1+2K_2+2K_3}, \tag{2.31}$$

with $\varrho_n\in\mathbb{C}_+$ and $\bar{\varrho}_n\in\mathbb{C}_-$. The distribution of $\mathcal{Z}$ on the z-plane is shown in Fig.1.

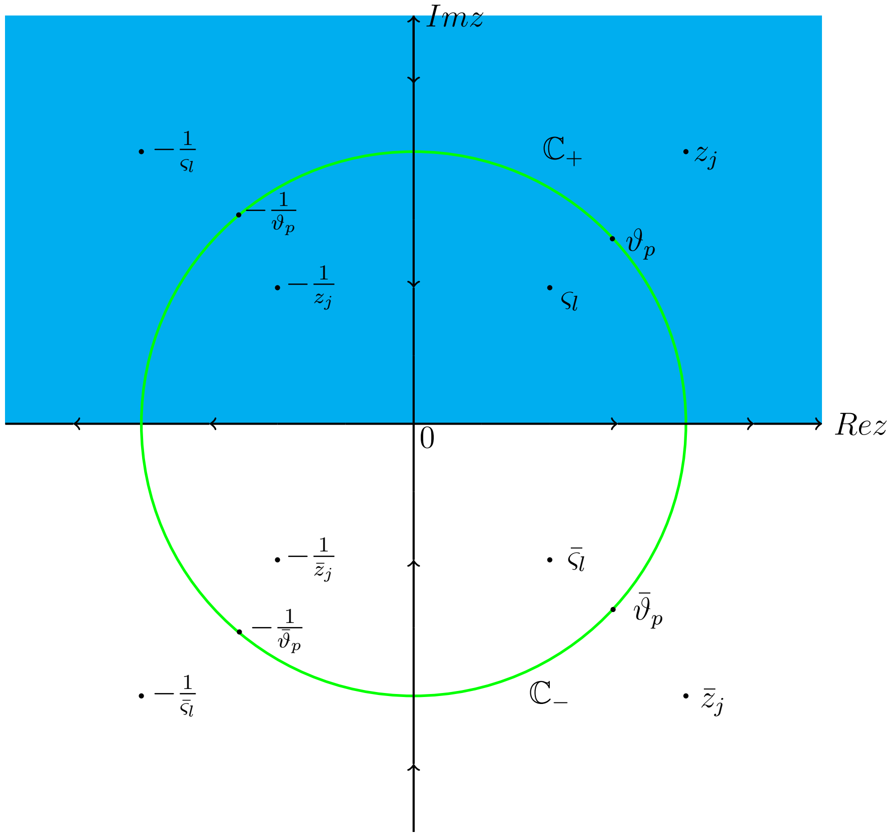


Fig.1 Analytical domains and distribution of the discrete spectrum $\mathcal{Z}$. There are $4K_2$ discrete spectrum on the green unit circle, while $4K_1 + 4K_3$ discrete spectrum are not on the circle.

Remark 2.2. The symmetry of $a(z)$ in the complex cubic Camassa-Holm equation is not the same as that in the cubic Camassa-Holm equation [27]. Therefore, the characteristics of the zeros of $a(z)$ are not the same.

Case II : $z = \pm i$ (corresponding to $\lambda = 0$).

Lax pair (2.2)-(2.3) has a Jost solution of the following asymptotic form

$$\phi_\pm(z) \sim e^{-\frac{\alpha}{2}x\sigma_3 - \frac{\alpha}{2\lambda^2}t\sigma_3}, \quad x \to \pm\infty. \tag{2.32}$$

Therefore, we make the transformation

$$\mu^0_\pm(z) = \phi_\pm(z) e^{\left(\frac{\alpha}{2}x + \frac{\alpha}{2\lambda^2}t\right)\sigma_3}, \tag{2.33}$$

then the matrix function $\mu^0_\pm(z)$ has the following asymptotics

$$\mu^0_\pm(z) \sim I, \quad x \to \pm\infty, \tag{2.34}$$

and satisfy Lax pair

$$\left(\mu_{\pm}^{0}\right)_{x}=-\frac{\alpha}{2}\left[\sigma_{3},\mu_{\pm}^{0}\right]+P_{0}\mu_{\pm}^{0}, \tag{2.35}$$

$$\left(\mu_{\pm}^{0}\right)_{t}=-\frac{\alpha}{2\lambda^{2}}\left[\sigma_{3},\mu_{\pm}^{0}\right]+Q_{0}\mu_{\pm}^{0}, \tag{2.36}$$

where

$P_0=\begin{pmatrix} 0 & \frac{\lambda m}{2} \\ -\frac{\lambda \bar{m}}{2} & 0 \end{pmatrix}$,

$Q_0=-\frac{1}{2}\begin{pmatrix} \frac{\alpha}{2}(|u|^2-u_x\bar{u}_x)+\frac{1}{2}(u\bar{u}_x-u_x\bar{u}) & -\lambda^{-1}(u-\alpha u_x)-\frac{1}{2}\lambda m(|u|^2-u_x\bar{u}_x) \\ \lambda^{-1}(\bar{u}+\alpha u_x)+\frac{1}{2}\lambda\bar{m}(|u|^2-u_x\bar{u}_x) & -\frac{\alpha}{2}(|u|^2-u_x\bar{u}_x)-\frac{1}{2}(u\bar{u}_x-u_x\bar{u}) \end{pmatrix}$.

From Lax pair (2.35)-(2.36), we assume that

$$\mu_{\pm}^{0}=I+\mu_{\pm,1}^{0}(z-i)+\mathcal{O}\left((z-i)^{2}\right), \tag{2.37}$$

where

$$\mu_{\pm,1}^{0}=\begin{pmatrix} 0 & -\frac{1}{2}\left(u+u_{x}\right) \\ -\frac{1}{2}\left(\bar{u}-\bar{u}_{x}\right) & 0 \end{pmatrix}. \tag{2.38}$$

Considering relations (2.6), (2.9) and (2.33), we have

$$\mu(z)=e^{g_{-}\sigma_{3}}A^{-1}(x,t)\mu_{\pm}^{0}(z)e^{\frac{i}{4}\left(z-\frac{1}{z}\right)k_{\pm}(x,t)\sigma_{3}}e^{-g_{-}\sigma_{3}}e^{-g_{+}\sigma_{3}}, \tag{2.39}$$

where

$$k_{\pm}(x,t)=\int_{\pm\infty}^{x}(d-1)ds.$$

Substituting Eq.(2.39) into Eq.(2.37) , we have

$$\begin{aligned}
\mu_{-}^{11}(z) &= \sqrt{\tfrac{d+1}{2d}}e^{-g_{+}}e^{\frac{i}{4}(z-\frac{1}{z})k_{-}}\left(1-\tfrac{1}{2}(\bar{u}-\bar{u}_{x})\tfrac{im}{d+1}(z-i)\right),\\
\overline{\mu_{+}^{11}(\bar{z})} &= \sqrt{\tfrac{d+1}{2d}}e^{g_{+}}e^{-\frac{i}{4}(z-\frac{1}{z})k_{+}}\left(1+\tfrac{1}{2}(u-u_{x})\tfrac{i\bar{m}}{d+1}(z+i)\right),\\
\mu_{-}^{21}(z) &= \left(\tfrac{i\bar{m}}{d+1}-\tfrac{1}{2}(\bar{u}-\bar{u}_{x})(z-i)\right)e^{\frac{i}{4}(z-\frac{1}{z})k_{-}}e^{-g_{+}}\sqrt{\tfrac{d+1}{2d}}e^{-2g_{-}},\\
\overline{\mu_{+}^{21}(\bar{z})} &= \left(\tfrac{-im}{d+1}-\tfrac{1}{2}(u-u_{x})(z+i)\right)e^{-\frac{i}{4}(z-\frac{1}{z})k_{-}}e^{g_{+}}\sqrt{\tfrac{d+1}{2d}}e^{2g_{-}},
\end{aligned}$$

and from

$$a(z)=\mu_{-}^{11}(z)\overline{\mu_{+}^{11}(\bar{z})}+\mu_{-}^{21}(z)\overline{\mu_{+}^{21}(\bar{z})},$$

we obtain that

$$a(z)=e^{-\frac{1}{2}\int_{\mathbb{R}}(d-1)dx}(1+O((z-i)^{2})),\ z\to i. \tag{2.40}$$

Case III : $z=0$.

We can obtain the property of $\mu(z)$ as $z \to 0$ by the symmetry condition in Proposition 2.1. Furthermore, Eq.(2.24), Eq.(2.26), Eq.(2.27) and Eq.(2.30) lead to $a(z) \to 1$, $b(z) \to 0$, as $z \to 0$, which gives $r(z) \to 0$ as $z \to 0$.

3. Riemann-Hilbert problem of complex cubic Camassa-Holm equation

3.1. Construction of the Riemann-Hilbert problem

Define a sectionally meromorphic matrix

$$\tilde{M}(z;x,t) = \begin{cases} \left(\frac{\mu_{-,1}(z)}{a(z)}, \mu_{+,2}(z)\right), & \text{as } z \in \mathbb{C}^+, \\ \left(\mu_{+,1}(z), \frac{\mu_{-,2}(z)}{\overline{a(\bar{z})}}\right), & \text{as } z \in \mathbb{C}^-, \end{cases} \tag{3.1}$$

by using the analyticity and symmetry of the eigenfunction and the spectral matrix, we can obtain the RH problem to initial value problem of equation (2.1).

Riemann-Hilbert Problem 3.1. Find a matrix-valued function $\tilde{M}(z;x,t)$ satisfies:

• Analyticity: $\tilde{M}(z;x,t)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$;

• Symmetry: $\tilde{M}(z) = \sigma_2 \overline{\tilde{M}(\bar{z})} \sigma_2 = e^{-g_- \hat{\sigma}_3} A^{-2} \sigma_3 \tilde{M}(-\frac{1}{z}) \sigma_3$;

• Jump condition: $\tilde{M}_+(z) = \tilde{M}_-(z) \tilde{V}(z), \quad z \in \mathbb{R}$,

where

$$\tilde{V}(z) = \begin{pmatrix} 1 + |r(z)|^2 & \overline{r(z)} e^{-\alpha h} \\ r(z) e^{\alpha h} & 1 \end{pmatrix}; \tag{3.2}$$

• Asymptotic behaviors:

$$\tilde{M}(z) = I + \mathcal{O}\left(z^{-1}\right), \quad z \to \infty, \tag{3.3}$$

$$\tilde{M}(z) = e^{g_- \sigma_3} A^{-1} \left[I + \mu_1^0 (z - i)\right] e^{-\frac{1}{2} k_+ \sigma_3} e^{-g_- \sigma_3} e^{-g_+ \sigma_3} + \mathcal{O}\left((z-i)^2\right), \quad z \to i; \tag{3.4}$$

• Residue conditions:

$$\operatorname*{Res}_{z=\varrho_n} \tilde{M}(z) = \lim_{z \to \varrho_n} \tilde{M}(z) \begin{pmatrix} 0 & 0 \\ c_n e^{\alpha(\varrho_n) h(\varrho_n)} & 0 \end{pmatrix}, \tag{3.5}$$

$$\operatorname*{Res}_{z=\bar{\varrho}_n}\tilde{M}(z)=\lim_{z\to\bar{\varrho}_n}\tilde{M}(z)\begin{pmatrix}0 & -\bar{c}_n e^{-\alpha(\bar{\varrho}_n)h(\bar{\varrho}_n)}\\ 0 & 0\end{pmatrix}. \tag{3.6}$$

where $c_n=\frac{s_{21}(\varrho_n)}{s'_{11}(\varrho_n)}$.

Due to $h(x,t,z)$ is still unknown, the solution of equation (2.1) is rather difficult to reconstruct. Boutet de Monvel and Shepelsky proposed a method to solve the problem [32, 33]. According to the method, we introduce a new scale

$$y(x,t)=x-\int_x^{+\infty}(d(s)-1)ds,\quad y=x-k_+(x,t). \tag{3.7}$$

Define

$$\theta(z)=\frac{i}{2}\alpha(z)\left[\frac{y}{t}+\lambda^{-2}(z)\right], \tag{3.8}$$

then we can get a RH problem for the new variable (y,t).

Riemann-Hilbert Problem 3.2. Find a matrix-valued function $M(z;y,t)$ which satisfies:

- Analyticity: $M(z;y,t)$ is meromorphic in $\mathbb{C}\setminus\mathbb{R}$ and has single poles;
- Symmetry: $M(z;y,t)=\sigma_2\overline{M(\bar{z};y,t)}\sigma_2$;
- Jump condition: $M_+(z;y,t)=M_-(z;y,t)V(z),\quad z\in\mathbb{R}$,

where

$$V(z)=\begin{pmatrix}1+|r(z)|^2 & \overline{r(z)}e^{2it\theta(z)}\\ r(z)e^{-2it\theta(z)} & 1\end{pmatrix}; \tag{3.9}$$

- Asymptotic behaviors:

$$M(z;y,t)=I+\mathcal{O}\left(z^{-1}\right),\quad z\to\infty, \tag{3.10}$$

$$M(z;y,t)=e^{g_-\sigma_3}A^{-1}\left[I+\mu_1^0(z-i)\right]e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3}+\mathcal{O}\left((z-i)^2\right),\quad z\to i; \tag{3.11}$$

- Residue conditions:

$$\operatorname*{Res}_{z=\varrho_n}M(z;y,t)=\lim_{z\to\varrho_n}M(z;y,t)\begin{pmatrix}0 & 0\\ c_n e^{-2it\theta(\varrho_n)} & 0\end{pmatrix}, \tag{3.12}$$

$$\operatorname*{Res}_{z=\bar{\varrho}_n}M(z;y,t)=\lim_{z\to\bar{\varrho}_n}M(z;y,t)\begin{pmatrix}0 & -\bar{c}_n e^{2it\theta(\bar{\varrho}_n)}\\ 0 & 0\end{pmatrix}. \tag{3.13}$$

From the asymptotic behavior of the functions μ and Eq.(3.11), we give that

$$M(z)=\sqrt{\tfrac{d+1}{2d}}e^{g_-\sigma_3}\begin{bmatrix}1 & \frac{im}{d+1}\\ \frac{i\bar{m}}{d+1} & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}(z-i)(u+u_x)\\ -\frac{1}{2}(z-i)\left(\bar{u}-\bar{u}_x\right) & 1\end{bmatrix}e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3}$$
$$=\sqrt{\tfrac{d+1}{2d}}\begin{bmatrix}e^{-\frac{1}{2}k_+-g_+}\left(1-\frac{1}{2}(z-i)\left(\bar{u}-\bar{u}_x\right)\frac{im}{d+1}\right) & e^{2g_-+\frac{1}{2}k_++g_+}\left(-\frac{1}{2}(z-i)\left(u+u_x\right)+\frac{im}{d+1}\right)\\ e^{-2g_--\frac{1}{2}k_+-g_+}\left(\frac{i\bar{m}}{d+1}-\frac{1}{2}(z-i)\left(\bar{u}-\bar{u}_x\right)\right) & e^{\frac{1}{2}k_++g_+}\left(1-\frac{1}{2}(z-i)\left(u+u_x\right)\frac{i\bar{m}}{d+1}\right)\end{bmatrix}, \tag{3.14}$$

$$M(i)=\sqrt{\frac{d+1}{2d}}\begin{bmatrix}e^{-\frac{1}{2}k_+-g_-} & \frac{im}{d+1}e^{2g_-+\frac{1}{2}k_++g_+}\\ \frac{i\bar{m}}{d+1}e^{2g_-\frac{1}{2}k_+-g_+} & e^{\frac{1}{2}k_++g_+}\end{bmatrix}. \tag{3.15}$$

Then we can obtain the following reconstruction formula

$$-u(x,t)e^{2g_-}=\lim_{z\to i}\left(\frac{M_{12}(z)-M_{12}(i)}{(z-i)M_{22}(i)}+\overline{\left(\frac{M_{21}(z)-M_{21}(i)}{(z-i)M_{11}(i)}\right)}\right) \tag{3.16}$$

with

$$x=y+k_+=y+\ln\tfrac{M_{22}(i)}{M_{11}(i)}.$$

Remark 3.1. The symmetry and asymptotic behaviors of matrix function $\tilde{M}(z;x,t)$ in cubic Camassa-Holm equation [27] and complex cubic Camassa-Holm equation are different, so the reconstruction formula of potential is also different. Here we reconstruct the reconstruction formula of potential (3.16).

3.2. Existence and uniqueness of solution to the Riemann-Hilbert problem

In this section, we will prove the existence and uniqueness of solutions to the initial value problem (1.6)-(1.7). Firstly, we transform the basic RH problem 3.2 to an equivalent RH problem- RH problem 3.3, whose existence and uniqueness are affirmed by proving that the associated Beals-Coifman initial integral equation admits a unique solution for $r(z)\in H^{1,1}(\Sigma)$. Then we replace the residue conditions (3.12)-(3.13) of RH problem 3.2 with Schwartz invariant jump conditions. For all poles $z_n\in\mathbb{C}_+$ and $\bar{z}_n\in\bar{\mathbb{C}}_+$, we replace their residue by the corresponding jumps on the circle D_n centered at z_n and the circle $\overline{D}_n$ centered at $\bar{z}_n$, respectively. The radii of these circles have been chosen to be small enough so that they do not intersect all other circles. Please see Figure 2.

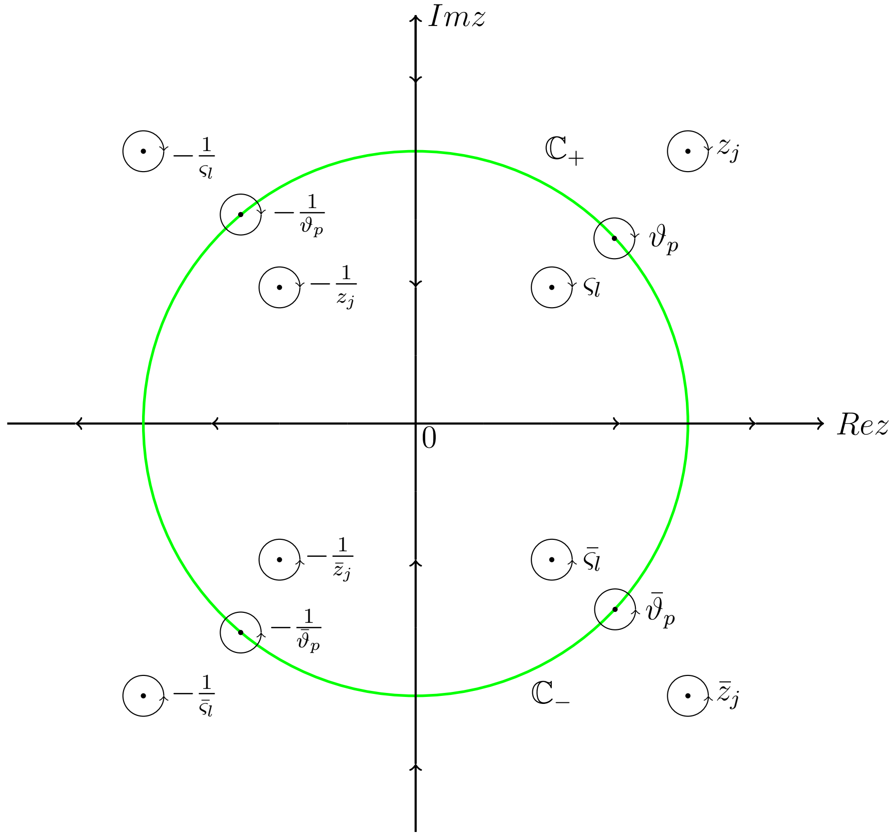


Fig.2 The jump contour Σ of RH problem for $N(z;y,t)$.

Let

$$\Sigma = \mathbb{R} \cup \{D_n\}_{n=1}^{2K_1+2K_2+2K_3} \cup \{\overline{D}_n\}_{n=1}^{2K_1+2K_2+2K_3},$$

we transform the RH problem 3.2 to an equivalent RH problem 3.3 on the new jump contour Σ.

Riemann-Hilbert Problem 3.3. Find a matrix-valued function $N(z) = N(z;y,t)$ which satisfies:

- Analyticity: $N(z;y,t)$ is analytic in $\mathbb{C} \setminus \Sigma$;
- Jump condition: $N(z;y,t)$ has continuous boundary values $N_\pm(z;y,t)$ on Σ and

$$N_+(z;y,t) = N_-(z;y,t)\check{V}(z), \tag{3.17}$$

where

$$\check{V}(z)=\begin{cases}\begin{pmatrix}1+|r(z)|^2 & \bar{r}(z)e^{2it\theta(z)}\\ r(z)e^{-2it\theta(z)} & 1\end{pmatrix}, & z\in\mathbb{R};\\ \begin{pmatrix}1 & 0\\ \frac{c_ne^{-2it\theta(\varrho_n)}}{z-\varrho_n} & 1\end{pmatrix} & z\in D_n;\\ \begin{pmatrix}1 & \frac{\bar{c}_ne^{2it\theta(\bar{\varrho}_n)}}{z-\bar{\varrho}_n}\\ 0 & 1\end{pmatrix}, & z\in\bar{D}_n;\end{cases}\tag{3.18}$$

• Asymptotic behaviors:

$$N(z;y,t)=I+\mathcal{O}\left(z^{-1}\right),\quad z\to\infty,\tag{3.19}$$

$$N(z;y,t)=e^{g_-\sigma_3}A^{-1}\left[I+\mu_1^0(z-i)\right]e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3}+\mathcal{O}\left((z-i)^2\right),\quad z\to i;\tag{3.20}$$

To emphasise the role of y, we suppress the dependence of all variables on t. Thus, the PH problem 3.3 can be reduced to the RH problem 3.4.

Riemann-Hilbert Problem 3.4. Find a matrix-valued function $N(z)=N(z;y)$ which satisfies:

• Analyticity: $N(z;y)$ is analytic in $\mathbb{C}\setminus\Sigma$;

• Jump condition: $N(z;y)$ has continuous boundary values $N_\pm(z;y)$ on Σ and

$$N_+(z;y)=N_-(z;y)\check{V}(z),\tag{3.21}$$

where

$$\check{V}(z)=\begin{cases}\begin{pmatrix}1+|r(z)|^2 & \bar{r}(z)e^{-\frac{i}{2}(z-\frac{1}{z})y}\\ r(z)e^{\frac{i}{2}(z-\frac{1}{z})y} & 1\end{pmatrix}, & z\in\mathbb{R};\\ \begin{pmatrix}1 & 0\\ \frac{c_ne^{\frac{i}{2}(\varrho_n-\frac{1}{\varrho_n})y}}{z-\varrho_n} & 1\end{pmatrix} & z\in D_n;\\ \begin{pmatrix}1 & \frac{\bar{c}_ne^{-\frac{i}{2}(\bar{\varrho}_n-\frac{1}{\bar{\varrho}_n})y}}{z-\bar{\varrho}_n}\\ 0 & 1\end{pmatrix}, & z\in\overline{D}_n;\end{cases}\tag{3.22}$$

• Normalization:

Type I. $N(z;y) = I + O(z^{-1})$ as $z \to \infty$,

Type II. $N(z;y) = O(z^{-1})$ as $z \to \infty$.

It can be seen that the existence and uniqueness of RH problem 3.4 are equivalent to the existence and uniqueness of the corresponding Beals-Coifman integral equation. Therefore, we only need to study the existence and uniqueness of the Beals-Coifman integral equation corresponding to RH problem 3.4. We decompose the jump matrix (3.22) as

$$\check{V}(z) = (I - \omega_-)^{-1}(I + \omega_+), \tag{3.23}$$

and let

$$\varphi = N_+(I + \omega_+)^{-1} = N_-(I - \omega_-)^{-1}, \tag{3.24}$$

from which we can write down φ explicitly: for $z \in \mathbb{R}$

$$\varphi(z,y) = \begin{cases} \begin{pmatrix} \frac{\mu_{-,1}(y,z)}{a(z)} & \mu_{+,2}(y,z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r(z)e^{\frac{i}{2}(z-\frac{1}{z})y} & 1 \end{pmatrix}, \\ \begin{pmatrix} \mu_{+,1}(y,z) & \frac{\mu_{-,2}(y,z)}{\overline{a(z)}} \end{pmatrix} \begin{pmatrix} 1 & \overline{r(z)}e^{-\frac{i}{2}(z-\frac{1}{z})y} \\ 0 & 1 \end{pmatrix}, \end{cases} \tag{3.25}$$

for $z \in D_n$,

$$\varphi(z,y) = \begin{cases} \begin{pmatrix} \frac{\mu_{-,1}(y,z)}{a(z)} & \mu_{+,2}(y,z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-c_n e^{\frac{i}{2}(\varrho_n - \frac{1}{\varrho_n})y}}{z-\varrho_n} & 1 \end{pmatrix}, \\ \begin{pmatrix} \frac{\mu_{-,11}(y,z)}{a(z)} - \frac{c_n e^{\frac{i}{2}(\varrho_n - \frac{1}{\varrho_n})y}\mu_{+,12}(y,z)}{z-\varrho_n} & \mu_{+,12}(y,z) \\ \frac{\mu_{-,21}(y,z)}{a(z)} - \frac{c_n e^{\frac{i}{2}(\varrho_n - \frac{1}{\varrho_n})y}\mu_{+,22}(y,z)}{z-\varrho_n} & \mu_{+,22}(y,z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{cases} \tag{3.26}$$

for $z \in \overline{D}_n$,

$$\varphi(z,y) = \begin{cases} \begin{pmatrix} \mu_{+,1}(y,z) & \frac{\mu_{-,2}(y,z)}{\overline{a(z)}} \end{pmatrix} \begin{pmatrix} 1 & \frac{-\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n - \frac{1}{\bar{\varrho}_n})y}}{z-\bar{\varrho}_n} \\ & 1 \end{pmatrix}, \\ \begin{pmatrix} \mu_{+,11}(y,z) & \frac{\mu_{-,12}(y,z)}{\overline{a(z)}} - \frac{\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n - \frac{1}{\bar{\varrho}_n})y}\mu_{+,11}(y,z)}{z-\bar{\varrho}_n} \\ \mu_{+,21}(y,z) & \frac{\mu_{-,22}(y,z)}{\overline{a(z)}} - \frac{\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n - \frac{1}{\bar{\varrho}_n})y}\mu_{+,22}(y,z)}{z-\bar{\varrho}_n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{cases} \tag{3.27}$$

From (3.24), we obtain $$N_+ - N_- = \varphi(\omega_+ + \omega_-),$$
the Plemelj formula gives the Beals-Coifman integral equation for the type I norm condition in RH problem 3.4.

$$\varphi = I + C_\omega \varphi = I + C_\Sigma^+(\varphi\omega_-) + C_\Sigma^-(\varphi\omega_+), \tag{3.28}$$

where I is the 2×2 identity matrix. And for type II normalization:

$$\varphi = C_\omega \varphi = C_\Sigma^+(\varphi\omega_-) + C_\Sigma^-(\varphi\omega_+). \tag{3.29}$$

In type I, for $z \in \mathbb{R}$,

$$\varphi_{11}(y,z) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{12}(y,s)r(s)e^{\frac{i}{2}(s-\frac{1}{s})y}}{s-z+i0}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{12}(y,\varrho_n)c_n e^{\frac{i}{2}(\varrho_n-\frac{1}{\varrho_n})y}}{z-\varrho_n},$$
$$\varphi_{12}(y,z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{11}(y,s)\bar{r}(s)e^{-\frac{i}{2}(s-\frac{1}{s})y}}{s-z+i0}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{11}(y,\bar{\varrho}_n)\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n-\frac{1}{\bar{\varrho}_n})y}}{z-\bar{\varrho}_n},$$
$$\varphi_{21}(y,z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{22}(y,s)r(s)e^{\frac{i}{2}(s-\frac{1}{s})y}}{s-z+i0}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{22}(y,\varrho_n)c_n e^{\frac{i}{2}(\varrho_n-\frac{1}{\varrho_n})y}}{z-\varrho_n},$$
$$\varphi_{22}(y,z) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{21}(y,s)\bar{r}(s)e^{-\frac{i}{2}(s-\frac{1}{s})y}}{s-z+i0}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{21}(y,\bar{\varrho}_n)\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n-\frac{1}{\bar{\varrho}_n})y}}{z-\bar{\varrho}_n},$$

and in order to close the system, we have

$$\varphi_{11}(y,\bar{\varrho}_n) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{12}(y,s)r(s)e^{\frac{i}{2}(s-\frac{1}{s})y}}{s-\bar{\varrho}_n}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{12}(y,\varrho_n)c_n e^{\frac{i}{2}(\varrho_n-\frac{1}{\varrho_n})y}}{\bar{\varrho}_n-\varrho_n},$$
$$\varphi_{12}(y,\varrho_n) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{11}(y,s)\bar{r}(s)e^{-\frac{i}{2}(s-\frac{1}{s})y}}{s-\varrho_n}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{11}(y,\bar{\varrho}_n)\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n-\frac{1}{\bar{\varrho}_n})y}}{\varrho_n-\bar{\varrho}_n},$$
$$\varphi_{21}(y,\bar{\varrho}_n) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{22}(y,s)r(s)e^{\frac{i}{2}(s-\frac{1}{s})y}}{s-\bar{\varrho}_n}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{22}(y,\varrho_n)c_n e^{\frac{i}{2}(\varrho_n-\frac{1}{\varrho_n})y}}{\bar{\varrho}_n-\varrho_n},$$
$$\varphi_{22}(y,\varrho_n) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi_{21}(y,s)\bar{r}(s)e^{-\frac{i}{2}(s-\frac{1}{s})y}}{s-\varrho_n}ds + \sum_{n=1}^{2K_1+2K_2+2K_3} \frac{\varphi_{21}(y,\bar{\varrho}_n)\bar{c}_n e^{-\frac{i}{2}(\bar{\varrho}_n-\frac{1}{\bar{\varrho}_n})y}}{\varrho_n-\bar{\varrho}_n}.$$

When $z \in \mathbb{R}$, the matrix $\check{V}(z)$ in (3.22) is Hermitian. We can obtain uniqueness of the solution to RH Problem 3.4 through Zhou's Vanishing Lemma [34]. Moreover, the jump matrices when $z \in D_n$ and $z \in \overline{D}_n$ are not Hermitian on the corresponding jump contours $\{D_n \cup \overline{D}_n\}_{n=1}^{2K_1+2K_2+2K_3}$. Then, we introduce an auxiliary function $F(z)$ to deal with this problem. The auxiliary function $F(z)$ should be analytic for $z \in \mathbb{C}\backslash\mathbb{R}$. We note that the jump contours $\{D_n \cup \overline{D}_n\}_{n=1}^{2K_1+2K_2+2K_3}$ are oriented to preserve Schwartz reflection symmetry in the real axis $\mathbb{R}$. In that case, these conditions also admit the framework of Zhou's vanishing lemma.

Proposition 3.1. The solution of the RH problem 3.4 with the normalization condition type II is identically zero.

Proof. Firstly, we know that for $z \in \{D_n \cup \overline{D}_n\}$, the jump matrix satisfies

$$\check{V}(z) = \check{V}(\bar{z})^H, \tag{3.30}$$

where the superscript H means the complex conjugation and transpose of a given matrix. Now, we introduce a matrix value auxiliary function

$$F(z) = N(z)N(\bar{z})^H. \tag{3.31}$$

We want to prove that

$$\int_{\mathbb{R}} F(z)dz = 0. \tag{3.32}$$

Obviously the auxiliary function $F(z)$ is analytic in $\mathbb{C}\backslash\Sigma$. For $z \in \mathbb{C}\backslash\mathbb{R}$, we get

$$\begin{aligned} F_+(z) &= N_+(z)N_-(\bar{z})^H \\ &= N_-(z)\check{V}(z)\left(\check{V}^{-1}(\bar{z})\right)^H N_+(\bar{z})^H. \\ &= N_-(z)N_+(\bar{z})^H = F_-(z). \end{aligned} \tag{3.33}$$

From Morera's theorem, we know that $F(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$. Due to $N_\pm(z) \in L^2(\mathbb{R})$, we get that the auxiliary function $F(z)$ is integrable. By the Beals-Coifman theorem, we get

$$\begin{aligned} N(y,z) =& \frac{1}{2\pi i}\int_\Sigma \frac{\varphi(y,s)\left(\omega_+(s)+\omega_-(s)\right)}{s-z}ds \\ =& \frac{1}{2z\pi i}\int_\Sigma \frac{s}{s-z}\varphi(y,s)\left(\omega_+(s)+\omega_-(s)\right)ds \\ &- \frac{1}{2z\pi i}\int_\Sigma \varphi(y,s)\left(\omega_+(s)+\omega_-(s)\right)ds. \end{aligned} \tag{3.34}$$

From the above analysis, we have

$$\begin{aligned} &N(y,z) = O(z^{-1}),\ z \to \infty,\ z \in \mathbb{C}\backslash\mathbb{R}, \\ &F(z) = O(z^{-2}),\ z \to \infty,\ z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned}$$

We can obtain the equality by using Jordon theorem and Cauchy integral theorem. For $z \in \mathbb{R}$, we have

$$\begin{aligned} F_+(z) &= N_+(z)N_-(z)^H \\ &= N_-(z)\check{V}(z)N_-(z)^H, \end{aligned} \tag{3.35}$$

$$\begin{aligned} F_-(z) &= N_-(z)N_+(z)^H \\ &= N_-(z)\check{V}(z)^H N_-(z)^H. \end{aligned} \tag{3.36}$$

From , we derive

$$\int_R N_-(z)(\check{V}(z) + \check{V}(z)^H)N_-(z)^H dz = 0. \tag{3.37}$$

For $z \in \mathbb{R}$,

$$\check{V}(z) + \check{V}^H(z) = 2\begin{pmatrix} 1+|r(z)|^2 & \bar{r}(z)e^{-\frac{i}{2}(z-\frac{1}{z})y} \\ r(z)e^{\frac{i}{2}(z-\frac{1}{z})y} & 1 \end{pmatrix}$$

which is positive definite and possesses positive eigenvalues. From (3.37), we obtain $N_-(z) = 0$ for $z \in \mathbb{R}$, then we have

$$N_+(z) = N_-(z)\check{V}(z) = 0,\ z \in \mathbb{R}.$$

From Morera's theorem, we can know that $N(z)$ is analytic in a neighborhood of every point on $\mathbb{R}$. Since $N(z) = 0$ for any $z \in \mathbb{R}$, the analytic continuation gives us that $N(z) = 0$ holds all the way up to the first complex part of $\mathbb{R}$. In the same way, we get the result that $N(z) = 0$ for $z \in \mathbb{C}$ in the remaining parts of Σ.

4. The reflection coefficient

In this section, we prove that scatter data $r(z) \in H^{1,1}(\mathbb{R})$ with a given initial value $u_0(x) \in H^{4,2}(\mathbb{R})$.

Firstly, we give some definitions. If I is an interval on the real line $\mathbb{R}$ and X is a Banach space, then $C^0(I,X)$ denotes the space of continuous functions on I taking values in X. It is equipped with the norm

$$\|f\|_{C^0(I,X)} = \sup_{x\in I}\|f(x)\|_X.$$

Denote $C_B^0(I)$ as a space of bounded continuous functions on I. In addition, denote $C_B^0\left(\mathbb{R}^{\pm}\times(1,+\infty)\right)$, $C^0\left(\mathbb{R}^{\pm}, L^2(1,+\infty)\right)$ and $L^2\left(\mathbb{R}^{\pm}\times(1,+\infty)\right)$ as C_B^0, C^0 and L^2 respectively.

If the entries f_1 and f_2 are in space X, then we say vector $\overrightarrow{f} = (f_1, f_2)^T$ is in space X with the norm $\|\vec{f}\|_X \triangleq \|f_1\|_X + \|f_2\|_X$. In the same way, if every entry of matrix A are in space X, then we say A is also in space X.

We introduce the normed spaces:
$L^{p,s}(\mathbb{R}) = \{f(x) \in L^p(\mathbb{R}) | |x|^s f(x) \in L^p(\mathbb{R})\}$;
$W^{k,p}(\mathbb{R}) = \{f(x) \in L^p(\mathbb{R}) \mid \partial^j f(x) \in L^p(\mathbb{R})$ for $j = 1, 2, \dots, k\}$;
$H^{k,s}(\mathbb{R}) = \{f(x) \in L^2(\mathbb{R}) \mid (1+|x|^s)\,\partial^j f(x) \in L^2(\mathbb{R}),$ for $j = 1, \dots, k\}$.
When $t = 0$, the matrix function $A(x,t)$ defined by (2.5) can be written as

$$A(x) = \sqrt{\frac{d(x)+1}{2d(x)}}\begin{pmatrix} 1 & \frac{-im}{d(x)+1} \\ \frac{-i\bar{m}}{d(x)+1} & 1 \end{pmatrix} \tag{4.1}$$

and

$$h(x,z)=x-\int_x^\infty (d-1)dy, \quad d=\sqrt{|m|^2+1}.$$

We make a transformation

$$\phi(x,z)=A(x)e^{g-\hat{\sigma}_3}\mu(x,z)e^{-g+\sigma_3}e^{-\frac{i}{4}(z-\frac{1}{z})h(x)\sigma_3}, \tag{4.2}$$

and μ satisfies a new spectral problem

$$\mu_x=-\frac{i}{4}\left(z-\frac{1}{z}\right)h_x\left[\sigma_3,\mu\right]+e^{-g-\hat{\sigma}_3}U_3\mu, \tag{4.3}$$

where

$$U_3=\begin{pmatrix} 0 & \frac{i(d+1)m_x}{4d^2}-\frac{im^2\bar{m}_x}{4d^2(d+1)} \\ \frac{i(d+1)\bar{m}_x}{4d^2}-\frac{i\bar{m}^2m_x}{4d^2(d+1)} & 0 \end{pmatrix}+\frac{1}{2zd}\begin{pmatrix} -i|m|^2 & m \\ -n & i|m|^2 \end{pmatrix}.$$

By above analysis, we know that μ satisfies asymptotics

$$\mu(z)\sim I, \quad x\to\pm\infty, \tag{4.4}$$

and have the following Volterra integral equations

$$\mu_\pm(x,t;z)=\mathbb{I}-\int_x^{\pm\infty} e^{-\frac{i}{4}\left(z-\frac{1}{z}\right)[h(x,t;z)-h(s,t;z)]\hat{\sigma}_3}e^{-g-\hat{\sigma}_3}U_3(s,t;z)\mu_-(s,t;z)ds. \tag{4.5}$$

In order to get our results, we need estimate the L^2-integral property of $\mu_\pm(z)$ and their derivatives. We can introduce some results [27].

Theorem 4.1. From $u_0(x)\in H^{4,2}(\mathbb{R})$, it can be deduced that $r(z)\in H^{1,1}(\mathbb{R})$.

Lemma 4.2. For $\psi(\eta)\in L^2(\mathbb{R})$, $f(x)\in L^{2,1/2}(\mathbb{R})$, following inequality hold

$$\left|\int_{\mathbb{R}}\int_x^{\pm\infty} f(s)e^{-\frac{i}{2}\eta(h(x)-h(s))}\psi(\eta)dsd\eta\right|\lesssim\left(\int_x^{\pm\infty}|f(s)|^2ds\right)^{1/2}\|\psi\|_2; \tag{4.6}$$

$$\int_0^{\pm\infty}\int_{\mathbb{R}}\left|\int_x^{\pm\infty} f(s)e^{-\frac{i}{2}\eta(h(x)-h(s))}ds\right|^2 d\eta dx\lesssim\|f\|^2_{2,1/2}. \tag{4.7}$$

By the symmetry reduction (2.19), we consider only $\mu_{\pm,1}(x,z)$ for $z\in(1,+\infty)$ here. We denote

$$\mu_{\pm,1}(x,z)-e_1\triangleq\nu_\pm(x,z), \tag{4.8}$$

where $e_1=(1,0)^T$, and define a integral operator $W_\pm$ as

$$W_\pm(f)(x,z)=\int_x^{\pm\infty}P_\pm(x,s,z)f(s,z)ds, \tag{4.9}$$

where integral kernel $P_{\pm}(x,s,z)$ is

$$\begin{aligned}P_{\pm}(x,s,z) &= \begin{pmatrix} 1 & 0 \\ 0 & -e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} e^{-g_-\hat{\sigma}_3} U_3 \\ &= e^{-g_-\hat{\sigma}_3} \begin{pmatrix} 0 & \frac{i(d+1)m_x}{4d^2} - \frac{im^2\bar{m}_x}{4d^2(d+1)} \\ \left(\frac{i\bar{m}^2 m_x}{4d^2(d+1)} - \frac{i(d+1)\bar{m}_x}{4d^2}\right) e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} & 0 \end{pmatrix} \\ &\quad + e^{-g_-\hat{\sigma}_3} \frac{1}{2zd} \begin{pmatrix} -i|m|^2 & m \\ \bar{m}e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} & -i|m|^2 e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix}.\end{aligned} \tag{4.10}$$

From (4.9), equation (4.5) can be written as

$$\nu_{\pm} = W_{\pm}(e_1) + W_{\pm}(\nu_{\pm}). \tag{4.11}$$

Taking the derivative of equation (4.11) with respect to z, we get

$$\nu_{\pm,z} = W_{\pm,z}(e_1) + W_{\pm,z}(\nu_{\pm}) + W_{\pm}(\nu_{\pm,z}). \tag{4.12}$$

$W_{\pm,z}$ are also a integral operators with integral kernel $P_{\pm,z}(x,s,z)$:

$$\begin{aligned}P_{\pm,z}(x,s,z) =& e^{-g_-\hat{\sigma}_3} \begin{pmatrix} 0 & 0 \\ \frac{i}{2}(1+\frac{1}{z^2})(h(x)-h(s))(\frac{-i(d+1)}{4d^2}\bar{m}_x + \frac{i\bar{m}^2 m_x}{4d^2(d+1)}) e^{\frac{i}{2}(z-\frac{1}{z})(h(x)-h(s))} & 0 \end{pmatrix} \\ & + \frac{i(h(x)-h(s))}{4zd}\left(1+\frac{1}{z^2}\right) e^{-g_-\hat{\sigma}_3} \begin{pmatrix} 0 & 0 \\ \bar{m}e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} & -i|m|^2 e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} \\ & - \frac{1}{2z^2 d} e^{-g_-\hat{\sigma}_3} \begin{pmatrix} -i|m|^2 & m \\ \bar{m}e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} & -i|m|^2 e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix}.\end{aligned} \tag{4.13}$$

Lemma 4.3. $W_{\pm}$ and $W_{\pm,z}$ are integral operators defined above, then $W_{\pm}(e_1)(x,z) \in C_B^0 \cap C^0 \cap L^2$ and $W_{\pm,z}(e_1)(x,z) \in C^0 \cap L^2$.

The proof is given in Appendix A.

Lemma 4.4. The integral operator $W_{\pm}$ maps $C_B^0 \cap C^0 \cap L^2$ to itself and $W_{\pm,z}$ is a integral operator on $C^0 \cap L^2$. In addition, $(I - W^{\pm})^{-1}$ exists as a bounded operator on $C_B^0 \cap C^0 \cap L^2$.

The proof is given in Appendix B.

From above analysis, we get

$$\begin{aligned}\|W_{\pm,z}(\nu_{\pm})\|_{C^0} &\le \|W_{\pm,z}\|_{\mathcal{B}(C^0)} \|\nu_{\pm}\|_{L^2}, \\ \|W_{\pm,z}(\nu_{\pm})\|_{L^2} &\le \|W_{\pm,z}\|_{\mathcal{B}(L^2)} \|\nu_{\pm}\|_{C_B^0},\end{aligned}$$

which implies $\nu_\pm \in C_0 \cap L_2$. So that the operator $(I-W_\pm)^{-1}$ exists, we can get the solutions of the equations (4.11) and (4.12)

$$\nu_\pm(x,z) = (I-W_\pm)^{-1}\left(W_\pm\left(e_1\right)\right)(x,z), \tag{4.14}$$

$$\nu_{\pm,z}(x,z) = (I-W_\pm)^{-1}\left(W_{\pm,z}\left(e_1\right)+W_\pm\left(\nu_{\pm,z}\right)\right)(x,z). \tag{4.15}$$

Combining the above Lemmas and the definition of $\nu_\pm$, the following properties of $\mu_\pm(x,z)$ can be directly obtained.

Proposition 4.1. Suppose that $u_0(x) \in H^{4,2}(\mathbb{R})$, then $\mu_\pm(x,z) - I \in C_B^0(1,+\infty) \cap L^2(1,+\infty)$, while its z-derivative $\mu_{\pm,z}(x,z) \in L^2(1,+\infty)$.

Proof. The steps of proof are similar to [27] and are omitted here for brevity.

Note: In fact, Proposition 4.1 still holds for $z \in (-\infty,-1)$.

As $z \in (0,1)$, we can get Proposition 4.2 via the similar analysis.

Proposition 4.2. Assume that $u_0 \in H^{(4,2)}(\mathbb{R})$, then $\mu_\pm(0,z) - I \in C_B^0(0,1) \cap L^2(0,1) \cap L^{2,1}(1,\infty)$, its z-derivative $\mu_{\pm,z}(0,z) \in L^2(0,1)$.

From (2.24), it is requisite to shown

$r(z) = \frac{b(z)}{a(z)}, \quad r'(z) = \frac{b'(z)}{a(z)} - \frac{b(z)a'(z)}{a^2(z)}, \quad zr(z) = \frac{zb(z)}{a(z)}$ in $L^2(\mathbb{R})$.

Rewrite (2.26) and (2.27) as

$$a(z) = \left(\nu_-^1(0,z)+1\right)\overline{\left(\nu_+^1(0,z)+1\right)} + \nu_-^2(0,z)\overline{\nu_+^2(0,z)}, \tag{4.16}$$

$$\overline{b(z)}e^{-\frac{i}{2}(z-\frac{1}{z})h(x,t,z)} = \nu_+^2(0,z) - \nu_-^2(0,z) + \nu_+^2(0,z)\nu_-^1(0,z) - \nu_-^2(0,z)\nu_+^1(0,z). \tag{4.17}$$

The Proposition 4.1 and 4.2 lead to the boundedness of $a(z)$, $a^{'}(z)$, $b(z)$, $b^{'}(z)$ and the L^2-integrability of $b(z)$, $b^{'}(z)$. We only have to show $zb(z) \in L^2(\mathbb{R})$. For $|z|>1$, Proposition 4.1 and 4.2 provide $zb(z) \in L^2(1,+\infty)$. For $|z|<1$,

$$\int_0^1 |z\nu_\pm(z)|^2\, dz = \int_1^{+\infty} \frac{|\nu_\pm(\gamma)|^2}{\gamma^4} d\gamma \leq \int_1^{+\infty} |\nu_\pm(\gamma)|^2\, d\gamma. \tag{4.18}$$

According to the symmetry (2.19), we can conclude that $zb(z) \in L^2(\mathbb{R})$. Theorem 4.1 has been proved.

5. Deformation of the Riemann-Hilbert problem

5.1. Renormalized Riemann-Hilbert problem

The oscillation term in the jump matrix (3.9) is

$$\theta(z) = -\frac{1}{4}\left(z - z^{-1}\right)\left[\frac{y}{t} + \frac{4}{\left(z + z^{-1}\right)^2}\right]. \tag{5.1}$$

The long-symptotic of RH problem 3.2 is affected by the growth and decay of the exponential function $e^{\pm 2it\theta}$. We convert $M(z)$ to $M^{(1)}(z)$ to ensure that $M^{(1)}(z)$ behaves well as $t \to \infty$ along any characteristic line. In order to obtain asymptotic behavior of $e^{2it\theta}$ as $t \to \infty$, we have to consider the real part of $2it\theta$:

$$\begin{aligned} &\mathrm{Re}(2it\theta) = -2t\,\mathrm{Im}\,\theta \\ &= -2t\,\mathrm{Im}\,z\left[-\frac{\xi}{4}\left(1 + |z|^{-2}\right) - \frac{-|z|^6 + 2|z|^4 + \left(3\,\mathrm{Re}^2 z - \mathrm{Im}^2 z\right)\left(1 + |z|^2\right) + 2|z|^2 - 1}{\left(\left(\mathrm{Re}^2 z - \mathrm{Im}^2 z + 1\right)^2 + 4\,\mathrm{Re}^2 z\,\mathrm{Im}^2 z\right)^2}\right], \end{aligned} \tag{5.2}$$

where $\xi = \frac{y}{t}$. In Fig.3, we divide half-plane $-\infty < y < \infty$, $t > 0$ in four space-time regions based on the signature of Imθ:

Case I: $\xi < -1$ in Fig.3 (a), there is no stationary phase point on the real axis;

Case II: $-1 < \xi \leq 0$ in Fig.3 (b), there is four stationary phase points on the real axis;

Case III: $0 < \xi < \frac{1}{8}$ in Fig.3 (c), there is eight stationary phase points on the real axis;

Case IV: $\xi > \frac{1}{8}$ in Fig.3 (d), there is no stationary phase points on the real axis.

Then we use these phase points to divide the real axis $\mathbb{R}$, denoted $\xi_0 = +\infty$, $\xi_{n(\xi)+1} = -\infty$ and introduce some interval when $j = 1, ..., n(\xi)$, where

$$n(\xi) = \begin{cases} 0, & \text{as } \xi < -1 \text{ and } \xi > \frac{1}{8}, \\ 4, & \text{as } -1 < \xi \leq 0, \\ 8, & \text{as } 0 < \xi < \frac{1}{8}, \end{cases} \tag{5.3}$$

is the number of stationary phase points for different cases.

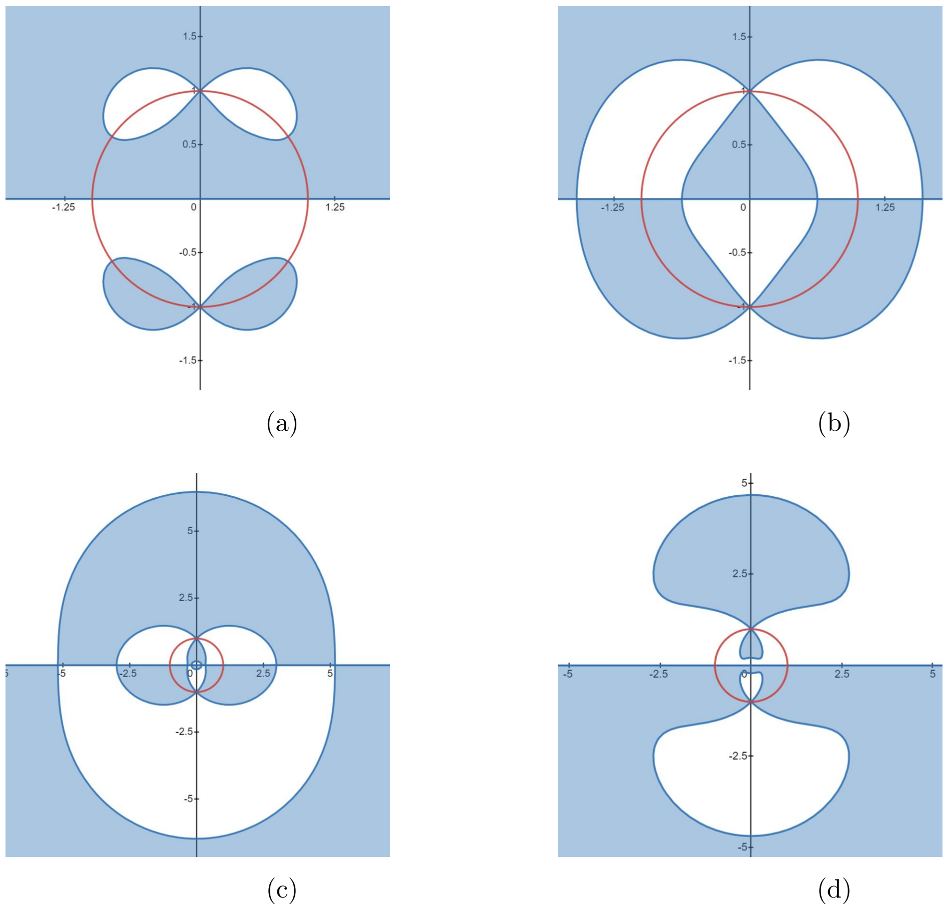


Fig.3 The classification of sign Imθ. In the white regions, Im$\theta < 0$, so $|e^{-2it\theta}| \to 0$ as $t \to \infty$. In the blue regions, Im$\theta > 0$, $|e^{2it\theta}| \to 0$ as $t \to \infty$. The red curve is unit circle. The blue curves Im$\theta = 0$ are critical lines.

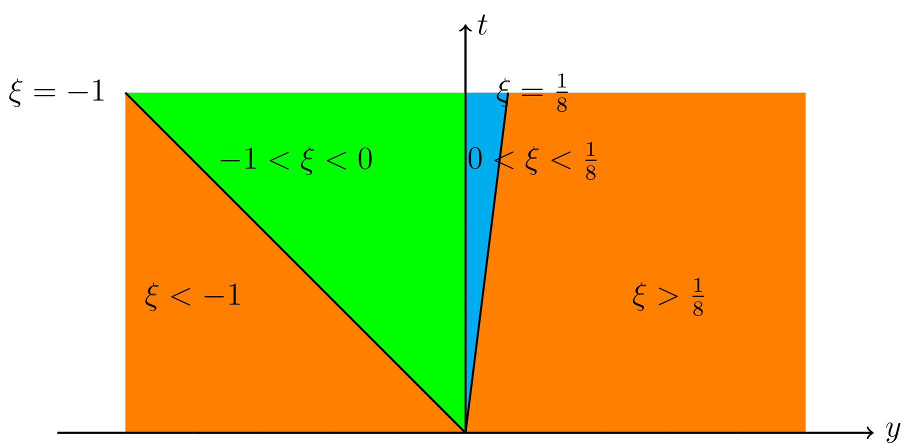


Fig.4 The half-plane $-\infty < y < \infty$, $t > 0$ is divided into four regions for cases I-IV.

Remark 5.1 : In comparison with the cubic Camassa-Holm equation [27], the positive and negative regions of the sign diagram of $\mathrm{Im}\theta$ are reversed. In addition, we know that $\xi = -1$, $\xi = 0$ and $\xi = \frac{1}{8}$ are the critical points by drawing them with the software.

As $-1 < \xi \leq 0$, we denote

$$J_{k1} = J_{k2} = \begin{cases} \left(\frac{\xi_k+\xi_{k+1}}{2}, \xi_k\right), & k \text{ is odd }, \\ \left(\xi_k, \frac{\xi_k+\xi_{k-1}}{2}\right), & k \text{ is even }, \end{cases} \tag{5.4}$$

$$J_{k3} = J_{k4} = \begin{cases} \left(\xi_k, \frac{\xi_k+\xi_{k-1}}{2}\right), & k \text{ is odd }, \\ \left(\frac{\xi_k+\xi_{k+1}}{2}, \xi_k\right), & k \text{ is even }. \end{cases} \tag{5.5}$$

As $0 < \xi < \frac{1}{8}$, we denote

$$J_{k1} = J_{k2} = \begin{cases} \left(\xi_k, \frac{\xi_k+\xi_{k-1}}{2}\right), & k \text{ is odd }, \\ \left(\frac{\xi_k+\xi_{k+1}}{2}, \xi_k\right), & k \text{ is even }, \end{cases} \tag{5.6}$$

$$J_{k3} = J_{k4} = \begin{cases} \left(\frac{\xi_k+\xi_{k+1}}{2}, \xi_k\right), & k \text{ is odd }, \\ \left(\xi_k, \frac{\xi_k+\xi_{k-1}}{2}\right), & k \text{ is even }. \end{cases} \tag{5.7}$$

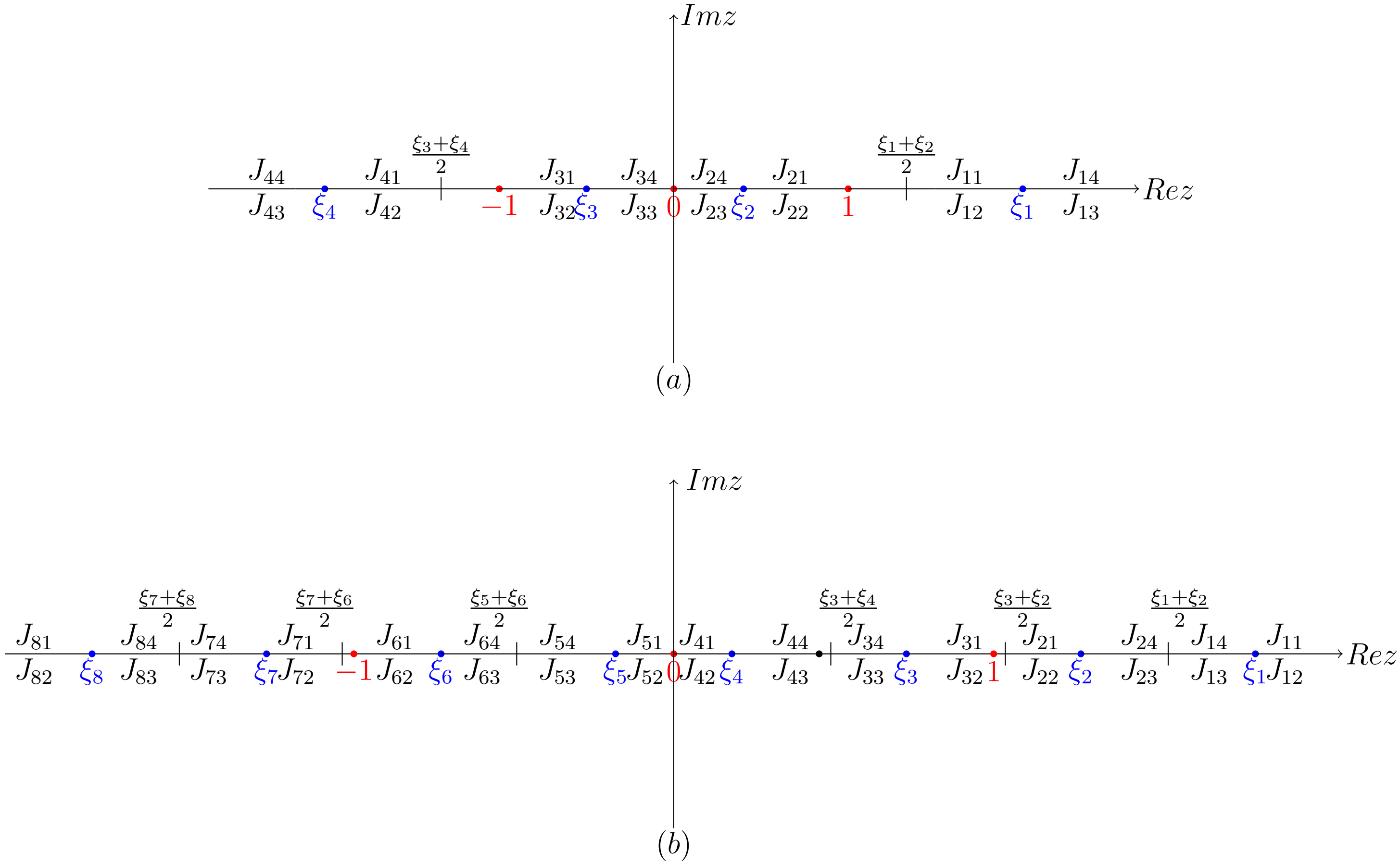


Figure 5. Panels (a) and (b) are corresponding to the $-1 < \xi \leq 0$ and $0 < \xi < \frac{1}{8}$ respectively. In (a), we note that $\xi_1 = -\xi_4 = \frac{1}{\xi_2} = -\frac{1}{\xi_3}$. And in (b), $\xi_1 = -\xi_8 = \frac{1}{\xi_4} = -\frac{1}{\xi_5}$ and $\xi_2 = -\xi_7 = \frac{1}{\xi_3} = -\frac{1}{\xi_6}$.

In the following we made a distinction between all the discrete spectral points. Take $\mathcal{N} \triangleq \{1, \ldots, 2K_1 + 2K_2 + 2K_3\}$. We give the partitions ∇, Δ and Λ of $\mathcal{N}$ based on a small positive constant δ_0:

$$\nabla = \{n \in \mathcal{N} : \operatorname{Im}\theta(\varrho_n) < 0\}, \Delta = \{n \in \mathcal{N} : \operatorname{Im}\theta(\varrho_n) > 0\}, \Lambda = \{n \in \mathcal{N} : |\operatorname{Im}\theta(\varrho_n)| \leq \delta_0\}. \tag{5.8}$$

AS $n \in \Delta$, the residue of $M(z)$ at ϱ_n in (3.12) grows as $t \to \infty$. However, the residues of $M(z)$ at ϱ_n decay as $n \in \nabla$. Denote two constants $\mathcal{N}(\Lambda) = |\Lambda|$ and

$$\rho_0 = \min_{n \in \mathcal{N} \setminus \Lambda} \{|\operatorname{Im}\theta(\varrho_n)|\} > \delta_0. \tag{5.9}$$

We denote

$\nabla_1 = \{i \in \{1, \ldots, K_1\} : \operatorname{Im}\theta(z_i) < 0\}, \Delta_1 = \{i \in \{1, \ldots, K_1\} : \operatorname{Im}\theta(z_i) > 0\}$,

$\nabla_2 = \{k \in \{1,\dots,K_2\} : \operatorname{Im}\theta(\vartheta_k) < 0\}, \Delta_2 = \{k \in \{1,\dots,K_2\} : \operatorname{Im}\theta(\vartheta_k) > 0\}$,
$\nabla_3 = \{j \in \{1,\dots,K_3\} : \operatorname{Im}\theta(\varsigma_j) < 0\}, \Delta_3 = \{j \in \{1,\dots,K_3\} : \operatorname{Im}\theta(\varsigma_j) > 0\}$,
$\Lambda_1 = \{i_0 \in \{1,\dots,K_1\} : |\operatorname{Im}\theta(z_{i_0})| \le \delta_0\}, \Lambda_2 = \left\{k_0 \in \{1,\dots,K_2\} :\mid \operatorname{Im}\theta\left(\vartheta_{k_0)}\right. \mid\le \delta_0\right\}$,
$\Lambda_3 = \left\{j_0 \in \{1,\dots,K_3\} :\mid \operatorname{Im}\theta\left(\varsigma_{l_0)}\right. \mid\le \delta_0\right\}$
to distinguish different type of zeros. For the poles ϱ_n with $n \notin \Lambda$, we have to convert them to jumps along small closed loops enclosing themselves respectively. And we also need to take the well known factorizations to restrict the jump matrix $V(z)$ in (3.9).

$$V(z) = \begin{pmatrix} 1 & 0 \\ \frac{r(z)e^{-2it\theta(z)}}{1+|r(z)|^2} & 1 \end{pmatrix} \left(1+|r(z)|^2\right)^{\sigma_3} \begin{pmatrix} 1 & \frac{\bar{r}(z)e^{2it\theta(z)}}{1+|r(z)|^2} \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma(\xi), \tag{5.10}$$

and

$$V(z) = \begin{pmatrix} 1 & \bar{r}(z)e^{2it\theta(z)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{-2it\theta(z)} & 1 \end{pmatrix}, \quad z \in \mathbb{R} \setminus \Sigma(\xi). \tag{5.11}$$

Via these factorisations, the oscillation factor $e^{\pm 2it\theta}$ decays in the corresponding regions respectively. We denote

$$\Sigma(\xi) = \begin{cases} \mathbb{R}, & \text{as } \xi < -1, \\ (-\infty, \xi_4) \cup (\xi_3, \xi_2) \cup (\xi_1, +\infty), & \text{as } -1 < \xi \le 0, \\ (\xi_8, \xi_7) \cup (\xi_6, \xi_5) \cup (\xi_4, \xi_3) \cup (\xi_2, \xi_1), & \text{as } 0 < \xi < \frac{1}{8}, \\ \emptyset, & \text{as } \xi > \frac{1}{8}. \end{cases} \tag{5.12}$$

where $\emptyset$ implies the integral interval is zero, it implies $\int_\emptyset f(x)dx = 0$ for any function $f(x)$.

In order to remove the middle diagonal matrix of the jump matrix (5.10), we need to introduce the function $\delta(z)$ and let it satisfy the following conditions

$$\delta_-(z) = \delta_+(z)(1+|r(z)|^2), \quad z \in \Sigma(\xi). \tag{5.13}$$

Further, we define

$$\delta_-(z) = \delta_+(z), \quad z \in \mathbb{R} \setminus \Sigma(\xi). \tag{5.14}$$

By using Pelemlj formula, we get

$$\delta(z) = \exp\left(-i\int_{\Sigma(\xi)} \frac{\nu(s)ds}{s-z}\right), \tag{5.15}$$

where

$$\nu(z) = -\frac{1}{2\pi}\log\left(1+|r(z)|^2\right). \tag{5.16}$$

Then we introduce the function

$$T(z)=\prod_{n\in\Delta}\frac{z-\varrho_n}{z-\bar{\varrho}_n}\delta(z,\xi), \tag{5.17}$$

and a sign

$$\eta\left(\xi,\xi_j\right)=\begin{cases}(-1)^{j+1}, & \text{as } 0<\xi<\frac{1}{8}\\ (-1)^{j}, & \text{as } -1<\xi\leq 0, \quad j=1,\cdots,n(\xi).\end{cases} \tag{5.18}$$

Proposition 5.1. The function $T(z)$ has following properties:

(a) $T(z)$ is meromorphic in $\mathbb{C}\setminus\mathbb{R}$. For each $n\in\Delta$, $T(z)$ has a simple pole at ϱ_n and a simple zero at $\bar{\varrho}_n$;

(b) $T(z)=T\left(-z^{-1}\right)=\overline{T(\bar{z})}^{-1}$;

(c) For $z\in\Sigma(\xi)$, $T(z)$ has boundary values $T_\pm(z)$ (as $z\to\mathbb{R}$ from above and below), which satisfy:

$$T_-(z)=\left(1+|r(z)|^2\right)T_+(z),\quad z\in\Sigma(\xi); \tag{5.19}$$

(d) $\lim_{z\to\infty}T(z)\triangleq T(\infty)$ with $T(\infty)=1$;

(e) As $z=i$,

$$T(i)=\prod_{i\in\Delta_1}\left(\frac{i-z_i}{i-\bar{z}_i}\right)\left(\frac{i+\frac{1}{z_i}}{i+\frac{1}{\bar{z}_i}}\right)\prod_{k\in\Delta_2}\left(\frac{i-\vartheta_k}{i-\bar{\vartheta}_k}\right)\left(\frac{i+\frac{1}{\vartheta_k}}{i+\frac{1}{\bar{\vartheta}_k}}\right)\prod_{j\in\Delta_3}\left(\frac{i-\varsigma_j}{i-\bar{\varsigma}_j}\right)\left(\frac{i+\frac{1}{\varsigma_j}}{i+\frac{1}{\bar{\varsigma}_j}}\right)\delta(i,\xi). \tag{5.20}$$

As $z\to i$, $T(z)$ has asymptotic expansion as

$$T(z)=T(i)\left(1-\Sigma_0(\xi)(z-i)\right)+\mathcal{O}\left((z-i)^2\right) \tag{5.21}$$

with

$$\Sigma_0(\xi)=\frac{1}{2\pi i}\int_{\Sigma(\xi)}\frac{\log\left(1+|r(s)|^2\right)}{(s-i)^2}ds; \tag{5.22}$$

(f) $T(z)$ is continuous at $z=0$, and

$$T(0)=T(\infty)=1; \tag{5.23}$$

(g) As $z\to\xi_j$ along any ray $z=\xi_j+e^{i\varpi}\mathbb{R}^+$ with $|\varpi|<\pi$,

$$\left|T(z,\xi)-T_j(\xi)\left(\eta\left(\xi,\xi_j\right)\left(z-\xi_j\right)\right)^{\eta(\xi,\xi_j)i\nu(\xi_j)}\right|\lesssim\|r\|_{H^1(\mathbb{R})}\left|z-\xi_j\right|^{1/2}, \tag{5.24}$$

where $T_j(\xi)$ is the complex unit

$$T_j(\xi)=\prod_{n\in\Delta}\frac{z-\varrho_n}{z-\bar{\varrho}_n}e^{i\beta_j(\xi_j,\xi)} \text{ for } j=1,...,n(\xi), \tag{5.25}$$

$$\beta_j(z,\xi)=-\int_{\Sigma(\xi)}\frac{\nu(s)}{s-z}ds-\eta\left(\xi,\xi_j\right)\log\left(\eta\left(\xi,\xi_j\right)\left(z-\xi_j\right)\right)\nu\left(\xi_j\right). \tag{5.26}$$

Proof. We can obtain (a), (b), (d) and (f) by simple calculation based on the definition of $T(z)$ in (5.17). (c) is obtained by Plemelj formula. We can obtain (e) via the Laurent expansion. And (g) is similar to [25]. Since

$$\delta(z,\xi)=\exp\left[i\beta_j(z,\xi)+i\eta\left(\xi,\xi_j\right)\log\left(\eta\left(\xi,\xi_j\right)\left(z-\xi_j\right)\right)\nu\left(\xi_j\right)\right], \tag{5.27}$$

we have

$$\left|\left(z-\xi_j\right)^{i\eta(\xi,\xi_j)\nu(\xi_j)}\right|\leq e^{-\pi\nu(\xi_j)}=\sqrt{1+\left|r\left(\xi_j\right)\right|^2}, \tag{5.28}$$

and

$$\left|\beta_j(z,\xi)-\beta_j\left(\xi_j,\xi\right)\right|\lesssim\|r\|_{H^1(\mathbb{R})}\left|z-\xi_j\right|^{1/2}.$$

Then (5.24) can be derived by computation. □

Next, we define

$$\zeta=\frac{1}{2}\min\left\{\min_{j\in\mathcal{N}}\left\{\left|\operatorname{Im}\varrho_j\right|\right\},\min_{j\in\mathcal{N}\setminus\Lambda,\operatorname{Im}\theta(z)=0}\left|\varrho_j-z\right|,\min_{j\in\mathcal{N}}\left|\varrho_j-i\right|,\min_{j,k\in\mathcal{N}}\left|\varrho_j-\varrho_k\right|\right\}.$$

According to the above definition, for every $n\in\mathcal{N}$, $D_n\triangleq D\left(\varrho_n,\zeta\right)$ are pairwise disjoint and are disjoint with $z\in\mathbb{C}|Im\theta(z)=0$ and $\mathbb{R}$. And $i\notin D_n$. Moreover, $\overline{D}_n\triangleq D\left(\overline{\varrho}_n,\zeta\right)$ have same property based on the symmetry of poles and function $\theta(z)$. Nextly, we introduce a piecewise matrix interpolation function

$$G(z)=\begin{cases}\begin{pmatrix}1 & 0\\ -c_n(z-\varrho_n)^{-1}e^{-2it\theta(\varrho_n)} & 1\end{pmatrix}, & as\ z\in D_n, n\in\nabla\setminus\Lambda;\\ \begin{pmatrix}1 & -c_n^{-1}(z-\varrho_n)e^{2it\theta(\varrho_n)}\\ 0 & 1\end{pmatrix}, & as\ z\in\ D_n, n\in\Delta\setminus\Lambda;\\ \begin{pmatrix}1 & \overline{c}_n(z-\overline{\varrho}_n)^{-1}e^{2it\theta(\overline{\varrho}_n)}\\ 0 & 1\end{pmatrix}, & as\ z\in\ \overline{D}_n,\ n\in\nabla\setminus\Lambda;\\ \begin{pmatrix}1 & 0\\ \overline{c}_n^{-1}(z-\overline{\varrho}_n)e^{-2it\theta(\overline{\varrho}_n)} & 1\end{pmatrix}, & as\ z\in\overline{D}_n, n\in\Delta\setminus\Lambda;\\ I, & as\ z\ in\ elsewhere;\end{cases} \tag{5.29}$$

By applying interpolation matrix $T(z)$ and (5.29), we define a new matrix $M^{(1)}(z)$:

$$M^{(1)}(z) \triangleq M^{(1)}(z;y,t) = M(z)G(z)T(z)^{\sigma_3}, \tag{5.30}$$

which satisfies the following RH problem.

Riemann-Hilbert Problem 5.1. Find a matrix-valued function $M^{(1)}(z)$ which satisfies

- Analyticity: $M^{(1)}(z)$ is meromorphic in $\mathbb{C}\backslash\Sigma^{(1)}$, where

$$\Sigma^{(1)} = \mathbb{R} \cup \left[\underset{n\in\mathcal{N}\backslash\Lambda}{\cup} \left(\overline{D}_n \cup D_n\right) \right], \tag{5.31}$$

- Symmetry: $M^{(1)}(z) = \sigma_2 \overline{M^{(1)}(\overline{z})} \sigma_2$;
- Jump condition: $M^{(1)}(z)$ has continuous boundary values $M_{\pm}^{(1)}(z)$ on $\Sigma^{(1)}$ and

$$M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z), z \in \Sigma^{(1)}, \tag{5.32}$$

where

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & \bar{r}(z)T^{-2}(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)T^2(z)e^{-2it\theta} & 1 \end{pmatrix}, & as\ z\ \in \mathbb{R}\setminus\Sigma(\xi); \\ \begin{pmatrix} 1 & 0 \\ \frac{r(z)T_-^2(z)e^{-2it\theta}}{1+|r(z)|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}(z)T_+^{-2}(z)e^{2it\theta}}{1+|r(z)|^2} \\ 0 & 1 \end{pmatrix}, & as\ z\ \in \Sigma(\xi); \\ \begin{pmatrix} 1 & 0 \\ -c_n(z-\varrho_n)^{-1}T^2(z)e^{-2it\theta(\varrho_n)} & 1 \end{pmatrix}, & as\ \ z \in \partial D_n, n \in \nabla\setminus\Lambda; \\ \begin{pmatrix} 1 & -c_n^{-1}(z-\varrho_n)T^{-2}(z)e^{2it\theta(\varrho_n)} \\ 0 & 1 \end{pmatrix}, & as\ \ z \in \partial D_n, n \in \Delta\setminus\Lambda; \\ \begin{pmatrix} 1 & \bar{c}_n(z-\bar{\varrho}_n)^{-1}T^{-2}(z)e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 1 \end{pmatrix}, & as\ \ z \in \partial\overline{D}_n, n \in \nabla\setminus\Lambda; \\ \begin{pmatrix} 1 & 0 \\ \bar{c}_n^{-1}(z-\bar{\varrho}_n)T^2(z)e^{-2it\theta(\bar{\varrho}_n)} & 1 \end{pmatrix}, & as\ \ z \in \partial\overline{D}_n, n \in \Delta\setminus\Lambda; \end{cases} \tag{5.33}$$

- Asymptotics

$$M^{(1)}(z;y,t) = I + O(z^{-1}), z \to \infty, \tag{5.34}$$

$$M^{(1)}(z;y,t) = e^{g_-\sigma_3}A^{-1}[I+\mu_1^0(z-i)]e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3}T(i)^{\sigma_3}\left(I-\Sigma_0(\xi)\sigma_3(z-i)\right) + \mathcal{O}\left((z-i)^2\right), z\to i; \tag{5.35}$$

• Residue conditions: $M^{(1)}(z)$ has simple poles at each point ϱ_n and $\bar{\varrho}_n$ for $n\in\Lambda$with:

$$\operatorname*{Res}_{z=\varrho_n} M^{(1)}(z;y,t) = \lim_{z\to\varrho_n} M^{(1)}(z;y,t)\begin{pmatrix} 0 & 0 \\ c_nT^2(\varrho_n)e^{-2it\theta(\varrho_n)} & 0 \end{pmatrix}, \tag{5.36}$$

$$\operatorname*{Res}_{z=\bar{\varrho}_n} M^{(1)}(z;y,t) = \lim_{z\to\bar{\varrho}_n} M^{(1)}(z;y,t)\begin{pmatrix} 0 & -\bar{c}_nT^{-2}(\bar{\varrho}_n)e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 0 \end{pmatrix}. \tag{5.37}$$

Proof. We omit it here for briefly, reader is referred to [27] for details. □

Remark 5.2 : Compared to the cubic Camassa-Holm equation, the symmetries and asymptotic expressions of solution to the RH problem are different, as are the residue conditions at the poles.

5.2. Contour deformation

In this subsection, we mainly remove the jump contour $\mathbb{R}$ of the jump matrix $V^{(1)}(z)$. Then the new problem is considered the decay/growth of $e^{2it\theta(z)}$ for $z\notin\mathbb{R}$. Based on the above, we introduce some new regions and contours relayed on ξ.

1. **For the region** $\xi\in(-\infty,-1)\cup(\frac{1}{8},\infty)$.

We denote

$$\Omega_{2n+1} = \{z\in\mathbb{C} \mid n\pi \le \arg z \le n\pi+\varpi\},$$
$$\Omega_{2n+2} = \{z\in\mathbb{C} \mid (n+1)\pi-\varpi \le \arg z \le (n+1)\pi\},$$

where $n=0,1$. And

$$\Sigma_k = e^{(k-1)i\pi/2+i\varpi}R_+, \quad k=1,3,$$
$$\Sigma_k = e^{ki\pi/2-i\varpi}R_+, \quad k=2,4,$$

is the boundary of Ω_k respectively. Futhermore, we take

$$\Omega(\xi) = \underset{k=1,\dots,4}{\cup}\Omega_k,$$
$$\Sigma^{(2)}(\xi) = \cup_{n\in\mathcal{N}\setminus\Lambda}\left(\partial\overline{D}_n\cup\partial D_n\right), \breve{\Sigma}(\xi) = \Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\Sigma_4.$$

Then we denote $\varpi\in(0,\frac{\pi}{8})$ to be a sufficiently small fixed angle, so that following conditions hold:

1.$-\frac{\xi}{4}-\frac{1}{2(cos2\varpi+1)}>0$, i.e. $\cos2\varpi > -\frac{2}{\xi}-1$ for $\xi\in(-\infty,-1)$;

2.$-\frac{\xi}{4}+\frac{1}{16(cos2\varpi+1)}<0$, i.e. $\cos2\varpi > \frac{1}{4\xi}-1$ for $\xi\in(\frac{1}{8},+\infty)$;

3. each Ω_i doesn't intersect $z \in \mathbb{C} | \operatorname{Im}\theta(z) = 0$ and any of D_n or $\overline{D}_n$.

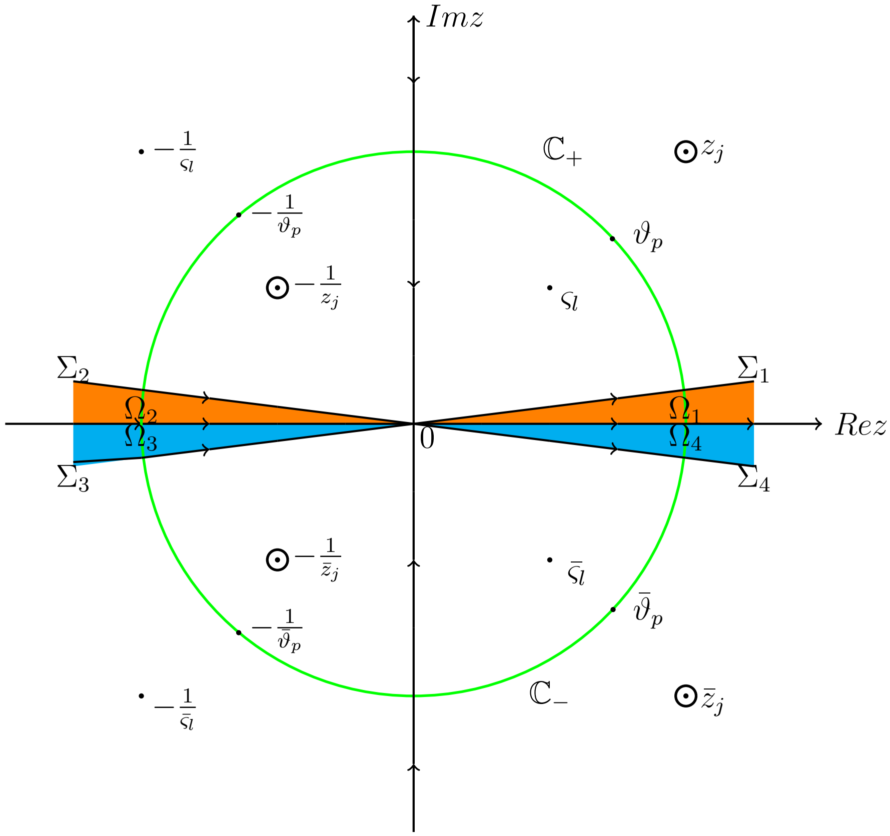


Fig.6 In the cases $\xi < -1$ and $\xi > \frac{1}{8}$, there is no stationary phase points. Regions $\Omega_1(\xi)$ and $\Omega_2(\xi)$ have same decay/grow properties; the same for regions $\Omega_3(\xi)$ and $\Omega_4(\xi)$. We note that the black circle around poles don't intersect with critical line $\operatorname{Im}\theta(z) = 0$ and boundary $\Sigma^{(2)}(\xi)$.

Lemma 5.1. Assume $p(s) = s + \frac{1}{s}$ is a real-valued function for $x \in \mathbb{R}$, then the imaginary part of phase function $\operatorname{Im}\theta(z)$ defined by (5.2) have following estimation:

For $\xi \in (-\infty, -1)$,

$$\operatorname{Im}\theta(z) \leq -|\sin\varpi| p(|z|) \left(-\frac{\xi}{4} - \frac{1}{2(\cos 2\varpi + 1)}\right), \quad \text{as } z \in \Omega_3, \Omega_4; \tag{5.38}$$

$$\operatorname{Im}\theta(z) \geq |\sin\varpi| p(|z|) \left(-\frac{\xi}{4} - \frac{1}{2(\cos 2\varpi + 1)}\right), \quad \text{as } z \in \Omega_1, \Omega_2. \tag{5.39}$$

For $\xi \in (\frac{1}{8}, +\infty)$,

$$\operatorname{Im}\theta(z) \leq |\sin\varpi| p(|z|) \left(-\frac{\xi}{4} + \frac{1}{16(\cos 2\varpi + 1)}\right), \quad \text{as } z \in \Omega_1, \Omega_2; \tag{5.40}$$

$$\operatorname{Im}\theta(z) \geq -|\sin\varpi|p(|z|)\left(-\frac{\xi}{4}+\frac{1}{16(\cos 2\varpi+1)}\right), \qquad \text{as } z\in\Omega_3,\Omega_4. \tag{5.41}$$

Proof. Here, we take $z\in\Omega_1$ as an example, and the other regions are proved by the similar method. Let $z=le^{i\varsigma}$, from (5.2), we rewrite $\operatorname{Im}\theta(z)$ as

$$\operatorname{Im}\theta(z)=p(|z|)\sin\varsigma\left(-\tfrac{\xi}{4}-\tfrac{2\cos 2\varsigma+6-p(|z|)^2}{(p(|z|)^2+2\cos 2\varsigma-2)^2}\right).$$

Denote

$$w(x,\varsigma)=\tfrac{2\cos 2\varsigma+6-p(|z|)^2}{(p(|z|)^2+2\cos 2\varsigma-2)^2},$$

then, since $x>4,\cos 2\varsigma<1$, we have

$$w(x,\varsigma)\in(-\tfrac{1}{16(\cos 2\varsigma+1)},\tfrac{1}{2(\cos 2\varsigma+1)}),$$

which implies the result. $\square$

Corollary 5.1. There exists a constant $\kappa(\xi)>0$ (respect to ξ), which ensure that the imaginary part of phase function (5.2) Im $\theta(z)$ has following property as $z=le^{i\varsigma}=\epsilon+i\nu$:

For $\xi\in(-\infty,-1)$,

$$\operatorname{Im}\theta(z)\geq\kappa(\xi)|\nu|, z\in\Omega_1,\Omega_2; \tag{5.42}$$

$$\operatorname{Im}\theta(z)\leq-\kappa(\xi)|\nu|, z\in\Omega_3,\Omega_4. \tag{5.43}$$

For $\xi\in(\frac{1}{8},+\infty)$,

$$\operatorname{Im}\theta(z)\geq\kappa(\xi)|\nu|, z\in\Omega_3,\Omega_4; \tag{5.44}$$

$$\operatorname{Im}\theta(z)\leq-\kappa(\xi)|\nu|, z\in\Omega_1,\Omega_2. \tag{5.45}$$

Lemma 5.2. There exists a constant $\kappa(\xi)>0$ relied on $\xi\in(-1,\frac{1}{8})$ that the imaginary part of phase function (5.2) $\operatorname{Im}\theta(z)$ has following estimation for $i=1,...,n(\xi)$:

$$\operatorname{Im}\theta(z)\geq\kappa(\xi)\operatorname{Im}z\frac{|z|^2-\xi_i^2}{4+|z|^2}, \qquad \text{as } z\in\Omega_{i2},\Omega_{i4}; \tag{5.46}$$

$$\operatorname{Im}\theta(z)\leq-\kappa(\xi)\operatorname{Im}z\frac{|z|^2-\xi_i^2}{4+|z|^2}, \qquad \text{as } z\in\Omega_{i1},\Omega_{i3}. \tag{5.47}$$

We then introduce functions $R^{(2)}(z,\xi)$ to remove jump on $\mathbb{R}$.

For $\xi\in(-\infty,-1)$,

$$R^{(2)}(z,\xi)=\begin{cases}\begin{pmatrix}1 & 0\\ R_j(z,\xi)e^{-2it\theta} & 1\end{pmatrix}, & z\in\Omega_j, j=3,4\\ \begin{pmatrix}1 & R_j(z,\xi)e^{2it\theta}\\ 0 & 1\end{pmatrix}, & z\in\Omega_j, j=1,2\\ I, & \text{elsewhere}\end{cases} \tag{5.48}$$

For $\xi \in (\frac{1}{8}, +\infty)$,

$$R^{(2)}(z,\xi) = \begin{cases} \begin{pmatrix} 1 & R_j(z,\xi)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_j, j = 3,4; \\ \begin{pmatrix} 1 & 0 \\ R_j(z,\xi)e^{-2it\theta} & 1 \end{pmatrix}, & z \in \Omega_j, j = 1,2; \\ I, & \text{elsewhere}; \end{cases} \tag{5.49}$$

where the functions R_j, $j = 1,2,3,4$, are defined in following Proposition.

Proposition 5.2. $R_j : \overline{\Omega}_j \to \mathbb{C}$, $j = 1,2,3,4$ have boundary values as follows:

For $\xi \in (-\infty, -1)$,

$$R_1(z,\xi) = \begin{cases} f_1(z,\xi)T_+^{-2}(z) & z \in \mathbb{R}^+, \\ 0 & z \in \Sigma_1, \end{cases} \quad R_2(z,\xi) = \begin{cases} 0 & z \in \Sigma_2, \\ f_2(z,\xi)T_+^{-2}(z) & z \in \mathbb{R}^-, \end{cases}$$

$$R_3(z,\xi) = \begin{cases} f_3(z,\xi)T_-^{2}(z) & z \in \mathbb{R}^-, \\ 0 & z \in \Sigma_3, \end{cases} \quad R_4(z,\xi) = \begin{cases} 0 & z \in \Sigma_4, \\ f_4(z,\xi)T_-^{2}(z) & z \in \mathbb{R}^+. \end{cases}$$

For $\xi \in (\frac{1}{8}, +\infty)$,

$$R_1(z,\xi) = \begin{cases} f_1(z,\xi)T^{2}(z) & z \in \mathbb{R}^+, \\ 0 & z \in \Sigma_1, \end{cases} \quad R_2(z,\xi) = \begin{cases} 0 & z \in \Sigma_2, \\ f_2(z,\xi)T^{2}(z) & z \in \mathbb{R}^-, \end{cases}$$

$$R_3(z,\xi) = \begin{cases} f_3(z,\xi)T^{-2}(z) & z \in \mathbb{R}^-, \\ 0 & z \in \Sigma_3, \end{cases} \quad R_4(z,\xi) = \begin{cases} 0 & z \in \Sigma_4, \\ f_4(z,\xi)T^{-2}(z) & z \in \mathbb{R}^+. \end{cases}$$

where

$$f_1(z,\xi) = f_2(z,\xi) = \begin{cases} -\frac{\bar{r}(z)}{1+|r(z)|^2}, & \text{for } \xi < -1, \\ -r(z), & \text{for } \xi > \frac{1}{8}, \end{cases} \tag{5.50}$$

$$f_3(z,\xi) = f_4(z,\xi) = \begin{cases} \frac{r(z)}{1+|r(z)|^2}, & \text{for } \xi < -1, \\ \bar{r}(z), & \text{for } \xi > \frac{1}{8}. \end{cases} \tag{5.51}$$

R_j have following property: for $j = 1,2,3,4$,

$$\left|\bar{\partial} R_j(z)\right| \lesssim \left|f_j'(|z|)\right| + |z|^{-1/2}, \text{ for all } z \in \Omega_j, \tag{5.52}$$

in addition,

$$\begin{aligned} &\left|\bar{\partial} R_j(z)\right| \lesssim \left|f_j'(|z|)\right| + |z|^{-1}, \text{ for all } z \in \Omega_j, \\ &\bar{\partial} R_j(z) = 0, \quad \text{if } z \in \text{ elsewhere}. \end{aligned} \tag{5.53}$$

Proof. We omit it here for briefly, the reader is referred to [27] for details. □

2. **For the region** $\xi \in (-1, \frac{1}{8})$.

We introduce deformation contours and domains

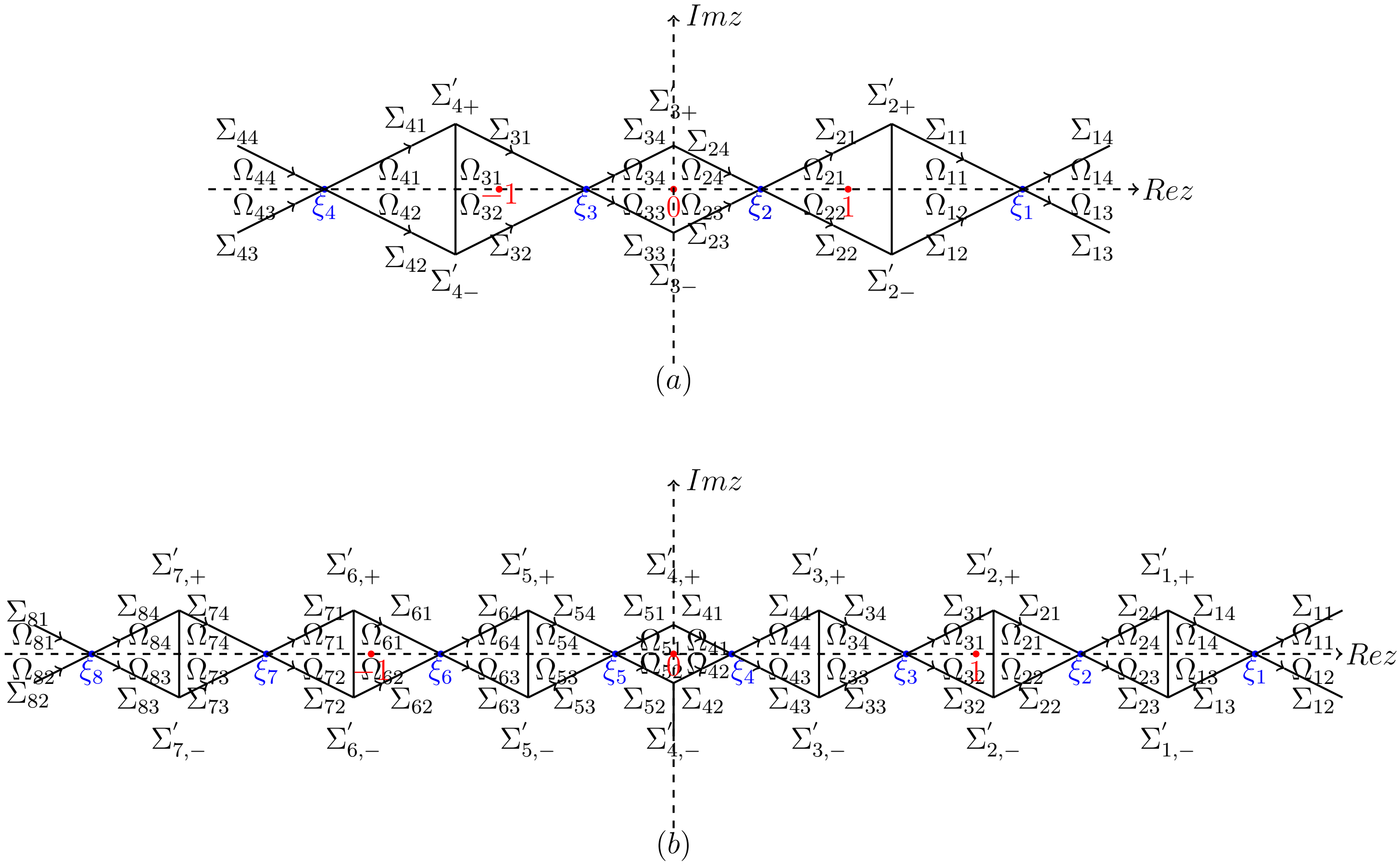


Figure 7. The stationary phase points of phase function $\theta(z)$. Panel (a): $-1 < \xi \leq 0$ with four phase points; panel (b):$0 < \xi < \frac{1}{8}$ with eight phase points. Σ_{ij} separate complex plane $\mathbb{C}$ into regions Ω_{ij}.

$$\Sigma_{jk}(\xi) = \begin{cases} \xi_j + e^{i\left[(\frac{k}{2}+\frac{1}{2}+j)\pi+(-1)^{j+1}\varpi\right]}l, & 0 < \xi < \frac{1}{8}, \\ \xi_j + e^{i\left[(\frac{k}{2}-\frac{1}{2}+j)\pi+(-1)^{j}\varpi\right]}l, & -1 < \xi \leq 0, \end{cases} \qquad k = 1, 3, \tag{5.54}$$

$$\Sigma_{jk}(\xi) = \begin{cases} \xi_j + e^{i\left[(\frac{k}{2}+j)\pi+(-1)^{j}\varpi\right]}l, & 0 < \xi < \frac{1}{8}, \\ \xi_j + e^{i\left[(\frac{k}{2}-1+j)\pi+(-1)^{j+1}\varpi\right]}l, & -1 < \xi \leq 0, \end{cases} \qquad k = 2, 4, \tag{5.55}$$

where $l \in \left(0, \frac{|\xi_{j+(-1)^j} - \xi_j|}{2cos\varpi}\right)$, $j = 2, \ldots, n(\xi) - 1$. For $j = 1$ or $n(\xi)$

$$\Sigma_{j1}(\xi)=\begin{cases}\xi_j+e^{(1+j)\pi i+(-1)^{j+1}i\varpi}\mathbb{R}^+, & 0<\xi<\frac{1}{8},\\ \xi_j+e^{j\pi i+(-1)^{j}i\varpi}l, & -1<\xi\leq 0,\end{cases}$$
$$\Sigma_{j2}(\xi)=\begin{cases}\xi_j+e^{(1+j)\pi i+(-1)^{j}i\varpi}\mathbb{R}^+, & 0<\xi<\frac{1}{8},\\ \xi_j+e^{j\pi i+(-1)^{j+1}i\varpi}l, & -1<\xi\leq 0,\end{cases}$$
$$\Sigma_{j3}(\xi)=\begin{cases}\xi_j+e^{j\pi i+(-1)^{j+1}i\varpi}l, & 0<\xi<\frac{1}{8},\\ \xi_j+e^{(1+j)\pi i+(-1)^{j}i\varpi}\mathbb{R}^+, & -1<\xi\leq 0,\end{cases}$$
$$\Sigma_{j4}(\xi)=\begin{cases}\xi_j+e^{j\pi i+(-1)^{j}i\varpi}l, & 0<\xi<\frac{1}{8},\\ \xi_j+e^{(1+j)\pi i+(-1)^{j+1}i\varpi}\mathbb{R}^+, & -1<\xi\leq 0.\end{cases}$$
$$\Sigma'_{j\pm}=\begin{cases}\frac{\xi_{j+1}+\xi_j}{2}+e^{i\pi}l, & 0<\xi<\frac{1}{8}, j=1,\dots,n(\xi)-1\\ \frac{\xi_{j-1}+\xi_j}{2}-e^{i\pi}l, & -1<\xi\leq 0,\quad j=2,\dots,n(\xi)\end{cases}$$

We denote

$$\breve{\Sigma}(\xi)=\left(\underset{k=1,\dots,4,j=1,..,n(\xi)}{\cup}\Sigma_{jk}\right)\cup\left(\underset{j=1,\dots,n(\xi)}{\cup}\Sigma'_{j\pm}\right),$$
$$\Sigma^{(2)}(\xi)=\breve{\Sigma}(\xi)\underset{n\in\mathcal{N}\setminus\Lambda}{\cup}\left(\partial\overline{D}_n\cup\partial D_n\right),$$
$$\Omega(\xi)=\underset{k=1,\dots,4,j=1,..,n(\xi)}{\cup}\Omega_{jk},\quad \Omega_\pm(\xi)=\mathbb{C}\setminus\Omega.$$

Denote $\Sigma'_{n(\xi)\pm}=\emptyset$ as $0<\xi<\frac{1}{8}$ and $\Sigma'_{1\pm}=\emptyset$ as $-1<\xi\leq 0$ for convenience. $\varsigma\in\left(0,\frac{\pi}{8}\right)$ is an fixed sufficiently small angle which ensure following conditions hold:

1. each Ω_i doesn't intersect $\{z\in\mathbb{C}; Im\theta(z)=0\}$ and any of D_n or $\overline{D}_n$,
2. $2\tan\varpi>\xi_{n(\xi)/2}-\xi_{n(\xi)/2+1}$.

This contours separate complex plane $\mathbb{C}$ into regions Ω_{ij}, as depicted in Fig. 7.

In this case, we denote $R^{(2)}(z,\xi)$ as

$$R^{(2)}(z,\xi)=\begin{cases}\begin{pmatrix}1 & R_{kj}(z,\xi)e^{2it\theta}\\ 0 & 1\end{pmatrix}, & z\in\Omega_{kj}, k=1,\dots,n(\xi), j=2,4;\\ \begin{pmatrix}1 & 0\\ R_{kj}(z,\xi)e^{-2it\theta} & 1\end{pmatrix}, & z\in\Omega_{kj}, k=1,\dots,n(\xi), j=1,3;\\ I, & \text{elsewhere };\end{cases}\tag{5.56}$$

so that the functions $R_{kj}(z,\xi)$, $k=1,...,n(\xi)$, $j=1,2,3,4$ have following property.

Proposition 5.3. As $\xi\in(-1,0]$ and $\xi\in\left(0,\frac{1}{8}\right)$, the functions $R_{kj}:\Omega_{kj}\to\mathbb{C}$, $j=1,2,3,4,k=1,...,n(\xi)$ have boundary values :

$$R_{k1}(z,\xi)=\begin{cases}f_{k1}(z,\xi)T^2(z), & z\in J_{k1},\\ f_{k1}\left(\xi_k,\xi\right)T_k^2(\xi)\left(\eta(\xi,k)\left(z-\xi_k\right)\right)^{2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k1},\end{cases}\tag{5.57}$$

$$R_{k2}(z,\xi)=\begin{cases} f_{k2}(z,\xi)T^{-2}(z), & z\in J_{k2},\\ f_{k2}\left(\xi_k,\xi\right)T_k^{-2}(\xi)\left(\eta(\xi,k)\left(z-\xi_k\right)\right)^{-2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k2}, \end{cases} \tag{5.58}$$

$$R_{k3}(z,\xi)=\begin{cases} f_{k3}(z,\xi)T_-^{2}(z), & z\in J_{k3},\\ f_{k3}\left(\xi_k,\xi\right)T_k^{2}(\xi)\left(\eta(\xi,k)\left(z-\xi_k\right)\right)^{2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k3}, \end{cases} \tag{5.59}$$

$$R_{k4}(z,\xi)=\begin{cases} f_{k4}(z,\xi)T_+^{-2}(z), & z\in J_{k4},\\ f_{k4}\left(\xi_k,\xi\right)T_k^{-2}(\xi)\left(\eta(\xi,k)\left(z-\xi_k\right)\right)^{-2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k4}, \end{cases} \tag{5.60}$$

where J_{kj} are defined in (5.4)-(5.7) and

$$f_{k1}(z,\xi)=-r(z),\quad f_{k2}(z,\xi)=\bar{r}(z), \tag{5.61}$$

$$f_{k3}(z,\xi)=\frac{r(z)}{1+|r(z)|^2},\quad f_{k4}(z,\xi)=-\frac{\bar{r}(z)}{1+|r(z)|^2}. \tag{5.62}$$

The functions R_{kj} hold following conditions:

$$|R_{kj}(z,\xi)|\lesssim\sin^2\left(k_0\arg\left(z-\xi_k\right)\right)+\left(1+\mathrm{Re}(z)^2\right)^{-1/2},\ \text{for all } z\in\Omega_{kj}, \tag{5.63}$$

$$\left|\bar{\partial}R_{kj}(z,\xi)\right|\lesssim|\ f_{kj}'(Rez)|+|z-\xi_k|^{-1/2},\ for\ all\ z\in\Omega_{kj}. \tag{5.64}$$

$$\bar{\partial}R_{kj}(z,\xi)=0,\quad if\ z\ at\ elsewhere. \tag{5.65}$$

Proof. We omit it here for briefly, the reader is referred to [27] for details. □

Then we can introduce a new matrix valued function $M^{(2)}(z)$ by transformation with $R^{(2)}$ that

$$M^{(2)}(z)=M^{(1)}(z)R^{(2)}(z). \tag{5.66}$$

Notably that $M^{(2)}(z)$ satisfies the following mixed $\bar{\partial}$-RH problem.

Riemann-Hilbert Problem 5.2. Find a matrix valued function $M^{(2)}(z)$ with following properties:

- Analyticity: $M^{(2)}(z)$ is continuous in $\mathbb{C}$, sectionally continuous first partial derivatives in $\mathbb{C}\backslash(\Sigma^{(2)}\cup(\varrho_n,\overline{\varrho}_n)_{n\in\Lambda})$ and meromorphic out $\overline{\Omega}$;
- Symmetry: $M^{(2)}(z)=e^{-g-\widehat{\sigma}_3}A^{-2}\sigma_3M^{(2)}(-1/z)\sigma_3$;
- Jump condition: $M^{(2)}(z)$ has continuous boundary values $M_\pm^{(2)}(z)$ on $\Sigma^{(2)}$ and

$$M_+^{(2)}(z)=M_-^{(2)}(z)V^{(2)}(z),\ z\in\Sigma^{(2)}, \tag{5.67}$$

For $\xi \in (-\infty, -1)$ or $\xi \in (\frac{1}{8}, +\infty)$

$$V^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -c_n(z-\varrho_n)^{-1}T^2(z)e^{-2it\theta(\varrho_n)} & 1 \end{pmatrix}, & as\ z \in \partial D_n, n \in \nabla; \\ \begin{pmatrix} 1 & -c_n^{-1}(z-\varrho_n)T^{-2}(z)e^{2it\theta(\varrho_n)} \\ 0 & 1 \end{pmatrix}, & as\ z \in\ \partial D_n, n \in \Delta; \\ \begin{pmatrix} 1 & \bar{c}_n(z-\bar{\varrho}_n)^{-1}T^{-2}(z)e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 1 \end{pmatrix}, & as\ z \in\ \partial \overline{D}_n,\ n \in \nabla; \\ \begin{pmatrix} 1 & 0 \\ \bar{c}_n^{-1}(z-\bar{\varrho}_n)T^2(z)e^{-2it\theta(\bar{\varrho}_n)} & 1 \end{pmatrix}, & as\ z \in \partial \overline{D}_n, n \in \Delta; \end{cases} \tag{5.68}$$

For $\xi \in (-1, \frac{1}{8})$,

$$V^{(2)}(z) = \begin{cases} R^{(2)}(z)^{-1}|_{\Sigma_{k1}\cup\Sigma_{k4}} & as\ z \in \Sigma_{k1}\cup\Sigma_{k4}; \\ R^{(2)}(z)|_{\Sigma_{k2}\cup\Sigma_{k3}} & as\ z \in \Sigma_{k2}\cup\Sigma_{k3}; \\ R^{(2)}(z)^{-1}|_{\Sigma_{k(7\pm1/2)}} R^{(2)}(z)|_{\Sigma_{(k-1)(7\pm1/2)}} & as\ z \in \Sigma'_{k\pm},\ k\ is\ even; \\ R^{(2)}(z)^{-1}|_{\Sigma_{k(3\mp1/2)}} R^{(2)}(z)|_{\Sigma_{(k-1)(3\mp1/2)}} & as\ z \in \Sigma'_{k\pm},\ k\ is\ odd; \\ \begin{pmatrix} 1 & 0 \\ -c_n(z-\varrho_n)^{-1}T^2(z)e^{-2it\theta(\varrho_n)} & 1 \end{pmatrix}, & as\ z \in \partial D_n, n \in \nabla; \\ \begin{pmatrix} 1 & -c_n^{-1}(z-\varrho_n)T^{-2}(z)e^{2it\theta(\varrho_n)} \\ 0 & 1 \end{pmatrix}, & as\ z \in\ \partial D_n, n \in \Delta; \\ \begin{pmatrix} 1 & \bar{c}_n(z-\bar{\varrho}_n)^{-1}T^{-2}(z)e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 1 \end{pmatrix}, & as\ z \in\ \partial \overline{D}_n,\ n \in \nabla; \\ \begin{pmatrix} 1 & 0 \\ \bar{c}_n^{-1}(z-\bar{\varrho}_n)T^2(z)e^{-2it\theta(\bar{\varrho}_n)} & 1 \end{pmatrix}, & as\ z \in \partial \overline{D}_n, n \in \Delta; \end{cases} \tag{5.69}$$

- Asymptotics:

$$M^{(2)}(z) = I + \mathcal{O}\left(z^{-1}\right), \quad z \to \infty, \tag{5.70}$$

$$\begin{aligned} M^{(2)}(z) =& e^{g_-\sigma_3}A^{-1}\left[I + \mu_1^0(z-i)\right]e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3} \\ & \left(T(i)^{\sigma_3}\left(I - \Sigma_0(\xi)(z-i)\sigma_3\right) + \mathcal{O}\left((z-i)^2\right)\right) + \mathcal{O}\left((z-i)^2\right), \quad z \to i. \end{aligned} \tag{5.71}$$

• $\bar{\partial}$-Derivative: For $z \in \mathbb{C}$, we have

$$\bar{\partial} M^{(2)} = M^{(2)} \bar{\partial} R^{(2)}, \tag{5.72}$$

for $\xi \in (\frac{1}{8}, +\infty)$,

$$\bar{\partial} R^{(2)}(z,\xi) = \begin{cases} \begin{pmatrix} 0 & 0 \\ \bar{\partial} R_j(z,\xi) e^{-2it\theta} & 0 \end{pmatrix}, & z \in \Omega_j, j = 1,2; \\ \begin{pmatrix} 0 & \bar{\partial} R_j(z,\xi) e^{2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_j, j = 3,4; \\ 0, & \text{elsewhere}; \end{cases} \tag{5.73}$$

for $\xi \in (-\infty, -1)$,

$$\bar{\partial} R^{(2)}(z,\xi) = \begin{cases} \begin{pmatrix} 0 & \bar{\partial} R_j(z,\xi) e^{2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_j, j = 1,2; \\ \begin{pmatrix} 0 & 0 \\ \bar{\partial} R_j(z,\xi) e^{-2it\theta} & 0 \end{pmatrix}, & z \in \Omega_j, j = 3,4; \\ 0, & \text{elsewhere}; \end{cases} \tag{5.74}$$

for $\xi \in (-1, 0] \cup (0, \frac{1}{8})$,

$$\bar{\partial} R^{(2)}(z,\xi) = \begin{cases} \begin{pmatrix} 0 & \bar{\partial} R_{kj}(z,\xi) e^{2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_{kj}, j = 2,4; \\ \begin{pmatrix} 0 & 0 \\ \bar{\partial} R_{kj}(z,\xi) e^{-2it\theta} & 0 \end{pmatrix}, & z \in \Omega_{kj}, j = 1,3; \\ 0, & \text{elsewhere}; \end{cases} \tag{5.75}$$

• Residue condition: $M^{(2)}$ has simple poles at each point ϱ_n and $\bar{\varrho}_n$ for $n \in \Lambda$ with:

$$\operatorname*{Res}_{z=\varrho_n} M^{(2)}(z) = \lim_{z \to \varrho_n} M^{(2)}(z) \begin{pmatrix} 0 & 0 \\ c_n T^2(\varrho_n) e^{-2it\theta(\varrho_n)} & 0 \end{pmatrix}, \tag{5.76}$$

$$\operatorname*{Res}_{z=\bar{\varrho}_n} M^{(2)}(z) = \lim_{z \to \bar{\varrho}_n} M^{(2)}(z) \begin{pmatrix} 0 & -\bar{c}_n T^{-2}(\bar{\varrho}_n) e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 0 \end{pmatrix}. \tag{5.77}$$

5.3. Decomposition of mixed $\overline{\partial}$-Riemann Hilbert problem

In this subsection, we have to decompose the $M^{(2)}(z)$ into a model RH problem for $M^R(z)$ with $\bar{\partial} R^{(2)} \equiv 0$ and a pure $\bar{\partial}$-Problem with $\bar{\partial} R^{(2)} \neq 0$. Firstly, we introduce a RH problem for the matrix-valued function $M^R(z)$.

Riemann-Hilbert Problem 5.3. Find a matrix-valued function $M^R(z)$ with following properties:

• Analyticity: $M^R(z)$ is meromorphic in $\mathbb{C}\backslash\Sigma^{(2)}$;

• Jump condition: $M^R(z)$ has continuous boundary values $M^R_{\pm}(z)$ on $\Sigma^{(2)}$ and

$$M_+^{(R)}(z) = M_-^{(R)}(z)V^{(2)}(z), \quad z\in\Sigma^{(2)}; \tag{5.78}$$

• Symmetry: $M^R(z)=e^{-g_-\hat{\sigma}_3}A^{-2}\sigma_3M^R(-1/z)\sigma_3$;

• $\bar{\partial}$-derivative: $\bar{\partial}R^{(2)}(z)=0$, for $z\in\mathbb{C}$;

• Asymptotic behaviors:

$$M^R(z) = I+\mathcal{O}\left(z^{-1}\right), \quad z\to\infty, \tag{5.79}$$

$$\begin{aligned} M^R(z) =& e^{g_-\sigma_3}A^{-1}\left[I+\mu_1^0(z-i)\right]e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3} \\ &\cdot T(i)^{\sigma_3}\left(I-\Sigma_0(\xi)\sigma_3(z-i)+\mathcal{O}\left((z-i)^2\right)\right)+\mathcal{O}\left((z-i)^2\right); \end{aligned} \tag{5.80}$$

• Residue conditions: $M^R(z)$ has simple poles at each point ϱ_n and $\bar{\varrho}_n$ for $n\in\Lambda$ with

$$\operatorname*{Res}_{z=\varrho_n} M^{(R)}(z) = \lim_{z\to\varrho_n} M^{(R)}(z)\begin{pmatrix} 0 & 0 \\ c_nT^2\left(\varrho_n\right)e^{-2it\theta(\varrho_n)} & 0\end{pmatrix}, \tag{5.81}$$

$$\operatorname*{Res}_{z=\bar{\varrho}_n} M^{(R)}(z) = \lim_{z\to\bar{\varrho}_n} M^{(R)}(z)\begin{pmatrix} 0 & -\bar{c}_nT^{-2}\left(\bar{\varrho}_n\right)e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 0\end{pmatrix}. \tag{5.82}$$

Denote $U(n(\xi))$ as the union set of neighborhood of ξ_j for $j=1,...,n(\xi)$

$$U(n(\xi)) = \underset{k=1,...,n(\xi)}{\cup} U_{\xi_k}, U_{\xi_k} = \left\{z:|z-\xi_j|\leq\min\left\{\zeta,\tfrac{1}{3}\min_{j\neq i\in\mathcal{N}}|\varrho_i-\varrho_j|\right\}\right\},$$

where $n(\xi)=0,4,8$ correspond to three cases $\xi\in(-\infty,-1)\cup(\frac{1}{8},\infty)$, $\xi\in(-1,0)$ and $\xi\in(0,\frac{1}{8})$ respectively. As $n(\xi)=0$, there is no phase point and $U(n(\xi))=\emptyset$. The jump matrix $V^{(2)}(z)$ outside of $U(n(\xi))$ holds following conditions.

Proposition 5.4. For $1\leq p\leq+\infty$, there exists a p-dependent positive constant τ_p such that the jump matrix $V^{(2)}(z)$ defined in (5.69) satisfies

$$\left\|V^{(2)}(z)-I\right\|_{L^p\left(\Sigma_{kj}\backslash U_{\xi_k}\right)} = \mathcal{O}\left(e^{-\tau_pt}\right), \qquad t\to\infty, \tag{5.83}$$

for $k=1,...,n(\xi)$ and $j=1,...,4$. There also exists a p-dependent positive constant τ_p' such that

$$\left\|V^{(2)}(z)-I\right\|_{L^p\left(\Sigma'_{k\pm}\right)} = \mathcal{O}\left(e^{-\tau_p't}\right), \quad t\to\infty, \tag{5.84}$$

for $k = 1, \ldots, n(\xi)$.

Proof. Here, we prove the case $\xi \in (0, \frac{1}{8})$, and the another case can be obtained by similar method. As $z \in \Sigma_{14} \backslash U_{\xi_1}$, $1 \leq p \leq +\infty$, from definition of $V^{(2)}(z)$ and (5.63),

$$\begin{aligned} \left\| V^{(2)}(z) - I \right\|_{L^p\left(\Sigma_{14} \backslash U_{\xi_1}\right)} &= \left\| f_{14}\left(\xi_1, \xi\right) T_1(\xi)^{-2} \left(z - \xi_1\right)^{-2i\nu(\xi_1)} e^{2it\theta} \right\|_{L^p\left(\Sigma_{14} \backslash U_{\xi_1}\right)} \\ &\lesssim \left\| e^{2it\theta} \right\|_{L^p\left(\Sigma_{14} \backslash U_{\xi_1}\right)}. \end{aligned} \tag{5.85}$$

As $z \in \Sigma_{14} \backslash U_{\xi_1}$, we denote $z = \xi_1 + l e^{i\varpi}$, $l \in (\zeta, +\infty)$. Since Lemma 5.2,

$$\begin{aligned} \left\| V^{(2)}(z) - I \right\|^p_{L^p\left(\Sigma_{14} \backslash U_{\xi_1}\right)} &\lesssim \int_{\Sigma_{14} \backslash U_{\xi_1}} \exp\left(-p\kappa(\xi) t \operatorname{Im} z \frac{|z|^2 - \xi_1^2}{4 + |z|^2} \right) dz \\ &\lesssim \int_{\zeta}^{+\infty} \exp\left(-p\kappa'(\xi) t l\right) dl \lesssim t^{-1} \exp\left(-p\kappa'(\xi) t \zeta\right). \end{aligned} \tag{5.86}$$

For $z \in \Sigma'_{k\pm}$, we give the details of Σ_{1+} for example. Obviously,

$$\begin{aligned} \left\| V^{(2)}(z) - I \right\|_{L^p\left(\Sigma'_{1+}\right)} &= \left\| \left(R_{21} - R_{11}\right) e^{-2it\theta} \right\|_{L^p\left(\Sigma'_{1+}\right)} \lesssim \left\| e^{-2it\theta} \right\|_{L^p\left(\Sigma'_{1+}\right)} \\ &\lesssim t^{-1/p} \exp\left(-\kappa''(\xi) t\right). \quad \square \end{aligned} \tag{5.87}$$

This proposition shows that the jump outside the $U(n(\xi))$ can be ignored. So that we construct the solution $M^{(R)}(z)$ as follows:

$$M^R(z) = \begin{cases} E(z,\xi) M^{(r)}(z) & z \notin U(n(\xi)), \\ E(z,\xi) M^{(r)}(z) M^{lo}(z) & z \in U(n(\xi)). \end{cases} \tag{5.88}$$

By the above construction, we have the following notes. As $\xi \in (-\infty, -1)$ or $\xi \in (\frac{1}{8}, +\infty)$, there is no phase point. So that $M^{(r)}(z)$ have no jump except the circles around poles not in Λ, $M^{(R)}(z) = M^{(r)}(z)$. As $\xi \in (-1, \frac{1}{8})$, we have added a localized model $M^{lo}(z)$ to match parabolic cylinder functions around each critical point ξ_j. In Section 6, we obtain $M^{(r)}(z)$ via a pure RH problem. Results of localized model $M^{lo}(z)$ and error function $E(z,\xi)$ are given in Section 7.

Then we use $M^R(z)$ to construct a new matrix function

$$M^{(3)}(z) = M^{(2)}(z) M^{(R)}(z)^{-1}, \tag{5.89}$$

which removes analytical component $M^R(z)$ to get a pure $\bar{\partial}$-problem.

Pure $\bar{\partial}$-Problem. Find $M^{(3)}(z)$ with following identities:

• Analyticity: $M^3(z)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C}$;

• Asymptotic behavior:

$$M^3(z) \sim I + O(z^{-1}), \quad z \to \infty; \tag{5.90}$$

• $\bar{\partial}$-Derivative: We have

$$\bar{\partial} M^3(z) = M^3(z) W^{(3)}(z), \quad z \in \mathbb{C}; \tag{5.91}$$

where

$$W^{(3)}(z) = M^R(z) \bar{\partial} R^2(z) M^R(z)^{-1}. \tag{5.92}$$

Proof. We omit it here for briefly, the reader is referred to [27] for details. □

6. Asymptotic analysis on soliton solutions

6.1. Soliton solutions $M^{(r)}(z)$ and $M^{(r)}_\Lambda(z)$

In this Section, we consider the case of $\bar{\partial} R^{(2)}(z) = 0$. The following Lemma shows that the contribution of the jump matrix on ∂D_n to the asymptotic behavior of the solution is negligible.

Lemma 6.1. The jump matrix $V^{(2)}(z)$ in (5.69) holds the condition

$$\left\| V^{(2)}(z) - I \right\|_{L^\infty\left(\Sigma^{(2)}\right)} = \mathcal{O}\left(e^{-2\rho_0 t}\right), \tag{6.1}$$

where ρ_0 is specified in (5.9).

Proof. We consider the case $z \in \partial D_n$, $n \in \nabla$ as an example.

$$\begin{aligned} \left\| V^{(2)}(z) - I \right\|_{L^\infty(\partial D_n)} &= \left| c_n \left(z - \varrho_n\right)^{-1} T^2(z) e^{-2it\theta(\varrho_n)} \right| \\ &\lesssim \zeta^{-1} e^{-\operatorname{Re}(2it\theta(\varrho_n))} \lesssim e^{2t \operatorname{Im}(\theta(\varrho_n))} \le e^{-2\rho_0 t}. \end{aligned} \tag{6.2}$$

Corollary 6.1. For $1 \le p \le +\infty$, the jump matrix $V^{(2)}(z)$ holds the condition

$$\left\| V^{(2)}(z) - I \right\|_{L^p\left(\Sigma^{(2)}\right)} \le \kappa_p e^{-2\rho_0 t}, \tag{6.3}$$

for constant $\kappa_p \ge 0$ (depend on p).

From the above Lemma, we can ignore all the jump condition on $M^{(r)}(z)$. So that $M^{(r)}(z)$ can be decomposed as

$$M^{(r)}(z) = \tilde{E}(z) M^{(r)}_\Lambda(z) \tag{6.4}$$

where $\tilde{E}(z)$ is a error function (small-norm RH problem). We will consider this small-norm RH problem in the following Subsection 6.2. With $V^{(2)}(z) \equiv I$, $M_\Lambda^{(r)}(z)$ satisfies the following RH problem.

Riemann-Hilbert Problem 6.1. Find a matrix-valued function $M_\Lambda^{(r)}(z)$ with following properties:

- Analyticity: $M_\Lambda^{(r)}(z)$ is analyticity in $\mathbb{C}\setminus\{\varrho_n, \bar{\varrho}_n\}_{n\in\Lambda}$;
- Symmetry: $M_\Lambda^{(r)}(z) = e^{-g_-\hat{\sigma}_3}A^{-2}\sigma_3 M_\Lambda^{(r)}\left(-z^{-1}\right)\sigma_3 = \sigma_2\overline{M_\Lambda^{(r)}(\bar{z})}\sigma_2$;
- Asymptotic behavior:

$$M_\Lambda^{(r)}(z) \sim I + O(z^{-1}), \quad z \to \infty; \tag{6.5}$$

$$\begin{aligned} M^R(z) =& e^{g_-\sigma_3}A^{-1}\left[I + M_1^0(z-i)\right]e^{-\frac{1}{2}k_+\sigma_3}e^{-g_-\sigma_3}e^{-g_+\sigma_3} \\ &\cdot T(i)^{\sigma_3}\left(I - \Sigma_0(\xi)\sigma_3(z-i)\right) + \mathcal{O}\left((z-i)^2\right), \quad z \to i; \end{aligned} \tag{6.6}$$

- Residue conditions: $M_\Lambda^{(r)}(z)$ has simple poles at each point ϱ_n and $\bar{\varrho}_n$ for $n \in \Lambda$ with:

$$\operatorname*{Res}_{z=\varrho_n} M_\Lambda^{(r)}(z) = \lim_{z\to\varrho_n} M_\Lambda^{(r)}(z)\begin{pmatrix} 0 & 0 \\ c_n T^2\left(\varrho_n\right)e^{-2it\theta(\varrho_n)} & 0 \end{pmatrix}, \tag{6.7}$$

$$\operatorname*{Res}_{z=\bar{\varrho}_n} M_\Lambda^{(r)}(z) = \lim_{z\to\bar{\varrho}_n} M_\Lambda^{(r)}(z)\begin{pmatrix} 0 & -\bar{c}_n T^{-2}\left(\bar{\varrho}_n\right)e^{2it\theta(\bar{\varrho}_n)} \\ 0 & 0 \end{pmatrix}. \tag{6.8}$$

In adddition, we denote the asymptotic expansion of $M_\Lambda^{(r)}(z)$ as $z \to i$ as:

$$M_\Lambda^{(r)}(z) = M_\Lambda^{(r)}(i) + M_{1,\Lambda}^{(r)}(z-i) + \mathcal{O}\left((z-i)^{-2}\right). \tag{6.9}$$

Proposition 6.1. RH problem 6.1 has an unique solution. Moreover, $M_\Lambda^{(r)}(z)$ is equivalent to a reflectionless solution of RH problem 3.2 with modified scattering data $\tilde{D}_\Lambda = \{0, \{\varrho_n, c_n T^2(\varrho_n)\}_{n\in\Lambda}\}$ as follows:

Case I : if $\Lambda = \varnothing$, then

$$M_\Lambda^{(r)}(z) = I; \tag{6.10}$$

Case II : if $\Lambda \neq \varnothing$ with $\Lambda = \{\varrho_{jk}\}_{k=1}^{\mathcal{N}}$, then

$$M_\Lambda^{(r)}(z) = I + \sum_{k=1}^{\mathcal{N}}\begin{pmatrix} \frac{\beta_k}{z-\varrho_{j_k}} & \frac{-\bar{\tau}_k}{z-\bar{\varrho}_{j_k}} \\ \frac{\tau_k}{z-\varrho_{j_k}} & \frac{\bar{\beta}_k}{z-\bar{\varrho}_{j_k}} \end{pmatrix}, \tag{6.11}$$

where $\beta_s=\beta_s(x,t)$ and $\tau_s=\tau_s(x,t)$ satisfy linearly dependant equations:

$$c_{j_k}^{-1}T\left(z_{j_k}\right)^{-2}e^{2it\theta\left(z_{j_k}\right)}\beta_k=\sum_{h=1}^{\mathcal{N}}\frac{-\bar{\tau}_h}{\varrho_{j_k}-\bar{\varrho}_{j_h}}, \tag{6.12}$$

$$c_{j_k}^{-1}T\left(z_{j_k}\right)^{-2}e^{2it\theta\left(z_{j_k}\right)}\tau_k=1+\sum_{h=1}^{\mathcal{N}}\frac{\bar{\beta}_h}{\varrho_{j_k}-\bar{\varrho}_{j_h}}, \tag{6.13}$$

for $k=1,...,\mathcal{N}$ respectively.

Proof. Similar to Proposition 9 in [27].

Corollary 6.2. The scattering matrices $S(z)\equiv I$ as $r(s)\equiv 0$. Then the $\mathcal{N}(\Lambda)$-soliton solution $u^r(x,t;\tilde{\mathcal{D}})$ of (1.6)-(1.7) with scattering data $\tilde{\mathcal{D}}_\Lambda=\{0,\{\varrho_n,c_nT^2(\varrho_n)\}_{n\in\Lambda}\}$ can be obtained via the reconstruction formula (3.16)

$$u^r(x,t;\tilde{\mathcal{D}})e^{2g_-}=-\lim_{z\to i}\left(\frac{M_{\Lambda,12}^{(r)}(z)-M_{\Lambda,12}^{(r)}(i)}{(z-i)M_{\Lambda,22}^{(r)}(i)}+\overline{\left(\frac{M_{\Lambda,21}^{(r)}(z)-M_{\Lambda,21}^{(r)}(i)}{(z-i)M_{\Lambda,11}^{(r)}(i)}\right)}\right). \tag{6.14}$$

where

$$x(y,t,\tilde{\mathcal{D}}_\Lambda)=y+k_+^r(x,t,\tilde{\mathcal{D}}_\Lambda)=y+\ln\left(\frac{M_{\Lambda,22}^r(i)}{\overline{M_{\Lambda,11}^r(i)}}\right).$$

In case I,

$$u^r\left(x,t;\tilde{\mathcal{D}}_\Lambda\right)=k_+^r\left(x,t;\tilde{\mathcal{D}}_\Lambda\right)=0. \tag{6.15}$$

As for case II,

$$u^r(x,t;\tilde{\mathcal{D}})e^{2g_-}=-\lim_{z\to i}\left(\frac{\sum_{k=1}^{\mathcal{N}}\left(\frac{-\bar{\tau}_k}{z-\bar{\varrho}_{j_k}}-\frac{-\bar{\tau}_k}{i-\bar{\varrho}_{j_k}}\right)}{(z-i)\left(1+\sum_{k=1}^{\mathcal{N}}\frac{\bar{\beta}_k}{i-\bar{\varrho}_{j_k}}\right)}+\overline{\left(\frac{\sum_{k=1}^{\mathcal{N}}\left(\frac{\tau_k}{z-\varrho_{j_k}}-\frac{\tau_k}{i-\varrho_{j_k}}\right)}{(z-i)\left(1+\sum_{k=1}^{\mathcal{N}}\frac{\beta_k}{i-\varrho_{j_k}}\right)}\right)}\right). \tag{6.16}$$

$$x=y+k_+^r(x,t,\tilde{\mathcal{D}}_\Lambda)=y+\ln\left(\frac{1+\sum_{k=1}^{N}\frac{\bar{\beta}_k}{i-\bar{\zeta}_{jk}}}{1+\sum_{k=1}^{N}\frac{\bar{\beta}_k}{-i-\bar{\zeta}_{jk}}}\right). \tag{6.17}$$

6.2. Error estimate between $M^{(r)}(z)$ and $M_\Lambda^{(r)}(z)$

In this subsection, we consider a RH problem for the matrix function $\tilde{E}(z)$ from the definition (6.4). The solution of this small-norm RH problem can be expanded asymptotically for large times.

Riemann-Hilbert Problem 6.2. Find a matrix-valued function $\tilde{E}(z)$ satisfies following identities:

- Analyticity: $\tilde{E}(z)$ is analyticity in $\mathbb{C}\backslash\Sigma^{(2)}$;
- Asymptotic behavior:

$$\tilde{E}(z) \sim I + O(z^{-1}), \quad |z| \to \infty; \tag{6.18}$$

- Jump condition: $\tilde{E}(z)$ has continuous boundary values $\tilde{E}_{\pm}(z)$ on $\Sigma^{(2)}$ satisfying

$$\tilde{E}_+(z) = \tilde{E}_-(z) V^{\tilde{E}}(z), \tag{6.19}$$

where the jump matrix $V^{\tilde{E}}(z)$ is given by

$$V^{\tilde{E}}(z) = M_{\Lambda}^{(r)}(z) V^{(2)}(z) M_{\Lambda}^{(r)}(z)^{(-1)}. \tag{6.20}$$

From Proposition 6.1, we can know that $M_{\Lambda}^{(r)}(z)$ is bound on $\Sigma^{(2)}$. Then via Lemma 6.1 and Corollary 6.1, we get

$$\left\| V^{\tilde{E}}(z) - I \right\|_p \lesssim \left\| V^{(2)}(z) - I \right\|_p = \mathcal{O}\left(e^{-2\rho_0 t}\right), \quad \text{for } 1 \leq p \leq +\infty \tag{6.21}$$

This uniformly vanishing bound $\|V^{\tilde{E}} - I\|$ implies that RH problem 6.2 is a small-norm RH problem. Therefore, the existence and uniqueness of the RH problem 6.2 can be derived via a small-norm RH problem [18, 35] with

$$\tilde{E}(z) = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{(I + \eta(s))\left(V^{\tilde{E}} - I\right)}{s - z} ds, \tag{6.22}$$

where the $\eta \in L^2(\Sigma^{(2)})$ is the unique solution of following equation:

$$(1 - C_{\tilde{E}})\eta = C_{\tilde{E}}(I). \tag{6.23}$$

Here $C_{\tilde{E}} : L^2(\Sigma^{(2)}) \to L^2(\Sigma^{(2)})$ is a integral operator satisfies

$$C_{\tilde{E}}(f)(z) = C_-\left(f(V^{\tilde{E}} - I)\right), \tag{6.24}$$

where the Cauchy projection operator C_- on $\Sigma^{(2)}$ holds

$$C_-(f)(s) = \lim_{z \to \Sigma_-^{(2)}} \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{f(s)}{s - z} ds. \tag{6.25}$$

From (6.20), we have

$$\|C_{\tilde{E}}\| \leq \|C_-\| \left\|V^{\tilde{E}} - I\right\|_\infty \lesssim \mathcal{O}\left(e^{-2\rho_0 t}\right). \tag{6.26}$$

So that $\| C_{\tilde{E}} \| < 1$ for sufficiently large t, which means $1 - C_{\tilde{E}}$ is invertible, and η exists and is unique. Moreover,

$$\|\eta\|_{L^2(\Sigma^{(2)})} \lesssim \frac{\|C_{\tilde{E}}\|}{1 - \|C_{\tilde{E}}\|} \lesssim \mathcal{O}\left(e^{-2\rho_0 t}\right). \tag{6.27}$$

So the existence and boundedness of $\tilde{E}(z)$ are derived. Then we start to reconstruct the solution $u(x,t)$ of (1.1) with the initial value condition (1.7). The asymptotic behavior of $\tilde{E}(z)$ as $z \to \infty$ and $\tilde{E}(i)$ as $t \to \infty$ are considerd.

Proposition 6.2. $\tilde{E}(z)$ (defined in (6.22)) satisfies

$$|\tilde{E}(z) - I| \lesssim \mathcal{O}\left(e^{-2\rho_0 t}\right). \tag{6.28}$$

As $z = i$,

$$\tilde{E}(i) = I + \frac{1}{2\pi i}\int_{\Sigma^{(2)}} \frac{(I + \eta(s))\left(V^{\tilde{E}} - I\right)}{s - i} ds. \tag{6.29}$$

As $z \to i$, $\tilde{E}(z)$ has expansion at $z = i$

$$\tilde{E}(z) = \tilde{E}(i) + \tilde{E}_1(z - i) + O((z - i)^2), \tag{6.30}$$

where

$$\tilde{E}_1 = \frac{1}{2\pi i}\int_{\Sigma^{(2)}} \frac{(I + \eta(s))\left(V^{\tilde{E}} - I\right)}{(s - i)^2} ds. \tag{6.31}$$

Moreover, $\tilde{E}(i)$ and $\tilde{E}_1$ hold long time asymptotic behavior condition:

$$\tilde{E}(i) - I \lesssim \mathcal{O}\left(e^{-2\rho_0 t}\right), \quad \tilde{E}_1 \lesssim \mathcal{O}\left(e^{-2\rho_0 t}\right). \tag{6.32}$$

Proof. We know that

$$|\tilde{E}(z) - I| = \left|\frac{1}{2\pi i}\int_{\Sigma^{(2)}} \frac{(I + \eta(s))\left(V^{\tilde{E}} - I\right)}{s - z} ds\right|, \tag{6.33}$$

then

$$\begin{aligned} |\tilde{E}(z) - I| &= |(1 - C_{\tilde{E}})(\eta) + C_{\tilde{E}}(\eta)| \\ &\leq |(1 - C_{\tilde{E}})(\eta)| + |C_{\tilde{E}}(\eta)| \\ &\leq \|1 - C_{\tilde{E}}\| \cdot \|\eta\| + \|C_{\tilde{E}}\| \cdot \|\eta\| \\ &\lesssim O(e^{-2\rho_0 t}). \end{aligned} \tag{6.34}$$

Since $\int_{\Sigma^{(2)}} \frac{1}{|s-z|^{-2}} ds$ is bounded, we have

$$|\tilde{E}_1| \leq \|V^{\tilde{E}} - I\|_{L^1} + \|\eta\|_{L^2}\|V^{\tilde{E}} - I\|_{L^2} \lesssim O(e^{-2\rho_0 t}). \tag{6.35}$$

7. Localized RH problem

7.1. A local solvable RH model

When $\xi \in (-1, \frac{1}{8})$, we consider the $M^{lo}(z)$ near the stationary phase points. First, we denote a new contour $\Sigma^{(lo)} = (\bigcup_{\substack{j=1,\ldots,n(\xi),\\ k=1,2,3,4}} \Sigma_{kj}) \cap U(n(\xi))$ shown in the Fig.8.

Then we consider following RH problem:

Riemann-Hilbert Problem 7.1. Find a matrix-valued function $M^{lo}(z)$ satisfies following properties:

• Analyticity: $M^{lo}(z)$ is analyticity in $\mathbb{C}\backslash\Sigma^{(lo)}$;

• Symmetry: $M^{lo}(z) = e^{-g_-\hat{\sigma}_3} A^{-2}\sigma_3 M^{lo}(-1/z)\sigma_3$;

• Jump condition: $M^{lo}(z)$ has continuous boundary values $M^{lo}_{\pm}(z)$ on $\Sigma^{(lo)}$ and

$$M^{lo}_{+}(z) = M^{lo}_{-}(z)V^{(2)}(z), \quad z \in \Sigma^{(lo)} \tag{7.1}$$

• Asymptotic behavior:

$$M^{lo}(z) \sim I + O(z^{-1}), \quad z \to \infty. \tag{7.2}$$

It is noted that the RH problem only has jump conditions without poles. Further, we denote

$$V^{(2)}(z) = \left(I - \omega_{kj}^{-}\right)\left(I + \omega_{kj}^{+}\right), \tag{7.3}$$

where $\omega_{kj}^{+}=0$, $\omega_{kj}^{-}=I-V^{(2)}(z)^{-1}$, and

$$w_{kj}^{-}(z)=\begin{cases}\begin{pmatrix}0 & 0\\ -R_{k1}(z,\xi)e^{-2it\theta} & 0\end{pmatrix}, & z\in\Sigma_{k1},\\ \begin{pmatrix}0 & R_{k2}(z,\xi)e^{2it\theta}\\ 0 & 0\end{pmatrix}, & z\in\Sigma_{k2},\\ \begin{pmatrix}0 & 0\\ R_{k3}(z,\xi)e^{-2it\theta} & 0\end{pmatrix}, & z\in\Sigma_{k3},\\ \begin{pmatrix}0 & -R_{k4}(z,\xi)e^{2it\theta}\\ 0 & 0\end{pmatrix}, & z\in\Sigma_{k4},\end{cases} \tag{7.4}$$

with $k=1,...,n(\xi)$. The Cauchy projection operator $C_{\pm}$ on $\Sigma^{(2)}$ is

$$C_{\pm}(f)(s)=\lim_{z\to\Sigma_{\pm}^{(2)}}\frac{1}{2\pi i}\int_{\Sigma^{(2)}}\frac{f(s)}{s-z}ds, \tag{7.5}$$

based on the above, we define the Beals-Coifman operator as

$$C_{\omega_{kj}}(f)=C_{-}\left(f\omega_{kj}^{+}\right)+C_{+}\left(f\omega_{kj}^{-}\right). \tag{7.6}$$

Additionally, we let

$$\Sigma_{k}^{(lo)}=\bigcup_{j=1,\dots,4}\Sigma_{kj},\quad \omega_{k}(z)=\sum_{j=1,\dots,4}\omega_{kj}(z),\quad \omega=\sum_{k=1}^{n(\xi)}\omega_{k},$$

then

$$C_{\omega}=\Sigma_{k=1}^{n(\xi)}C_{\omega_{k}}.$$

Lemma 7.1. The matrix functions ω_{kj}^{-} defined above admits following estimation:

$$\left\|\omega_{kj}^{-}\right\|_{L^{p}\left(\Sigma_{kj}\right)}=\mathcal{O}\left(t^{-1/2}\right),1\leq p<+\infty. \tag{7.7}$$

Lemma 7.2 Let $f\in I+L^{2}(\Sigma^{(lo)})$ be the solution of the singular integral equation

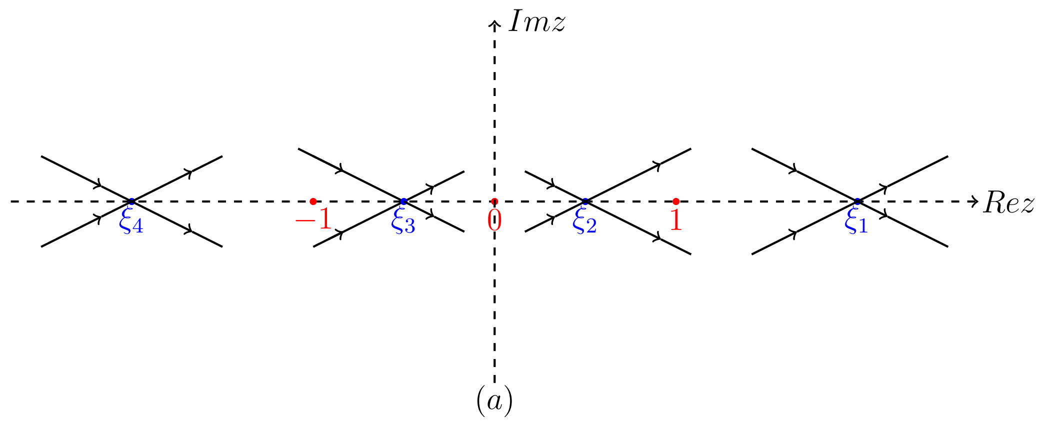
$$f=I+C_{\omega}(f), \tag{7.8}$$

then the solution to RH problem 7.1 has the form

$$M^{lo}(z)=I+\frac{1}{2\pi i}\int_{\Sigma^{(lo)}}\frac{(I-C_{\omega})^{-1}I\omega}{s-z}ds. \tag{7.9}$$

From Lemma 7.1, we know that $I-C_{\omega}$ and $I-C_{\omega_{k}}$ are reversible, then the solution of RH problem 7.1 is unique. The solution can be written as

$$M^{lo}(z)=I+\frac{1}{2\pi i}\int_{\Sigma^{(lo)}}\frac{\left(I-C_{w}\right)^{-1}I\cdot w}{s-z}ds. \tag{7.10}$$



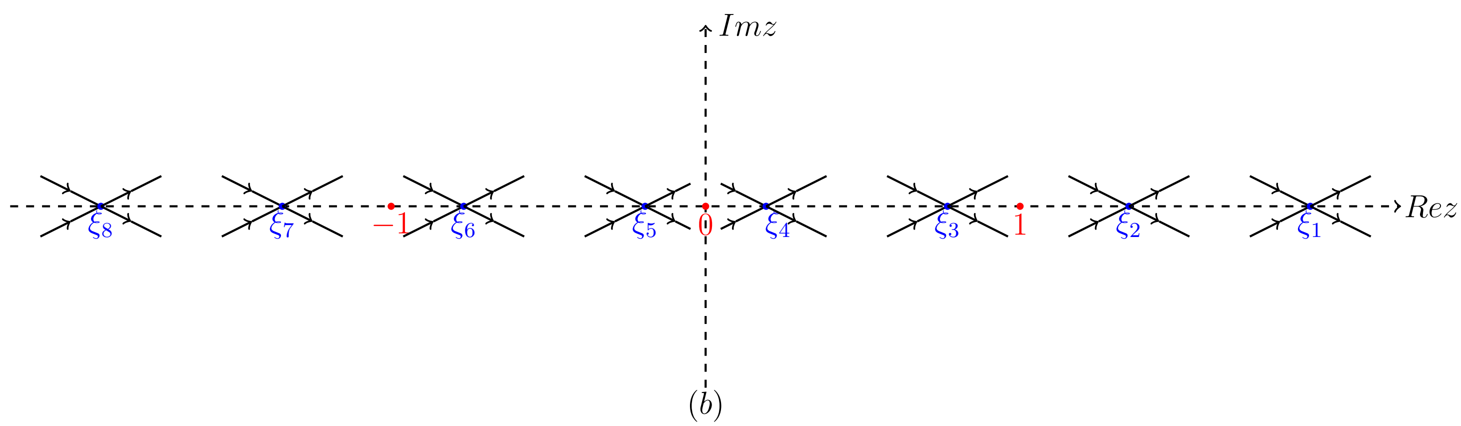


Figure 8. Jump contours $\Sigma^{(lo)}$ of $M^{lo}(z)$. Panels (a): $-1 < \xi \leq 0$. Panels (b): $0 < \xi < \frac{1}{8}$.

In the following we show that the contributions of every cross $\Sigma_k^{(lo)}$ can be considered separately.

Proposition 7.1. As $t \to +\infty$,

$$\int_{\Sigma^{(lo)}} \frac{(I - C_w)^{-1} I \cdot w}{s - z} ds = \sum_{k=1}^{n(\xi)} \int_{\Sigma_k^{(lo)}} \frac{(I - C_{w_k})^{-1} I \cdot w_k}{s - z} ds + \mathcal{O}\left(t^{-3/2}\right). \tag{7.11}$$

Proof. By direct calculation, we have

$$(I - C_w)\left(I + \textstyle\sum_{k=1}^{n(\xi)} C_{w_k} (I - C_{w_k})^{-1}\right) = I - \sum_{1 \leq k \neq j \leq n(\xi)} C_{w_j} C_{w_k} (I - C_{w_k})^{-1},$$

$$\left(I + \textstyle\sum_{k=1}^{n(\xi)} C_{w_k} (I - C_{w_k})^{-1}\right)(I - C_w) = I - \sum_{1 \leq k \neq j \leq n(\xi)} (I - C_{w_k})^{-1} C_{w_k} C_{w_j}.$$

Via the Lemma 3.5 [27], we get

$$\left\|C_{w_k}C_{w_j}\right\|_{B\left(L^2\left(\Sigma^{(lo)}\right)\right)} \lesssim t^{-1}, \quad \left\|C_{w_k}C_{w_j}\right\|_{L^\infty\left(\Sigma^{(lo)}\right)\to L^2\left(\Sigma^{(lo)}\right)} \lesssim t^{-1}. \tag{7.12}$$

According to [3], we have

$$\begin{aligned}(1-C_\omega)^{-1} I = & I + \sum_{k=1}^{n(\xi)} C_{\omega_k}(1-C_\omega)^{-1} I + \left[1+\sum_{k=1}^{n(\xi)} C_{\omega_k}(1-C_\omega)^{-1}\right] \\ & \times \left[1-\sum_{1\le k\ne j\le n(\xi)} C_{\omega_k}C_{\omega_j}\left((1-C_\omega)^{-1}\right)\right]^{-1}\left(\sum_{1\le k\ne j\le n(\xi)} C_{\omega_k}C_{\omega_j}(1-C_\omega)^{-1}\right) I \\ & \triangleq I + \sum_{=k1}^{n(\xi)} C_{\omega_k}(1-C_\omega)^{-1} I + W_1W_2W_3 I,\end{aligned} \tag{7.13}$$

where

$$\begin{aligned} W_1 &= 1+\textstyle\sum_{k=1}^{n(\xi)} C_{\omega_k}(1-C_\omega)^{-1}, \\ W_2 &= \left[1-\sum_{1\le k\ne j\le n(\xi)} C_{\omega_k}C_{\omega_j}\left((1-C_\omega)^{-1}\right)\right]^{-1}, \\ W_3 &= \sum_{1\le k\ne j\le n(\xi)} C_{\omega_k}C_{\omega_j}(1-C_\omega)^{-1}. \end{aligned}$$

From Cauchy-Schwartz inequality, we get

$$\left|\int_{\Sigma^{(0)}} W_1W_2W_3\omega\right| \lesssim t^{-1}t^{-\frac{1}{2}} = O(t^{-\frac{3}{2}}). \tag{7.14}$$

□

Thus, the RH problem 7.1 is reduced to a model RH problem, and each contour $\Sigma_k^{(lo)}$ can be considered separated. We take $\Sigma_1^{(lo)}$ as an example here, and denote $\hat{\Sigma}_1^{(lo)}$ as the contour $\{z=\xi_1+le^{\pm\varpi i}, l\in\mathbb{R}\}$ oriented from $\Sigma_1^{(lo)}$. $\hat{\Sigma}_{1j}$ is the extension of Σ_{1j}. Then we rewrite phase function as z near ξ_1:

$$\theta(z) = \theta(\xi_1) + (z-\xi_1)^2\theta''(\xi_1) + \mathcal{O}\left((z-\xi_1)^3\right). \tag{7.15}$$

It is noted that $\theta''(\xi_1)>0$ as $\xi\in(-1,0]$, and $\theta''(\xi_1)<0$ as $\xi\in(0,\frac{1}{8})$.

Then we consider following local RH problem:

Riemann-Hilbert Problem 7.2. Find a matrix-valued function $M^{lo,1}(z)$ satisfies following properties:

- Analyticity: $M^{lo,1}(z)$ is analytical in $\mathbb{C} \setminus \hat{\Sigma}_1^{(lo)}$;
- Jump condition: $M^{lo,1}(z)$ has continuous boundary values $M_{\pm}^{lo,1}(z)$ on $\hat{\Sigma}_1^{(lo)}$ and

$$M_{+}^{lo,1}(z) = M_{-}^{lo,1}(z) V^{lo,1}(z), \ z \in \hat{\Sigma}_1^{(lo)}, \tag{7.16}$$

where

$$V^{lo,1}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r(\xi_1)T_1(\xi)^2 \left(\eta(\xi,1)\,(z-\xi_1)\right)^{2i\eta(\xi,1)\nu(\xi_1)} e^{-2it\theta} & 1 \end{pmatrix}, & z \in \hat{\Sigma}_{11}, \\ \begin{pmatrix} 1 & \bar{r}(\xi_1)T_1(\xi)^{-2} \left(\eta(\xi,1)\,(z-\xi_1)\right)^{-2i\eta(\xi,1)\nu(\xi_1)} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \hat{\Sigma}_{12}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r(\xi_1)}{1+|r(\xi_1)|^2} T_1(\xi)^2 \left(\eta(\xi,1)\,(z-\xi_1)\right)^{2i\eta(\xi,1)\nu(\xi_1)} e^{-2it\theta} & 1 \end{pmatrix}, & z \in \hat{\Sigma}_{13}, \\ \begin{pmatrix} 1 & \frac{\bar{r}(\xi_1)}{1+|r(\xi_1)|^2} T_1(\xi)^{-2} \left(\eta(\xi,1)\,(z-\xi_1)\right)^{-2i\eta(\xi,1)\nu(\xi_1)} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \hat{\Sigma}_{14}, \end{cases} \tag{7.17}$$

- Asymptotic behaviors:

$$M^{lo,1}(z) = I + O(z^{-1}), \quad z \to \infty; \tag{7.18}$$

$\xi \in (-1, 0)$

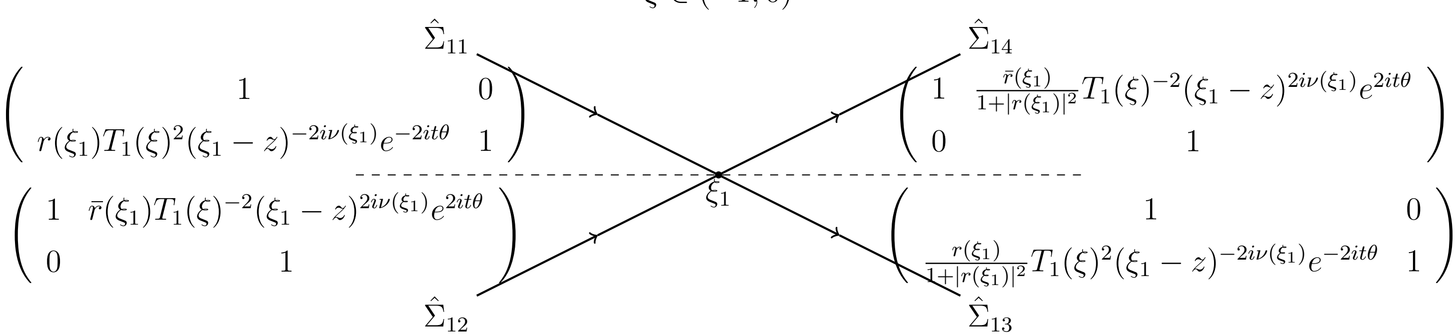


Figure 9. The contour $\hat{\Sigma}_1^{(lo)}$ and the jump matrix on it in case $-1 < \xi \leq 0$.

$\xi \in (0, \frac{1}{8})$

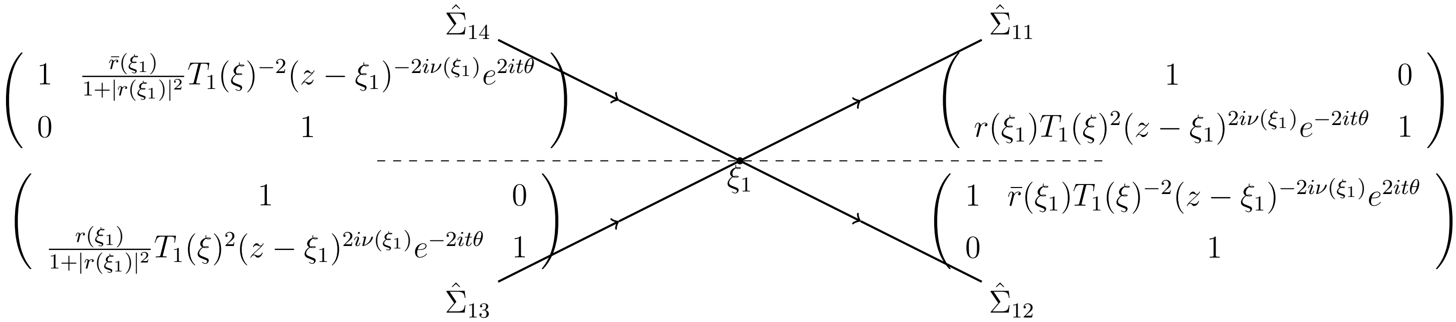


Figure 10. The contour $\hat{\Sigma}_1^{(lo)}$ and the jump matrix on it in case $0 < \xi < \frac{1}{8}$.

Since RH problem 7.2 is a local model, it does not hold the symmetry condition. So we make a transformation to motivate the model. We denote $\rho = \rho(z)$, and rescaled the local variable

$$\rho(z) = t^{1/2}\sqrt{-4\eta(\xi,\xi_1)\theta''(\xi_1)}\,(z-\xi_1)\,, \tag{7.19}$$

where $\eta(\xi,1)$ is defined in (5.18). This transformation maps U_{ξ_1} to an expanding neighborhood of $\rho = 0$. Furthermore, we take

$$r_{\xi_1} = r(\xi_1)\,T_1(\xi)^2 e^{-2it\theta(\xi_1)}\eta(\xi,\xi_1)^{2i\eta\nu(\xi_1)}\exp\{-i\nu\eta(\xi_1)\log(-4t\theta''(\xi_1)\,\eta(\xi,\xi_1))\}.$$

Obviously $|r_{\xi_1}| = |r(\xi_1)|$. In the above expression, we choose the branch of the logarithm with $-\pi < \arg\rho < \pi$ as $\xi \in [0, \frac{1}{8})$, and $0 < \arg\rho < 2\pi$ as $\xi \in (-1, 0]$.

Then the jump $V^{lo,1}(z)$ approximates to the jump of a parabolic cylinder model problem as follows after transformation:

Riemann-Hilbert Problem 7.3. Find a matrix-valued function $M^{pc}(\rho;\xi)$ with following properties:

- Analyticity: $M^{pc}(\rho;\xi)$ is analytical in $\mathbb{C}\setminus\Sigma^{pc}$ with $\Sigma^{pc} = \{\mathbb{R}e^{\varpi i}\}\cup\{\mathbb{R}e^{(\pi-\varpi)i}\}$;
- Jump condition: M^{pc} has continuous boundary values $M^{pc}_{\pm}$ on Σ^{pc} and

$$M^{pc}_{+}(\rho,\xi) = M^{pc}_{-}(\rho,\xi)V^{pc}(\rho,\xi), \quad \rho\in\Sigma^{pc}. \tag{7.20}$$

In the above expression, when $\xi \in [0, \frac{1}{8})$

$$V^{pc}(\rho,\xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_{\xi_1}\rho^{2i\nu(\xi_1)}e^{\frac{i}{2}\rho^2} & 1 \end{pmatrix}, & \rho\in\mathbb{R}^+e^{i\varpi}, \\ \begin{pmatrix} 1 & \overline{r}_{\xi_1}\rho^{-2i\nu(\xi_1)}e^{-\frac{i}{2}\rho^2} \\ 0 & 1 \end{pmatrix}, & \rho\in\mathbb{R}^+e^{-i\varpi}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1+|r_{\xi_1}|^2}\rho^{2i\nu(\xi_1)}e^{\frac{i}{2}\rho^2} & 1 \end{pmatrix}, & \rho\in\mathbb{R}^+e^{i(-\pi+\varpi)}, \\ \begin{pmatrix} 1 & \frac{\overline{r}_{\xi_1}}{1+|r_{\xi_1}|^2}\rho^{-2i\nu(\xi_1)}e^{-\frac{i}{2}\rho^2} \\ 0 & 1 \end{pmatrix}, & \rho\in\mathbb{R}^+e^{i(\pi-\varpi)}, \end{cases} \tag{7.21}$$

and when $\xi \in (-1, 0]$,

$$V^{pc}(\rho, \xi) = \begin{cases} \begin{pmatrix} 1 & \frac{\overline{r}_{\xi_1}}{1+|r_{\xi_1}|^2}\rho^{2i\nu(\xi_1)}e^{\frac{i}{2}\rho^2} \\ 0 & 1 \end{pmatrix}, & \rho \in \mathbb{R}^+ e^{i\varpi}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1+|r_{\xi_1}|^2}\rho^{-2i\nu(\xi_1)}e^{-\frac{i}{2}\rho^2} & 1 \end{pmatrix}, & \rho \in \mathbb{R}^+ e^{i(2\pi-\varpi)}, \\ \begin{pmatrix} 1 & \overline{r}_{\xi_1}\rho^{2i\nu(\xi_1)}e^{\frac{i}{2}\rho^2} \\ 0 & 1 \end{pmatrix}, & \rho \in \mathbb{R}^+ e^{i(\pi+\varpi)}, \\ \begin{pmatrix} 1 & 0 \\ r_{\xi_1}\rho^{-2i\nu(\xi_1)}e^{-\frac{i}{2}\rho^2} & 1 \end{pmatrix}, & \rho \in \mathbb{R}^+ e^{i(\pi-\varpi)}, \end{cases} \tag{7.22}$$

• Asymptotic behaviors:

$$M^{pc}(\rho, \xi) = I + M_1^{pc}\rho^{-1} + O(\rho^{-2}), \quad \rho \to \infty. \tag{7.23}$$

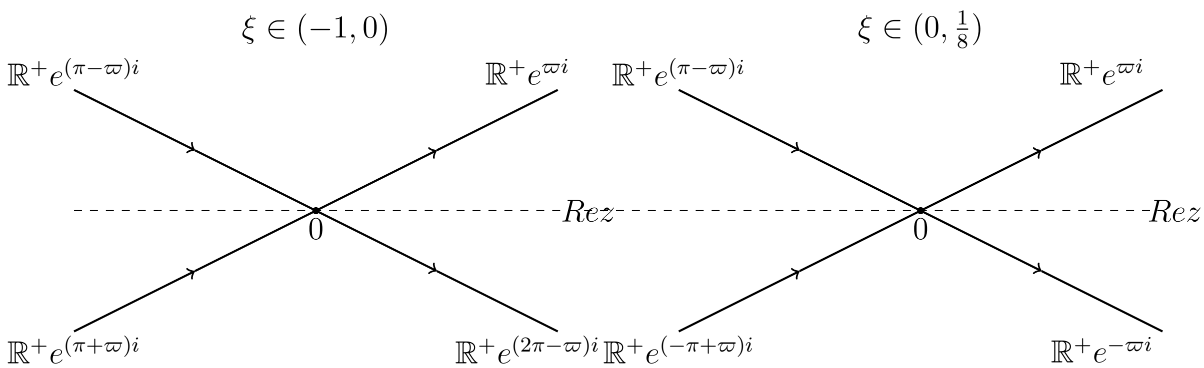


Figure 11. The contour Σ^{pc} in case $-1 < \xi \leq 0$ and $0 < \xi < \frac{1}{8}$ respectively.

From [21] Theorem A.1-A.6, we prove

$$M^{lo,1}(z) = I + \frac{t^{-1/2}}{z - \xi_1} \frac{1}{\sqrt{-4\eta\theta''(\xi_1)}} \begin{pmatrix} 0 & [M_1^{pc}]_{12} \\ [M_1^{pc}]_{21} & 0 \end{pmatrix} + \mathcal{O}\left(t^{-1}\right). \tag{7.24}$$

Then RH problem 7.3 can be expressed in terms of solutions of the parabolic cylinder equation with an explicit solution $M^{pc}(\rho)$, which is

$$\left(\frac{\partial^2}{\partial z^2} + \left(\frac{1}{2} - \frac{z^2}{2} + a\right)\right) D_a(z) = 0. \tag{7.25}$$

Actually, we take

$$M^{pc}(\rho; \xi) = \Psi(\rho; \xi) P(\xi) e^{\frac{i}{4}\eta\rho^2\sigma_3} \rho^{i\eta\nu(\xi_1)\sigma_3}, \tag{7.26}$$

where in the case $\xi \in (0, \frac{1}{8})$

$$P(\xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r_{\xi_1} & 1 \end{pmatrix}, & \arg \rho \in (0, \varpi), \\ \begin{pmatrix} 1 & \bar{r}_{\xi_1} \\ 0 & 1 \end{pmatrix}, & \arg \rho \in (-\varpi, 0), \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1+\left|r_{\xi_1}\right|^2} & 1 \end{pmatrix}, & \arg \rho \in (-\pi, -\pi + \varpi), \\ \begin{pmatrix} 1 & -\frac{\bar{r}_{\xi_1}}{1+|r_{\xi_1}|^2} \\ 0 & 1 \end{pmatrix}, & \arg \rho \in (\pi - \varpi, \pi), \\ I, & \text{else} \end{cases} \tag{7.27}$$

and in the case $\xi \in (-1, 0]$

$$P(\xi) = \begin{cases} \begin{pmatrix} 1 & -\frac{\bar{r}_{\xi_1}}{1+\left|r_{\xi_1}\right|^2} \\ 0 & 1 \end{pmatrix}, & \arg \rho \in (0, \varpi), \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1+\left|r_{\xi_1}\right|^2} & 1 \end{pmatrix}, & \arg \rho \in (2\pi - \varpi, 2\pi), \\ \begin{pmatrix} 1 & \bar{r}_{\xi_1} \\ 0 & 1 \end{pmatrix}, & \arg \rho \in (\pi, \pi + \varpi), \\ \begin{pmatrix} 1 & 0 \\ -r_{\xi_1} & 1 \end{pmatrix}, & \arg \rho \in (\pi - \varpi, \pi), \\ I, & \text{else} \end{cases} \tag{7.28}$$

After construction, we note that the matrix Ψ is continuous along Σ^{pc}. And there are identical (constant) jump matrices along the negative real axis and the positive real axis for Ψ, since the branch cut of the logarithmic function along $\eta\mathbb{R}^+$. Then the function $\Psi(\rho; \xi)$ satisfies the following model RH problem.

Riemann-Hilbert Problem 7.4. Find a matrix-valued function $\Psi(\rho; \xi)$ satisfies following properties:

- Analyticity: $\Psi(\rho; \xi)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$;
- Jump condition: $\Psi(\rho; \xi)$ has continuous boundary values $\Psi_\pm(\rho; \xi)$ on $\mathbb{R}$ and

$$\Psi_+(\rho; \xi) = \Psi_-(\rho; \xi) V^\Psi(\rho), \quad \Psi \in \mathbb{R}, \tag{7.29}$$

where

$$V^{\Psi}(\xi)=\left(\begin{array}{cc} 1+|r_{\xi_1}|^2 & \bar{r}_{\xi_1} \\ r_{\xi_1} & 1 \end{array}\right) \tag{7.30}$$

• Asymptotic behaviors:

$$\Psi(\rho;\xi)\sim\left(I+M_1^{pc}\rho^{-1}\right)\rho^{-i\eta\nu(\xi_1)\sigma_3}e^{-\frac{i}{4}\eta\rho^2\sigma_3},\quad \rho\to\infty. \tag{7.31}$$

We denote $\beta_{12}^1=i[M_1^{pc}]_{12}$ and $\beta_{21}^1=-i[M_1^{pc}]_{21}$ here for brevity. The unique solution to RH problem 7.4 is:

1. as $\xi\in\left(0,\frac{1}{8}\right)$, when $\rho\in\mathbb{C}^+$,

$$\Psi(\rho;\xi)=\left(\begin{array}{cc} e^{\frac{3}{4}\pi\nu(\xi_1)}D_{-i\nu(\xi_1)}\left(e^{-\frac{3}{4}\pi i}\rho\right) & \frac{i\nu(\xi_1)}{\beta_{21}^1}e^{-\frac{\pi}{4}(\nu(\xi_1)+i)}D_{i\nu(\xi_1)-1}\left(e^{-\frac{\pi i}{4}}\rho\right) \\ -\frac{i\nu(\xi_1)}{\beta_{12}^1}e^{\frac{3\pi}{4}(\nu(\xi_1)-i)}D_{-i\nu(\xi_1)-1}\left(e^{-\frac{3\pi i}{4}}\rho\right) & e^{-\frac{\pi}{4}\nu(\xi_1)}D_{i\nu(\xi_1)}\left(e^{-\frac{\pi}{4}i}\rho\right) \end{array}\right),$$

when $\rho\in\mathbb{C}^-$,

$$\Psi(\rho;\xi)=\left(\begin{array}{cc} e^{\frac{-\pi}{4}\nu(\xi_1)}D_{-i\nu(\xi_1)}\left(e^{\frac{\pi}{4}i}\rho\right) & \frac{i\nu(\xi_1)}{\beta_{21}^1}e^{\frac{3\pi}{4}(\nu(\xi_1)+i)}D_{i\nu(\xi_1)-1}\left(e^{\frac{3\pi i}{4}}\rho\right) \\ -\frac{i\nu(\xi_1)}{\beta_{12}^1}e^{\frac{-\pi}{4}(\nu(\xi_1)-i)}D_{-i\nu(\xi_1)-1}\left(e^{-\frac{\pi i}{4}}\rho\right) & e^{\frac{3}{4}\pi\nu(\xi_1)}D_{i\nu(\xi_1)}\left(e^{\frac{3}{4}\pi i}\rho\right) \end{array}\right).$$

2. as $\xi\in[-1,0)$, when $\rho\in\mathbb{C}^+$,

$$\Psi(\rho;\xi)=\left(\begin{array}{cc} e^{-\frac{\pi}{4}\nu(\xi_1)}D_{i\nu(\xi_1)}\left(e^{-\frac{\pi}{4}i}\rho\right) & \frac{i\nu(\xi_1)}{\beta_{21}^1}e^{\frac{3\pi}{4}(\nu(\xi_1)-i)}D_{-i\nu(\xi_1)-1}\left(e^{-\frac{3\pi i}{4}}\rho\right) \\ \frac{-i\nu(\xi_1)}{\beta_{12}^1}e^{-\frac{\pi}{4}(\nu(\xi_1)+i)}D_{i\nu(\xi_1)-1}\left(e^{-\frac{\pi i}{4}}\rho\right) & e^{\frac{3}{4}\pi\nu(\xi_1)}D_{-i\nu(\xi_1)}\left(e^{-\frac{3}{4}\pi i}\rho\right) \end{array}\right),$$

when $\rho\in\mathbb{C}^-$,

$$\Psi(\rho;\xi)=\left(\begin{array}{cc} e^{\frac{3}{4}\pi\nu(\xi_1)}D_{i\nu(\xi_1)}\left(e^{\frac{3}{4}\pi i}\rho\right) & \frac{i\nu(\xi_1)}{\beta_{21}^1}e^{-\frac{\pi}{4}(\nu(\xi_1)-i)}D_{-i\nu(\xi_1)-1}\left(e^{\frac{\pi i}{4}}\rho\right) \\ \frac{-i\nu(\xi_1)}{\beta_{12}^1}e^{\frac{3\pi}{4}(\nu(\xi_1)+i)}D_{i\nu(\xi_1)-1}\left(e^{\frac{3\pi i}{4}}\rho\right) & e^{-\frac{\pi}{4}\nu(\xi_1)}D_{-i\nu(\xi_1)}\left(e^{\frac{\pi}{4}i}\rho\right) \end{array}\right).$$

In above expressions, when $\xi\in(0,\frac{1}{8})$,

$$\begin{aligned} &\beta_{12}^1=\frac{\sqrt{2\pi}e^{\frac{1}{2}\pi\nu(\xi_1)}e^{\frac{\pi}{4}i}}{r_{\xi_1}\Gamma(i\nu(\xi_1))},\quad \beta_{21}^1\beta_{12}^1=-\nu\left(\xi_1\right), \\ &|\beta_{12}^1|=\sqrt{\frac{-\nu(\xi_1)}{1+|r|^2}}, \\ &\arg\left(\beta_{12}^1\right)=\frac{\pi}{4}-\arg r_{\xi_1}-\arg\Gamma\left(i\nu(\xi_1)\right); \end{aligned}$$

when $\xi \in (-1, 0)$,

$$\begin{aligned}
&\beta_{12}^{1} = \frac{\sqrt{2\pi} e^{\frac{\pi}{2}\nu(\xi_1)} e^{-\frac{\pi}{4}i}}{r_{\xi_1}\Gamma(-i\nu(\xi_1))}, \quad \beta_{21}^{1}\beta_{12}^{1} = -\nu\left(\xi_1\right), \\
&|\beta_{12}^{1}| = \sqrt{-\frac{\nu(\xi_1)}{1+|r|^2}}, \\
&\arg\left(\beta_{12}^{1}\right) = -\frac{\pi}{4} - \arg r_{\xi_1} - \arg\Gamma\left(-i\nu\left(\xi_1\right)\right).
\end{aligned}$$

The literature [3] has the derivation of this result, [24] has direct verification of the solution. We substitute above consequence into (7.24) and obtain:

$$M^{lo,1}(z) = I + \frac{t^{-1/2}}{2i\sqrt{\eta\theta''\left(\xi_1\right)}\left(z-\xi_1\right)} \begin{pmatrix} 0 & -i\beta_{12}^{1} \\ i\beta_{21}^{1} & 0 \end{pmatrix} + \mathcal{O}\left(t^{-1}\right). \tag{7.32}$$

As $k = 2, ..., n(\xi)$, RH problem models around other stationary phase points also holds

$$M^{lo,k}(z) = I + \frac{t^{-1/2}}{2i\sqrt{\eta\theta''\left(\xi_k\right)}\left(z-\xi_k\right)} \begin{pmatrix} 0 & -i\beta_{12}^{k} \\ i\beta_{21}^{k} & 0 \end{pmatrix} + \mathcal{O}\left(t^{-1}\right). \tag{7.33}$$

We note that $\xi \in (0, \frac{1}{8})$, k is a odd number, or $\xi \in (-1, 0)$, k is an even number,

$$r_{\xi_k} = r\left(\xi_k\right) T_k(\xi)^2 e^{-2it\theta(\xi_k)} \exp\left\{-i\nu\left(\xi_k\right)\log\left(-4t\theta''\left(\xi_k\right)\right)\right\}, \tag{7.34}$$

and

$$\begin{aligned}
&\beta_{12}^{k} = \frac{\sqrt{2\pi} e^{\frac{1}{2}\pi\nu(\xi_k)} e^{\frac{\pi}{4}i}}{r_{\xi_k}\Gamma(i\nu(\xi_k))}, \quad \beta_{21}^{k}\beta_{12}^{k} = -\nu\left(\xi_k\right), \\
&\left|\beta_{12}^{k}\right| = \sqrt{-\frac{\nu(\xi_k)}{\left(1+|r|^2\right)}}, \\
&\arg\left(\beta_{12}^{k}\right) = \frac{\pi}{4} - \arg r_{\xi_k} - \arg\Gamma\left(i\nu\left(\xi_k\right)\right);
\end{aligned}$$

and when $\xi \in (0, \frac{1}{8})$, k is an even number or $\xi \in (-1, 0)$, k is a odd number,

$$r_{\xi_k} = r\left(\xi_k\right) T_k(\xi)^2 e^{-2it\theta(\xi_k)} \exp\left\{i\nu\left(\xi_k\right)\log\left(4t\theta''\left(\xi_k\right)\right)\right\}, \tag{7.35}$$

and

$$\begin{aligned}
&\beta_{12}^{k} = \frac{\sqrt{2\pi} e^{\frac{\pi}{2}\nu(\xi_k)} e^{-\frac{\pi}{4}i}}{r_{\xi_k}\Gamma(-i\nu(\xi_k))}, \quad \beta_{21}^{k}\beta_{12}^{k} = -\nu\left(\xi_k\right), \\
&\left|\beta_{12}^{k}\right| = \sqrt{-\frac{\nu(\xi_k)}{\left(1+|r|^2\right)}},
\end{aligned}$$

$$\arg\left(\beta_{12}^{k}\right) = -\tfrac{\pi}{4} - \arg r_{\xi_k} - \arg\Gamma\left(-i\nu\left(\xi_k\right)\right).$$

Then we final derive the following Proposition from above properties and Proposition 7.1.

Proposition 7.2. As $t \to +\infty$, we have

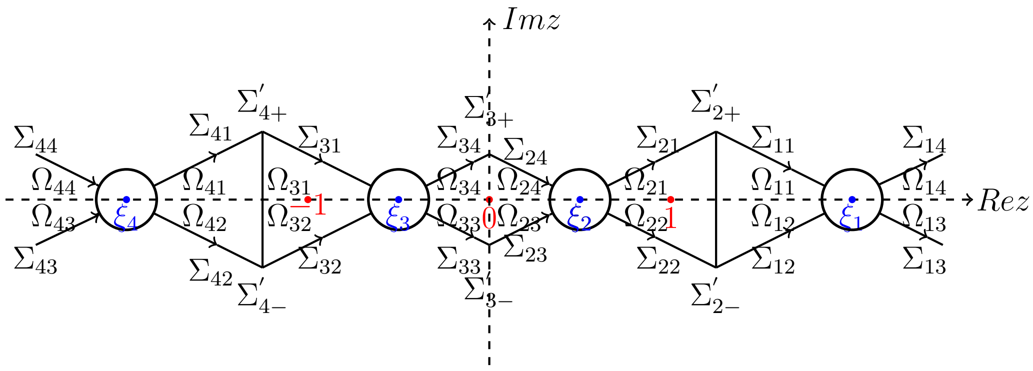
$$M^{lo}(z) = I + t^{-1/2}\sum_{k=1}^{n(\xi)}\frac{H_k(\xi)}{z-\xi_k} + \mathcal{O}\left(t^{-1}\right), \tag{7.36}$$

where

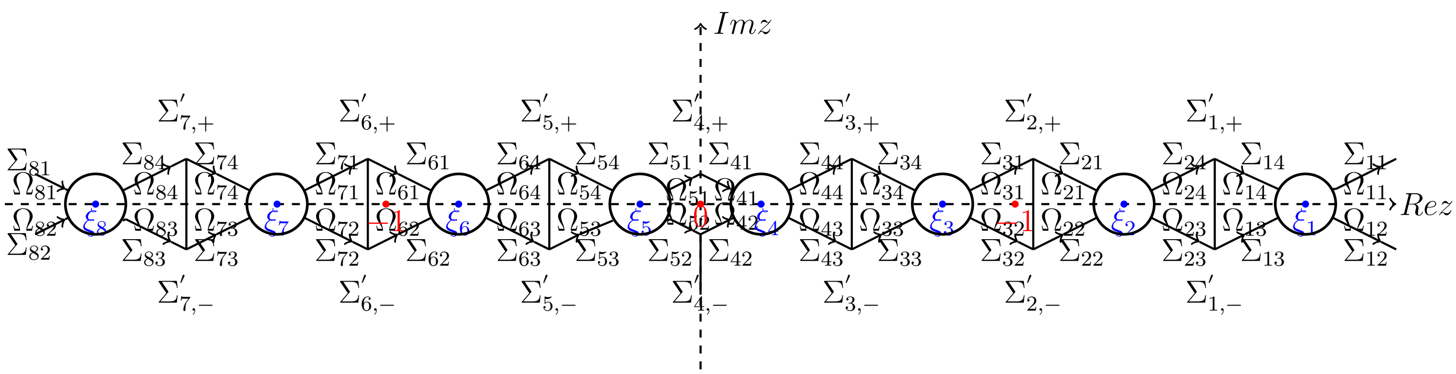
$$H_k(\xi) = \frac{1}{2i\sqrt{\eta\theta''(\xi_k)}}\begin{pmatrix} 0 & -i\beta_{12}^{k} \\ i\beta_{21}^{k} & 0 \end{pmatrix}.$$

7.2. Small norm RH problem

In this subsection, since $E(z;\xi) \equiv I$ as $\xi \in (-\infty,-1)$ or $\xi \in (\frac{1}{8},+\infty)$, we only consider the error matrix-function $E(z;\xi)$ in the case $\xi \in (-1,\frac{1}{8})$ here. In this case, $E(z;\xi)$ satisfies the following RH problem.



(a)



(b)

Figure 12. The jump contour $\Sigma^{(E)}$ for the $E(z;\xi)$. The circles are $U(n(\xi))$.

Riemann-Hilbert Problem 7.5. Find a matrix-valued function $E(z;\xi)$ satisfies following properties:

• Analyticity: $E(z;\xi)$ is analytical in $\mathbb{C}\setminus\Sigma^{(E)}$, where

$$\Sigma^{(E)}=\partial U(n(\xi))\cup(\Sigma^{(2)}\backslash U(n(\xi)));$$

• Asymptotic behaviors:

$$E(z;\xi)\sim I+O(z^{-1}),\quad |z|\to\infty; \tag{7.37}$$

• Jump condition: $E(z;\xi)$ has continuous boundary values $E_\pm(z;\xi)$ on $\Sigma^{(E)}$ (shown in Fig. 12) satisfying

$$E_+(z;\xi)=E_-(z;\xi)V^{(E)}(z),$$

where the jump matrix $V^{(E)}(z)$ is given by

$$V^{(E)}(z)=\begin{cases} M^{(r)}(z)V^{(2)}(z)M^{(r)}(z)^{-1}, & z\in\Sigma^{(2)}\backslash U(n(\xi)),\\ M^{(r)}(z)M^{lo}(z)M^{(r)}(z)^{-1}, & z\in\partial U(n(\xi)). \end{cases} \tag{7.38}$$

From Proposition 5.4, we get:

$$\left\|V^{(E)}(z)-I\right\|_p\lesssim\begin{cases} \exp\left\{-t\tau_p\right\}, & z\in\Sigma_{kj}\backslash U(n(\xi)),\\ \exp\left\{-t\tau_p'\right\}, & z\in\Sigma_{k\pm}'. \end{cases} \tag{7.39}$$

So that $M^{(r)}(z)$ is bounded for $z\in\partial U(n(\xi))$, by using Proposition 7.2, we get

$$|V^{(E)}(z)-I|=|M^{(r)}(z)^{-1}(M^{lo}(z)-I)M^{(r)}(z)|=O(t^{-1/2}). \tag{7.40}$$

Then we can obtain that the existence and uniqueness of the RH problem 7.5 via a small-norm RH problem. In addition, from Beal-Coifman theory, we get the solution of the RH problem 7.5

$$E(z;\xi)=I+\frac{1}{2\pi i}\int_{\Sigma^{(E)}}\frac{(I+\varpi(s))(V^{(E)}(s)-I)}{s-z}ds, \tag{7.41}$$

where the $\varpi\in L^\infty(\Sigma^{(E)})$ is the unique solution of

$$(1-C_E)\varpi=C_E(I). \tag{7.42}$$

C_E is a integral operator: $L^\infty(\Sigma^{(E)})\to L^2(\Sigma^{(E)})$, and is defined as

$$C_E(f)(z)=C_-\left(f(V^{(E)}(z)-I)\right),$$

where the C_- is the usual Cauchy projection operator on $\Sigma^{(E)}$

$$C_-(f)(s) = \lim_{z\to\Sigma_-^{(E)}} \frac{1}{2\pi i}\int_{\Sigma^{(E)}} \frac{f(s)}{s-z}ds.$$

From (7.40), we get

$$\|C_E\| \leq \|C_-\|\|V^{(E)}(z) - I\|_2 \lesssim O(t^{-1/2}). \tag{7.43}$$

So that $1 - C_E$ is invertible for as t is sufficiently large. Therefore, ϖ exists and is unique. Additionally,

$$\|\varpi\|_{L^\infty(\Sigma^{(E)})} \lesssim \frac{\|C_E\|}{1-\|C_E\|} \lesssim t^{-1/2}. \tag{7.44}$$

Then we have to reconstruct the solution $u(y,t)$ of (1.1). The asymptotic behavior of $E(z;\xi)$ as $z \to i$and the long time asymptotic behavior of $E(i)$ are needed. It is worth noting that when we estimate its asymptotic behavior, we only have to consider the computation on $\partial U(n(\xi))$ from (7.41) and (7.39), since it tends to zero exponentially on the other bounds.

Proposition 7.3. As $z \to i$,

$$E(z;\xi) = E(i) + E_1(z-i) + O((z-i)^2), \tag{7.45}$$

where

$$E(i) = I + \frac{1}{2\pi i}\int_{\Sigma^{(E)}} \frac{(I+\varpi(s))(V^{(E)}-I)}{s-i}ds. \tag{7.46}$$

And $E(i)$ has long time asymptotic behavior

$$E(i) = I + t^{-1/2}E^{(0)} + O(t^{-1}), \tag{7.47}$$

where

$$\begin{aligned} E^{(0)} &= \sum_{k=1}^{n(\xi)} \frac{1}{2\pi i}\int_{\partial U_{\xi_k}} \frac{M^{(r)}(s)^{-1}H_k(\xi)M^{(r)}(s)}{(s-i)\,(s-\xi_k)}ds \\ &= \sum_{k=1}^{n(\xi)} \frac{1}{\xi_k - i} M^{(r)}\left(\xi_k\right)^{-1} H_k(\xi) M^{(r)}\left(\xi_k\right). \end{aligned} \tag{7.48}$$

The last equality follows from a residue calculation. Furthermore,

$$E_1 = -\frac{1}{2\pi i}\int_{\Sigma^{(E)}} \frac{(I+\varpi(s))\left(V^{(E)}-I\right)}{(z-i)^2}ds \tag{7.49}$$

holds long time asymptotic behavior condition

$$E_1 = t^{-1/2}E^{(1)} + O(t^{-1}), \tag{7.50}$$

where

$$
\begin{aligned}
E^{(1)} &= -\sum_{k=1}^{n(\xi)} \frac{1}{2\pi i} \int_{\partial U_{\xi_k}} \frac{M^{(r)}(s)^{-1} H_k(\xi) M^{(r)}(s)}{(s-i)^2 (s-\xi_k)} ds \\
&= -\sum_{k=1}^{n(\xi)} \frac{1}{(\xi_k - i)^2} M^{(r)}(\xi_k)^{-1} H_k(\xi) M^{(r)}(\xi_k).
\end{aligned} \tag{7.51}
$$

8. Pure $\overline{\partial}$-problem

In this section, we will research the asymptotics behavior of $M^{(3)}(z)$. The solutions of the $\overline{\partial}$-problem for $M^{(3)}(z)$ satisfies the integral equation.

$$
M^{(3)}(z) = I + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s) W^{(3)}(s)}{s-z} dm(s), \tag{8.1}
$$

where $m(s)$ is the Lebesgue measure on the $\mathbb{C}$. C_z is the left Cauchy-Green integral operator

$$
f C_z(z) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s) W^{(3)}(s)}{s-z} dm(s), \tag{8.2}
$$

then we can rewrite equation (8.1) as

$$
M^{(3)}(z) = I \cdot (I - C_z)^{-1}. \tag{8.3}
$$

To reconstruct the solution of equation (1.1), we need to prove the existence of operator $(I - C_z)^{-1}$ and derive the asymptotic expansion of $M^{(3)}(z)$ as $z \to i$. These properties will be demonstrated in the subsequent parts of this section. Since $W^{(3)}$ has different properties and structures in the regions $\xi \in (-\infty, -1) \cup (1, +\infty)$ and $\xi \in (-1, \frac{1}{8})$, we consider these cases respectively in subsection 8.1 and subsection 8.2.

8.1. Region $\xi \in (-\infty, -1) \cup (\frac{1}{8}, +\infty)$

First we give the proof of the existence of operator $(I - C_z)^{-1}$.

Lemma 8.1. When $t \to \infty$, the estimation to norm of the integral operator C_z is

$$
\| C_z \|_{L^\infty \to L^\infty} \lesssim t^{-1/2}, \tag{8.4}
$$

it means that $(I - C_z)^{-1}$ exists.

Proof. For any $f \in L^\infty$,

$$\|fC_z\|_{L^\infty} \leq \|f\|_{L^\infty}\tfrac{1}{\pi}\iint_{\mathbb{C}}\tfrac{\left|W^{(3)}(s)\right|}{|z-s|}dm(s).$$

So we only have to consider the integral $\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\left|W^{(3)}(s)\right|}{|z-s|}dm(s)$. Since $W^{(3)}(s)$ is a sectorial function, we start to consider it on ever sector. It is known that $W^{(3)}(s) = M^{(r)}(z)\bar{\partial}R^{(2)}(z)M^{(r)}(z)^{-1}$, and $W^{(3)}(s) \equiv 0$ out of $\overline{\Omega}$. We take the case Ω_1 as $\xi < -1/4$ for example here. Proposition 6.1 and 7.3 shows the boundedness of $M^{(r)}(z)$ and $M^{(r)}(z)^{-1}$ for $z \in \overline{\Omega}$, so

$$\frac{1}{\pi}\iint_{\Omega_1}\frac{\left|W^{(3)}(s)\right|}{|z-s|}dm(s) \lesssim \frac{1}{\pi}\iint_{\Omega_1}\frac{\left|\bar{\partial}R_1(s)e^{2it\theta}\right|}{|z-s|}dm(s) \tag{8.5}$$

Similar to (5.52) in Proposition 5.2, we can divide the integral $\iint_{\Omega_1}\frac{\left|\bar{\partial}R_1(s)\right|}{|z-s|}dm(s)$ into two parts:

$$\iint_{\Omega_1}\frac{\left|\bar{\partial}R_1(s)\right|e^{-2t\operatorname{Im}\theta}}{|z-s|}dm(s) \lesssim \iint_{\Omega_1}\frac{\left|f_1'(s)\right|e^{-2t\operatorname{Im}\theta}}{|z-s|}dm(s) + \iint_{\Omega_1}\frac{|s|^{-1/2}e^{-2t\operatorname{Im}\theta}}{|z-s|}dm(s). \tag{8.6}$$

If we take $s = \epsilon + i\nu = le^{i\varsigma}$, $z = x + yi$. Then,

$$\begin{aligned}
&\left\||s-z|^{-1}\right\|_{L^q(\nu,+\infty)} = \left(\int_{\Omega_1}|s-z|^{-q}d\epsilon\right)^{\frac{1}{q}}\\
&= \left(\int_{\Omega_1}\left((\epsilon-x)^2+(\nu-y)^2\right)^{-\frac{q}{2}}(\nu-y)d\frac{\epsilon-x}{\nu-y}\right)^{\frac{1}{q}}\\
&= \left(\int_{\Omega_1}\left(\left(\frac{\epsilon-x}{\nu-y}\right)^2+1\right)^{-\frac{q}{2}}(\nu-y)^{1-q}d\frac{\epsilon-x}{\nu-y}\right)^{\frac{1}{q}}\\
&= \left\{\int_{z_0}^{+\infty}\left[\left(\frac{\epsilon-x}{\nu-y}\right)^2+1\right]^{-q/2}d\left(\frac{\epsilon-x}{\nu-y}\right)\right\}^{1/q}|\nu-y|^{1/q-1}\\
&\lesssim |\nu-y|^{1/q-1},
\end{aligned} \tag{8.7}$$

with $1 \leq q < +\infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. In addition, $e^{-2tIm\theta} \leq e^{-c(\xi)t\nu}$ is known from Corollary 5.1.

Thus, we have

$$\begin{aligned}
\iint_{\Omega_1}\frac{\left|f_1'(s)\right|e^{-2t\operatorname{Im}\theta}}{|z-s|}dm(s) &\leq \int_0^{+\infty}\int_\nu^{\infty}\frac{\left|f_1'(s)\right|e^{-c(\xi)t\nu}}{|z-s|}d\epsilon d\nu\\
&\lesssim \int_0^{+\infty}\left(\int_\nu^{+\infty}\frac{1}{|z-s|^2}d\epsilon\right)^{\frac{1}{2}}\left(|f_1'(s)|^2d\epsilon\right)^{\frac{1}{2}}e^{-c(\xi)t\nu}d\nu\\
&\lesssim t^{-\frac{1}{2}}\int_0^{+\infty}\frac{1}{\sqrt{t|\nu-y|}}e^{-c(\xi)t\nu}d(t\nu) \lesssim t^{-\frac{1}{2}}.
\end{aligned} \tag{8.8}$$

On the other hand, from Cauchy-Schwarz inequality, we have

$$\begin{aligned}\iint_{\Omega_1} \frac{|s|^{-1/2}e^{-2t\operatorname{Im}\theta}}{|z-s|}dm(s) &\le \int_0^{+\infty} \left\||s-z|^{-1}\right\|_{L^q(\mathbb{R}^+)} \left\||z|^{-\frac{1}{2}}\right\|_{L^p(\mathbb{R}^+)} e^{-c(\xi)t\nu}d\nu \\ &\lesssim \int_0^{+\infty} |\nu-y|^{\frac{1}{q}-1}\nu^{-\frac{1}{2}+\frac{1}{p}}e^{-c(\xi)t\nu}d\nu \\ &\lesssim \int_0^{+\infty} \nu^{-\frac{1}{2}}e^{-c(\xi)t\nu}d\nu \lesssim t^{-1/2},\end{aligned} \tag{8.9}$$

It is worth noted that for $p>2$,

$$\begin{aligned}&\left\||z|^{-\frac{1}{2}}\right\|_{L^p} = \left(\int_\nu^{+\infty} \left|\sqrt{\epsilon^2+\nu^2}\right|^{-\frac{p}{2}} d\epsilon\right)^{\frac{1}{p}} = \left(\int_\nu^{+\infty} |l|^{-\frac{p}{2}+1}\epsilon^{-1}dl\right)^{\frac{1}{p}} \\ &\lesssim \left(\int_\nu^{+\infty} |l|^{-\frac{p}{2}}dl\right)^{\frac{1}{p}} \lesssim \left(|l|^{-\frac{p}{2}+1}\right)^{\frac{1}{p}} \lesssim \left(\nu^{-\frac{p}{2}+1}\right)^{\frac{1}{p}}.\end{aligned} \tag{8.10}$$

The proof is completed from (8.8)-(8.10). □

Then we start to derive the asymptotic expansion of $M^{(3)}(z)$ as $z \to i$. Take $z=i$ in (8.1), we note that

$$M^{(3)}(i) = I + \frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s-i}dm(s). \tag{8.11}$$

Proposition 8.1. There exists a small positive constant $\varepsilon < \frac{1}{4}$, the solution $M^{(3)}(z)$ of $\bar{\partial}$-problem has the following estimation

$$\left\|M^{(3)}(i) - I\right\| = \left\|\frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s-i}dm(s)\right\| \lesssim t^{-1+2\varepsilon}. \tag{8.12}$$

As $z \to i$, $M^{(3)}(z)$ has asymptotic expansion

$$M^{(3)}(z) = M^{(3)}(i) + M_1^{(3)}(x,t)(z-i) + O((z-i)^2), \tag{8.13}$$

where $M_1^{(3)}(x,t)$ is a z-independent coefficient with

$$M_1^{(3)}(x,t) = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{(s-i)^2}dm(s). \tag{8.14}$$

Furthermore, there exist constants t_1, such that for all $t > t_1$, $M_1^{(3)}(x,t)$ satisfies

$$|M_1^{(3)}(x,t)| \lesssim t^{-1+2\varepsilon}. \tag{8.15}$$

Proof. First we prove (8.12), The reader can prove (8.15) by a similar step. From Lemma 8.1 and (8.3), we have $\|M^{(3)}\|_\infty \lesssim 1$ for large t. And for same reason, we take the case Ω_1 as $\xi < -1/4$ for example here, and let $s = \epsilon + i\nu = le^{i\varsigma}$. $M^{(3)}(i) - I$ can also be divided into two parts from (5.53):

$$\frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s-i}dm(s) \lesssim \iint_{\Omega_1} \frac{|f_1'(s)|\, e^{-2t\,\mathrm{Im}\,\theta}}{|i-s|}dm(s) + \iint_{\Omega_1} \frac{|s|^{-1}e^{-2t\,\mathrm{Im}\,\theta}}{|i-s|}dm(s). \quad (8.16)$$

We can derive that

$$\begin{aligned}\iint_{\Omega_1} \frac{|f_1'(s)|\, e^{-2t\,\mathrm{Im}\,\theta}}{|i-s|}dm(s) &\leq \int_0^{+\infty}\int_\nu^{+\infty} \frac{|f_1'(s)|\, e^{-c(\xi)t\nu}}{|i-s|}d\epsilon d\nu \\ &\leq \int_0^{+\infty} \|f_1'\|_{L^1(\mathbb{R}^+)}\sqrt{2}e^{-c(\xi)t\nu}d\nu \lesssim \int_0^{+\infty} e^{-c(\xi)t\nu}d\nu \leq t^{-1},\end{aligned} \quad (8.17)$$

where $f_1' \in L^1(\mathbb{R})$ since $|p_1'| \lesssim |r'|, r' \in L^1(\mathbb{R})$, and $|i-s| \geq \frac{1}{\sqrt{2}}$ for $s \in \Omega_1$.

The second integral is divided it into two parts:

$$\begin{aligned}\iint_{\Omega_1} \frac{|s|^{-1}e^{-2t\,\mathrm{Im}\,\theta}}{|i-s|}dm(s) &\leq \int_0^{+\infty}\int_\nu^{+\infty} \frac{|s|^{-1}e^{-2tIm\theta}}{|i-s|}dm(s) \\ &\leq \int_0^{\frac{1}{2}}\int_\nu^{+\infty} \frac{|s|^{-1}e^{-c(\xi)t\nu}}{|i-s|}d\epsilon d\nu + \int_{\frac{1}{2}}^{+\infty}\int_\nu^{+\infty} \frac{|s|^{-1}e^{-c(\xi)t\nu}}{|i-s|}d\epsilon d\nu.\end{aligned} \quad (8.18)$$

Then we have

$$\begin{aligned}&\int_0^{\frac{1}{2}}\int_\nu^{+\infty} \tfrac{|s|^{-1}e^{-c(\xi)t\nu}}{|i-s|}d\epsilon d\nu \\ &\leq \int_0^{\frac{1}{2}}\int_\nu^{+\infty} (\epsilon^2+\nu^2)^{-\frac{1}{2}-\varepsilon}\left(\epsilon^2+(\nu-1)^2\right)^{-\frac{1}{2}+\varepsilon} d\epsilon e^{-c(\xi)t\nu}d\nu \\ &\leq \int_0^{\frac{1}{2}}\left[\int_\nu^{+\infty}\left(1+\left(\tfrac{\epsilon}{\nu}\right)^2\right)^{-\frac{1}{2}-\varepsilon}\nu^{-2\varepsilon}d\tfrac{\epsilon}{\nu}\right]\left(\nu^2+(\nu-1)^2\right)^{-\frac{1}{2}+\varepsilon} e^{-c(\xi)t\nu}d\nu \\ &\lesssim \int_0^{\frac{1}{2}} \nu^{-2\varepsilon}\left(\tfrac{1}{2}\right)^{-\frac{1}{2}+\varepsilon} e^{-c(\xi)t\nu}d\nu \lesssim t^{-1+2\varepsilon}.\end{aligned} \quad (8.19)$$

since $0<\nu<\frac{1}{2}$, $|s-i|^2 = \epsilon^2+(\nu-1)^2 > \epsilon^2+\nu^2 = |s|^2$, while as $\nu > \frac{1}{2}$, $|s-i|^2 < |s|^2$.

Similarly,

$$\begin{aligned}&\int_{\frac{1}{2}}^{+\infty}\int_\nu^{+\infty} \tfrac{|s|^{-1}e^{-c(\xi)t\nu}}{|i-s|}d\epsilon d\nu \\ &\leq \int_{\frac{1}{2}}^{+\infty} \left(2\nu^2\right)^{-\frac{1}{2}+\varepsilon}|\nu-1|^{-2\varepsilon}e^{-c(\xi)t\nu}d\nu \\ &\lesssim \int_{\frac{1}{2}}^1 (1-\nu)^{-2\varepsilon}e^{-c(\xi)t\nu}d\nu + \int_1^{+\infty}(\nu-1)^{-2\varepsilon}e^{-c(\xi)t\nu}d\nu \\ &\leq e^{-\frac{c(\xi)}{2}t}\int_{\frac{1}{2}}^1(1-\nu)^{-2\varepsilon}d\nu + e^{-c(\xi)t}\int_1^{+\infty}(\nu-1)^{-2\varepsilon}e^{-c(\xi)t(\nu-1)}d(\nu-1) \lesssim e^{-\frac{c(\xi)}{2}t}.\end{aligned} \quad (8.20)$$

The proof of (8.12) is completed. □

8.2. Region $\xi \in (-1, \frac{1}{8})$

Lemma 8.2. When $t \to \infty$, the estimation on the norm of the integral operator C_z is:

$$\|C_z\|_{L^\infty \to L^\infty} \lesssim t^{-1/4}, \tag{8.21}$$

it means that $(I - C_z)^{-1}$ exists.

Proof. For any $f \in L^\infty$,

$$\|fC_z\|_{L^\infty} \leq \|f\|_{L^\infty} \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\left|W^{(3)}(s)\right|}{|z-s|} dm(s). \tag{8.22}$$

Based on the same reason of Lemma 8.1, we only have to consider the integral $\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\left|W^{(3)}(s)\right|}{|z-s|} dm(s)$ on ever sector. Also we have $W^{(3)}(s) = M^{(r)}(z)\bar{\partial}R^{(2)}(z)M^{(r)}(z)^{-1}$, and $W^{(3)}(s) \equiv 0$ out of $\overline{\Omega}$. We take the case Ω_{11} as $\xi \in (-1, 0)$ for example here, and let $z = x + yi$, $s = \xi_1 + \epsilon + i\nu$ with $x, y, \epsilon, \nu \in \mathbb{R}$.

From the boundedness of $M^{(R)}(z)$ and $M^{(R)}(z)^{-1}$ for $z \in \overline{\Omega}$ (Proposition 6.2, 7.2 and 7.3),

$$\frac{1}{\pi} \iint_{\Omega_{11}} \frac{\left|W^{(3)}(s)\right|}{|z-s|} dm(s) \lesssim \frac{1}{\pi} \iint_{\Omega_{11}} \frac{\left|\bar{\partial}R_{11}(s)e^{2it\theta}\right|}{|z-s|} dm(s). \tag{8.23}$$

According to (5.64) in Proposition 5.3, we can divide the integral $\iint_{\Omega_{11}} \frac{\left|\bar{\partial}R_{11}(s)\right|}{|z-s|} dm(s)$ into two parts:

$$\iint_{\Omega_{11}} \frac{\left|\bar{\partial}R_{11}(s)e^{-2tIm\theta}\right|}{|z-s|} dm(s) \lesssim \iint_{\Omega_{11}} \frac{|f_{11}'(s)|\, e^{-2t\operatorname{Im}\theta}}{|z-s|} dm(s) + \iint_{\Omega_{11}} \frac{|s-\xi_1|^{-1/2}\, e^{-2t\operatorname{Im}\theta}}{|z-s|} dm(s). \tag{8.24}$$

Then Lemma 5.2 gives that

$$\begin{aligned}
&\iint_{\Omega_{11}} \frac{\left|f_{11}'(s)\right|e^{-2t\operatorname{Im}\theta}}{|z-s|} dm(s) \leq \int_0^{+\infty} \int_\nu^{+\infty} \frac{\left|f_{11}'(s)\right|}{|z-s|} e^{-c(\xi)t\nu \frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\epsilon d\nu \\
&\leq \int_0^{+\infty} \|f_{11}'\|_2 \, \||z-s|^{-1}\|_2 \, e^{-2c(\xi)t\nu^2 \frac{\epsilon+\xi_1}{4+\xi_1^2+\nu^2}} d\nu \\
&\lesssim \int_0^{+\infty} |\nu - y|^{-1/2} e^{-2c(\xi)t\nu^2 \frac{\epsilon+\xi_1}{4+\xi_1^2+\nu^2}} d\nu \\
&= \left(\int_0^y + \int_y^{+\infty}\right) |\nu - y|^{-1/2} e^{-2c(\xi)t\nu^2 \frac{\nu+\xi_1}{4+\xi_1^2+\nu^2}} d\nu.
\end{aligned} \tag{8.25}$$

For $0 < \nu < y$, since the inequality $e^{-z} \lesssim z^{-1/4}$, we have

$$\int_0^y (y-\nu)^{-1/2} e^{-2c(\xi)t\nu^2 \frac{u+\xi_1}{4+\xi_1^2+\nu^2}} dv \lesssim \int_0^y (y-\nu)^{-1/2}\nu^{-1/2} d\nu t^{-1/4} \lesssim t^{-1/4}. \tag{8.26}$$

For $\nu > y$, we make the translate $w = \nu - y : 0 \to +\infty$, then

$$\begin{aligned}
&\int_y^{+\infty} (\nu - y)^{-1/2} e^{-2c(\xi)t\nu^2 \frac{\epsilon+\xi_1}{4+\xi_1^2+\nu^2}} d\nu \\
&= \int_0^{+\infty} w^{-1/2} e^{-2c(\xi)t(y+w)^2} \frac{y+w+\xi_1}{4+\xi_1^2+(y+w)^2} dw \\
&= \int_0^{+\infty} w^{-1/2} e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2}(w+y)} dw \\
&= \int_0^{+\infty} w^{-1/2} e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2} w} dw \cdot e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2} y} \\
&\lesssim e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2} y}.
\end{aligned} \tag{8.27}$$

On the other hand, for $p > 2$, $1/p + 1/q = 1$, we have

$$\begin{aligned}
&\iint_{\Omega_{11}} \frac{|s-\xi_1|^{-1/2} e^{-2t \operatorname{Im}\theta}}{|z-s|} dm(s) \leq \int_0^{+\infty} \left\| |s-\xi_1|^{-1/2} \right\|_p \left\| |z-s|^{-1} \right\|_q e^{-c(\xi)t\nu \frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\nu \\
&\lesssim \int_0^{+\infty} \nu^{1/p-1/2} |y-\nu|^{1/q-1} e^{-c(\xi)t\nu \frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\nu \\
&= \left(\int_0^y + \int_y^{+\infty} \right) \nu^{1/p-1/2} |y-\nu|^{1/q-1} e^{-c(\xi)t\nu \frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\nu.
\end{aligned} \tag{8.28}$$

In the same way, as $0 < \nu < y$,

$$\begin{aligned}
&\int_0^y \nu^{1/p-1/2} |y-\nu|^{1/q-1} e^{-c(\xi)t\nu \frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\nu \\
&\lesssim \int_0^y \nu^{1/p-1} (y-\nu)^{1/q-1} d\nu t^{-1/4} \\
&\lesssim t^{-1/4},
\end{aligned} \tag{8.29}$$

as $\nu > y$,

$$\begin{aligned}
&\int_y^{+\infty} \nu^{1/p-1/2} |y-\nu|^{1/q-1} e^{-c(\xi)t\nu \frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\nu \\
&\leq \int_y^{+\infty} (\nu - y)^{-1/2} e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2}(\nu-y)} d\nu e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2} y} \\
&\lesssim e^{-2c(\xi)ty \frac{y+\xi_1}{4+\xi_1^2+y^2} y}.
\end{aligned} \tag{8.30}$$

Above all, the Lemma is confirmed. $\square$

Then we derive the asymptotic expansion of $M^{(3)}(z)$ as $z \to i$.

Proposition 8.2. The solution $M^{(3)}(z)$ of $\overline{\partial}$-problem has the following estimation:

$$\left\| M^{(3)}(i) - I \right\| = \left\| \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s) W^{(3)}(s)}{s - i} dm(s) \right\| \lesssim t^{-3/4}. \tag{8.31}$$

As $z \to i$, $M^{(3)}(z)$ has asymptotic expansion:

$$M^{(3)}(z) = M^{(3)}(i) + M_1^{(3)}(y,t)(z-i) + O((z-i)^2), \tag{8.32}$$

where $M_1^{(3)}(y,t)$ is a z-independent coefficient with

$$M_1^{(3)}(y,t) = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{(s-i)^2} dm(s). \tag{8.33}$$

Furthermore, there exist constants T_1, such that for all $t > T_1$, $M_1^{(3)}(y,t)$ satisfies

$$|M_1^{(3)}(y,t)| \lesssim t^{-3/4}. \tag{8.34}$$

Proof. First we prove (8.31), The reader can prove (8.34) by a similar step. From Lemma 8.2 and (8.3), we have $\|M^{(3)}\|_\infty \lesssim 1$ for large t. And for same reason, we take the case Ω_{11} as $\xi \in (-1,0)$ for example here. $M^{(3)}(i) - I$ can also be divided into two parts from (5.64) in Proposition 5.3:

$$\frac{1}{\pi}\iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{(s-i)^2} dm(s) \lesssim \iint_{\Omega_{11}} \frac{|f_{11}'(s)|\, e^{-2t\operatorname{Im}\theta}}{|i-s|} dm(s) + \iint_{\Omega_{11}} \frac{|s-\xi_1|^{-1/2}\, e^{-2t\operatorname{Im}\theta}}{|i-s|} dm(s). \tag{8.35}$$

Since that $|i-s|^{-1}$ has nonzero maximum, we have

$$\begin{aligned}
&\iint_{\Omega_{11}} \frac{|f_{11}'(s)|\, e^{-2t\operatorname{Im}\theta}}{|i-s|} dm(s) \le \int_0^{+\infty}\int_\nu^{+\infty} |f_{11}'(s)|\, e^{-c(\xi)t\nu\frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}} d\epsilon d\nu \\
&\le \int_0^{+\infty} e^{-c(\xi)t\frac{\nu^3}{4+\nu^2+(\nu+\xi_1)^2}} \|f_{11}'(s)\|_2 \left(\int_\nu^{+\infty} e^{-\epsilon 2c(\xi)t\nu}\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2} d\epsilon\right)^{1/2} d\nu \\
&\lesssim t^{-1/2}\int_0^{+\infty}\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/2} e^{-2c(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}} d\nu \\
&= t^{-1/2}\left(\int_0^1 + \int_1^{+\infty}\right)\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/2} e^{-2c(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}} d\nu.
\end{aligned} \tag{8.36}$$

As $0 < \nu < 1$, from $\frac{4+\nu^2+(\nu+\xi_1)^2}{\nu(\nu+2\xi_1)} \lesssim \nu^{-1}$, then

$$\begin{aligned}
&\int_0^1 \left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/2} e^{-2c(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}} d\nu \\
&\lesssim \int_0^1 \nu^{-1/2} e^{-c't\nu^2} d\nu \lesssim t^{-1/4}.
\end{aligned} \tag{8.37}$$

As $\nu > 1$,

$$\begin{aligned}&\int_1^{+\infty}\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/2}e^{-2c(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}}d\nu\\ \lesssim&\int_1^{+\infty}e^{-c't(\nu+\xi_1)}d\nu\lesssim t^{-1}e^{-c't(1+\xi_1)}.\end{aligned}\tag{8.38}$$

So we have $\iint_{\Omega_{11}}\frac{|p'_{11}(s)|e^{-2t\operatorname{Im}\theta}}{|i-s|}dm(s)\lesssim t^{-3/4}$.

On the other hand, we take $2<\rho<4$, and $1/p+1/q=1$, the second integral holds

$$\begin{aligned}&\iint_{\Omega_{11}}\frac{|s-\xi_1|^{-1/2}e^{-2t\operatorname{Im}\theta}}{|i-s|}dm(s)\leq\int_0^{+\infty}\int_\nu^{+\infty}|s-\xi_1|^{-1/2}e^{-c(\xi)t\nu\frac{\epsilon^2+2\epsilon\xi_1+\nu^2}{4+|s|^2}}d\epsilon d\nu\\ &\leq\int_0^{+\infty}e^{-c(\xi)t\frac{\nu^3}{4+\nu^2+(\nu+\xi_1)^2}}\left\||s-\xi_1|^{-1/2}\right\|_p\left(\int_\nu^{+\infty}e^{-\epsilon qc(\xi)t\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}}d\epsilon\right)^{1/q}d\nu\\ &\lesssim t^{-1/q}\int_0^{+\infty}\nu^{2/p-1/2}\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/q}e^{-qc(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}}d\nu\\ &=t^{-1/q}\left(\int_0^1+\int_1^{+\infty}\right)\nu^{1/p-1/2}\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/q}e^{-qc(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}}d\nu.\end{aligned}\tag{8.39}$$

As $0<\nu<1$,

$$\begin{aligned}&\int_0^1\nu^{1/p-1/2}\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/q}e^{-qc(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}}d\nu\\ \lesssim&\int_0^1\nu^{\frac{2}{p}-\frac{3}{2}}e^{-c_pt\nu}d\nu\\ =&\int_0^1\nu^{\frac{2}{p}-\frac{3}{2}}t^{\frac{2}{p}-\frac{3}{2}}e^{-c_pt\nu}d(\nu t)\cdot t^{\frac{1}{2}-\frac{2}{p}}\\ \lesssim&\,t^{\frac{1}{2}-\frac{2}{p}}\lesssim t^{\frac{1}{4}-\frac{1}{p}}.\end{aligned}\tag{8.40}$$

And as $\nu>1$,

$$\begin{aligned}&\int_1^{+\infty}\nu^{1/p-1/2}\left(\nu\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}\right)^{-1/q}e^{-qc(\xi)t\nu^2\frac{\nu+2\xi_1}{4+\nu^2+(\nu+\xi_1)^2}}d\nu\\ \lesssim&\int_1^{+\infty}e^{-c'_pt(\nu+\xi_1)}d\nu\lesssim t^{-1}e^{-c'_pt(1+\xi_1)}.\end{aligned}\tag{8.41}$$

So $\iint_{\Omega_{11}}\frac{|s-\xi_1|^{-1/2}e^{-2t\operatorname{Im}\theta}}{|i-s|}dm(s)\lesssim t^{1/4-1/p-1/q}=t^{-3/4}$.

Finally we note that $|(s-i)|^{-1}$ is bounded. Via (8.33) and (8.31),

$$\begin{aligned}\left|M_1^{(3)}(y,t)\right| &= \frac{1}{\pi}\iint_{\mathbb{C}}\left|\frac{M^{(3)}(s)W^{(3)}(s)}{|s-i|^2}\right| dm(s)\\ &\leq \frac{1}{\pi}\iint_{\mathbb{C}}\left|\frac{M^{(3)}(s)W^{(3)}(s)}{|s-i|}\right| dm(s)\\ &\lesssim t^{-3/4}.\end{aligned} \tag{8.42}$$

The prove of (8.31) is completed. $\square$

9. Long-time asymptotics for the mCH equation

In this section, we construct the long time asymptotics of equation (1.1). Inverting the sequence of transformations (5.30), (5.66), (5.89) and (6.4), we have

$$M(z) = M^{(3)}(z)E(z;\xi)M^{(r)}(z)R^{(2)}(z)^{-1}T(z)^{-\sigma_3}. \tag{9.1}$$

In order to reconstruct $u(x,t)$ via (3.16), in above equation (9.1) we take $z\to i$ out of $\overline{\Omega}$. In this case, $R^{(2)}(z)=I$. In addition, we use Propositions 5.1 and 8.1, can obtain behavior when $z\to i$

$$\begin{aligned}M(z) = &\left(M^{(3)}(i)+M_1^{(3)}(z)(z-i)\right)E(z;\xi)M_\Lambda^{(r)}(z)\\ &\cdot T(i)^{-\sigma_3}\left(I-\Sigma_0(\xi)(z-i)\right)^{-\sigma_3}+\mathcal{O}\left((z-i)^2\right),\end{aligned} \tag{9.2}$$

where $\Sigma_0(\xi)$ is defined by (5.22).

For $\xi\in(-\infty,-1)\cup(\frac{1}{8},+\infty)$,

$$M(z) = M_\Lambda^{(r)}(z)T(i)^{-\sigma_3}\left(I-\Sigma_0(\xi)(z-i)\right)^{-\sigma_3}+\mathcal{O}\left((z-i)^{-2}\right)+\mathcal{O}\left(t^{-1+2\varepsilon}\right), \tag{9.3}$$

and

$$M(i) = M_\Lambda^{(r)}(i)T(i)^{-\sigma_3}+O(t^{-1+2\varepsilon}). \tag{9.4}$$

For $\xi\in(-1,\frac{1}{8})$,

$$M(z) = \left(I+E^{(0)}t^{-\frac{1}{2}}+E^{(1)}t^{-\frac{1}{2}}(z-i)\right)M_\Lambda^{(r)}(z)T(i)^{-\sigma_3}\left(I-\Sigma_0(\xi)(z-i)\right)^{-\sigma_3}+O((z-i)^2)+O(t^{-\frac{3}{4}}), \tag{9.5}$$

$$M(i) = \left(I+E^{(0)}t^{-\frac{1}{2}}\right)M_\Lambda^{(r)}(i)T(i)^{-\sigma_3}+O(t^{-\frac{3}{4}}), \tag{9.6}$$

then we have

$$M(z)=M_\Lambda^{(r)}(z)T(i)^{-\sigma_3}\left(I-\Sigma_0(\xi)(z-i)\right)^{-\sigma_3}\left(I+E^{(0)}t^{-\frac{1}{2}}\right)$$
$$+M_\Lambda^{(r)}(z)T(i)^{-\sigma_3}\left(I-\Sigma_0(\xi)(z-i)\right)^{-\sigma_3}E^{(1)}t^{-\frac{1}{2}}(z-i)+O((z-i)^2)+O(t^{-\frac{3}{4}}), \tag{9.7}$$

where $E^{(0)}$ is shown in (7.48).

We substitute the properties of $M(z)$ (9.3) and (9.5) into the reconstruction formula (3.16) to get the following Theorem.

Theorem 9.1. Let $u(x,t)$ be the solution for the initial-value problem (1.6)-(1.7) with generic data $u_0(x)\in H^{4,2}(\mathbb{R})$ and scatting data $\{r(z),\{\xi_n,c_n\}_{n=1}^{2K_1+2K_2+2K_3}\}$. And $u_\Lambda^r(x,t)$ denote the $\mathcal{N}(\Lambda)$-soliton solution corresponding to scattering data $\tilde{\mathcal{D}}_\Lambda=\{0,\{\xi_n,c_n\delta^2(\xi_n)\}_{n\in\Lambda}\}$ shown in Corollary 6.2. And Λ is defined in (5.8).

1. For $\xi\in(-\infty,-1)\cup(\frac{1}{8},+\infty)$, as $t\to+\infty$, we have

$$u(x,t)e^{2g_-}=u^r(x,t;\tilde{\mathcal{D}})e^{2\tilde{g}_-}+O(t^{-1+2\varepsilon}). \tag{9.8}$$

with

$$x(y,t)=y+2\ln|T(i)|+k_+^r(x,t;\tilde{\mathcal{D}}_\Lambda)+O(t^{-1+2\varepsilon}),$$

where $T(i)$, $u^r(x,t;\tilde{\mathcal{D}}_\Lambda)$ and $k_+^r(x,t;\tilde{\mathcal{D}}_\Lambda)$ are shown in Proposition 5.1 and Corollary 6.2 respectively. Replacing u in equation (2.10) with $u^r(x,t;\tilde{\mathcal{D}})$ yields $\tilde{g}$. Here g and $\tilde{g}$ are pure imaginary numbers.

2. For $\xi\in(-1,\frac{1}{8})$, as $t\to+\infty$, we have

$$u(x,t)e^{2g_-}=u^r(x,t;\tilde{\mathcal{D}})e^{2\tilde{g}_-}+k_{11}t^{-\frac{1}{2}}+O(t^{-\frac{3}{4}}), \tag{9.9}$$

with

$$x(y,t)=y+2\ln|T(i)|+k_+^r(x,t;\tilde{\mathcal{D}}_\Lambda)+k_{12}t^{-\frac{1}{2}}+O(t^{-\frac{3}{4}}), \tag{9.10}$$

where

$$k_{11} = -\frac{\left[-E^{(0)}M_{\Lambda}^{(r)}(i)\Sigma_0(\xi) + E^{(0)}M_{1,\Lambda}^{(r)}(i) + E^{(1)}M_{\Lambda}^{(r)}(i)\right]_{12}}{M_{\Lambda,22}^{(r)}(i)} - \overline{\left(\frac{\left[E^{(0)}M_{\Lambda}^{(r)}(i)\Sigma_0(\xi) + E^{(0)}M_{1,\Lambda}^{(r)}(i) + E^{(1)}M_{\Lambda}^{(r)}(i)\right]_{21}}{M_{\Lambda,11}^{(r)}(i)}\right)}$$
$$\frac{-M_{\Lambda,12}^{(r)}(i)\Sigma_0(\xi)\left(E^{(0)}M_{\Lambda}^{(r)}(i)\right)_{22} + M_{1,\Lambda,12}^{(r)}(i)\left(E^{(0)}M_{\Lambda}^{(r)}(i)\right)_{22}}{[M_{\Lambda,22}^{(r)}(i)]^2} + \overline{\left(\frac{(M_{\Lambda,21}^{(r)}(i)\Sigma_0(\xi) + M_{1,\Lambda,21}^{(r)}(i))(E^{(0)}M_{\Lambda}^{(r)}(i))_{11}}{[M_{\Lambda,11}^{(r)}(i)]^2}\right)}, \tag{9.11}$$

$$k_{12} = E_{22}^{0} - \overline{E_{11}^{0}} + E_{21}^{0}\frac{M_{\Lambda,12}^{r}(i)}{M_{\Lambda,22}^{r}(i)} - \overline{E_{12}^{0}\frac{M_{\Lambda,21}^{r}(i)}{M_{\Lambda,11}^{r}(i)}}. \tag{9.12}$$

Here $\Sigma_0(\xi)$ is (5.22), $E^{(0)}$ is (7.48) and $E^{(1)}$ is (7.51).

Remark : When u is real, Theorem 9.1 degenerates to Theorem 2 in [27].

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Appendix A. Proof of Lemma 4.3

Proof. $W_{\pm}(e_1)(x,z)$ is given by

$$W_{\pm}(e_1)(x,z) = \int_x^{\pm\infty} e^{2g_-}\left(\frac{i\bar{m}^2 m_x}{4d^2(d+1)} - \frac{i(d+1)\bar{m}_x}{4d^2}\right)\begin{pmatrix} 0 \\ e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} ds + \frac{1}{2z}\int_x^{\pm\infty}\frac{1}{d}\begin{pmatrix} -i|m|^2 \\ e^{2g_-}\bar{m}e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} ds. \tag{A.1}$$

We can derive that

$$\left|W_{\pm}\left(e_{1}\right)(x, z)\right| \lesssim\|m\|_{2}^{2}+\|m\|_{1}. \tag{A.2}$$

According to Lemma 1 in [27], we only consider the first integral of (A.1). Let $\eta=z-\frac{1}{z}$, we have $dz=\left(\frac{1}{2}+\frac{\eta}{2 \sqrt{4+\eta^{2}}}\right) d\eta$. Then from Lemma 2 in [27], we denote

$$H_{1}(x, \eta) \triangleq \int_{x}^{\pm \infty} e^{2 g_{-}}\left(\frac{i \bar{m}^{2} m_{x}}{4 d^{2}(d+1)}-\frac{i(d+1) \bar{m}_{x}}{4 d^{2}}\right) e^{\frac{i}{2} \eta(h(x)-h(s))} ds. \tag{A.3}$$

From Lemma 4.2, we get

$$\left\|H_{1}(x, \eta)\right\|_{C^{0}} \lesssim\left\|m_{x}\right\|_{2}, \quad\left\|H_{1}(x, \eta)\right\|_{L^{2}} \lesssim\left\|m_{x}\right\|_{2,1/2}. \tag{A.4}$$

On the other hand,

$$\begin{aligned}
& W_{\pm, z}\left(e_{1}\right)(x, z) \\
= & \int_{x}^{\pm \infty} e^{2 g_{-}}\left(\frac{i}{2}\left(1+\frac{1}{z^{2}}\right)(h(x)-h(s))\left(\frac{i \bar{m}^{2} m_{x}}{4 d^{2}(d+1)}-\frac{i(d+1) \bar{m}_{x}}{4 d^{2}}\right)\right)\begin{pmatrix} 0 \\ e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} ds \\
& +\int_{x}^{\pm \infty} e^{2 g_{-}} \frac{i \bar{m}(h(x)-h(s))}{4 z d}\left(1+\frac{1}{z^{2}}\right)\begin{pmatrix} 0 \\ e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} ds \\
& -\int_{x}^{\pm \infty} \frac{1}{2 z^{2} d}\begin{pmatrix} -i|m|^{2} \\ e^{2 d_{-}} \bar{m} e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} \end{pmatrix} ds.
\end{aligned} \tag{A.5}$$

Similarly, we denote

$H_{2}(x, s)=\int_{x}^{\pm \infty} \frac{i}{2}(h(x)-h(s))\left(\frac{i \bar{m}^{2} m_{x}}{4 d^{2}(d+1)}-\frac{i(d+1) \bar{m}_{x}}{4 d^{2}}\right) e^{2 g_{-}} e^{\frac{i}{2}(z-1/z)(h(x)-h(s))} ds.$

Since $|h(x)-h(s)| \leq|x-s|+\|m\|_{1}$, we can obtain

$$\left\|H_{2}(x, \eta)\right\|_{C^{0}} \lesssim\left\|m_{x}\right\|_{2,1}+\left\|m_{x}\right\|_{2}\|m\|_{1}, \tag{A.6}$$

$$\left\|H_{2}(x, \eta)\right\|_{L^{2}} \lesssim\left\|m_{x}\right\|_{2,3/2}+\left\|m_{x}\right\|_{2,1/2}\|m\|_{1}. \quad \square \tag{A.7}$$

Appendix B. Proof of Lemma 4.4

Proof. From equation (4.10), we have

$$\left|P_{\pm}(x, s, z)\right|=\left|U_{3}(s, z)\right|. \tag{B.1}$$

Therefore, for any $f(x,s) \in C_B^0(\mathbb{R}^\pm \times (1,+\infty))$, we have

$$|W_\pm(f)(x,z)| \leq \int_x^{\pm\infty} |U_3(s,z)|ds\|f\|_{C_B^0}. \tag{B.2}$$

Denote $P_\pm^n$ is the integral kernel of Volterra operator $[W_\pm]^n$ as

$$P_\pm^n(x,s,z) = \int_x^s \int_{s_1}^s \cdots \int_{s_{n-2}}^z P_\pm(x,s_1,z) P_\pm(s_1,s_2,z) \ldots P_\pm(s_{n-1},s,z)\, ds_{n-1} \ldots ds_1, \tag{B.3}$$

with

$$\left|P_\pm^n(x,s,z)\right| \leq \frac{1}{(n-1)!}\left(\int_x^{\pm\infty} |U_3(s,z)|dy\right)^{n-1} |U_3(s,z)|. \tag{B.4}$$

The standard Volterra theory gives the following operator norm:

$$\left\|(I - W_\pm)^{-1}\right\|_{\mathcal{B}\left(C_B^0\right)} \leq e^{\int_0^{\pm\infty} |U_3(s,1)|ds}. \tag{B.5}$$

Similarly, $W_\pm$ is a bounded operator on C_0,

$$\left\|(I - W_\pm)^{-1}\right\|_{\mathcal{B}(C^0)} \leq e^{\int_{\mathbb{R}} |U_3(s,1)|ds}. \tag{B.6}$$

In addition, $W_\pm$ is a bounded operator on L_2,

$$\|W_\pm\|_{\mathcal{B}(L^2)} \leq \int_{\mathbb{R}} \left|sU_3^2(s,1)\right| ds, \tag{B.7}$$

$$\left\|(I - W_\pm)^{-1}\right\|_{\mathcal{B}(L^2)} \leq e^{\int_{\mathbb{R}} |U_3(s,1)|ds} \int_{\mathbb{R}} \left|sU_3^2(s,1)\right| ds. \quad \square \tag{B.8}$$

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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