

University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

School of Mathematical and Statistical
Sciences Faculty Publications and
Presentations

College of Sciences

12-19-2022

Borsuk–Ulam type theorems for G-spaces with applications to Tucker type lemmas

Oleg R. Musin

The University of Texas Rio Grande Valley

Alexey Yu. Volovikov

Follow this and additional works at: https://scholarworks.utrgv.edu/mss_fac



Part of the [Mathematics Commons](#)

Recommended Citation

Musin, O.R., Volovikov, A.Y. Borsuk–Ulam type theorems for G-spaces with applications to Tucker type lemmas. *J. Fixed Point Theory Appl.* 25, 32 (2023). <https://doi.org/10.1007/s11784-022-01035-7>

This Article is brought to you for free and open access by the College of Sciences at ScholarWorks @ UTRGV. It has been accepted for inclusion in School of Mathematical and Statistical Sciences Faculty Publications and Presentations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact justin.white@utrgv.edu, william.flores01@utrgv.edu.

Borsuk–Ulam type theorems for G -spaces with applications to Tucker type lemmas

Oleg R. Musin and Alexey Yu. Volovikov

Abstract

In this paper we consider several generalizations of the Borsuk–Ulam theorem for G -spaces and apply these results to Tucker type lemmas for G -simplicial complexes and PL -manifolds.

Keywords: Borsuk–Ulam theorem, Tucker’s lemma, G -space, G -index

1 Introduction

The classical Borsuk–Ulam theorem [4] states that *for any continuous mapping $f : \mathbb{S}^d \rightarrow \mathbb{R}^d$ there is a point $x \in \mathbb{S}^d$ such that $f(-x) = f(x)$* . In the same paper [4] Borsuk showed that this result is equivalent to the following statement:

Theorem A. *For any continuous mapping $f : \mathbb{B}^d \rightarrow \mathbb{R}^d$ such that f is odd on the boundary $\partial \mathbb{B}^d = \mathbb{S}^{d-1}$, there exists a point $x \in \mathbb{B}^d$ such that $f(x) = 0 \in \mathbb{R}^d$.*

In [19] it was shown that similar statement holds in a case when $\mathbb{S}^{d-1} = \partial M^d$ where M^d is a manifold. In [23] we extended this result for more general spaces. Namely we considered a space X with a closed subspace $Y \subset X$ which is a free \mathbb{Z}_2 -space (such a space is called bounded). In [23] we discussed conditions on X and Y under which for any map $f : X \rightarrow \mathbb{R}^d$ such that $f|_Y : Y \rightarrow \mathbb{R}^d$ is equivariant there exists a point $x \in X$ such that $f(x) = 0 \in \mathbb{R}^d$.

Note that Theorem A follows from **odd mapping theorem** that states “*Every continuous odd mapping $h : \mathbb{S}^d \rightarrow \mathbb{S}^d$ has odd degree*”, in particular, any odd mapping $h : \mathbb{S}^d \rightarrow \mathbb{S}^d$ can not be homotopic to zero (see [1], [13], [14], [22, Sect. 2]). This theorem follows also from the Conner–Floyd generalization of the Borsuk–Ulam theorem given in [8]. In [22] the odd mapping theorem has been extended for BUT-manifolds.

One more Conner–Floyd’s generalization of odd mapping theorem is in their earlier paper [7], where they proved that if there is an equivariant map $h : X \rightarrow Y$ of free \mathbb{Z}_2 -spaces X and Y with the same Yang’s cohomological index [35] which equals n , then $h^* : H^n(Y; \mathbb{Z}_2) \rightarrow H^n(X; \mathbb{Z}_2)$ is a nontrivial homomorphism (of nonzero groups). In [23] we used this fact as well as other properties of indexes of the free \mathbb{Z}_2 -spaces.

In this paper as a main tool in our considerations we use indexes of G -spaces where G is a finite group. For groups other than \mathbb{Z}_2 indexes were introduced by Yang, Wu, Schwarz, Conner–Floyd and others.

In this paper we prove G -analogs of the above results for maps of G -spaces where G is a finite group. We use G -analogs of the odd mapping theorem such as the Krasnosel’skii theorem on the degree of G -maps of a sphere and G -analogs of the Conner–Floyd result on cohomological indexes of free \mathbb{Z}_2 -spaces.

Let X be a free G -space. As a main tool in our consideration we use indexes of G -spaces, *topological index* $\text{t-ind}^G X$ and *cohomological index* $\text{ind}^G X$, whose definitions and properties are given in Sections 2 and 3 respectively.

In Section 4 we give first applications of cohomological index and present generalizations of Dold’s results from his highly cited paper [10]. In particular we discuss nonexistence of the equivariant map $X * G \rightarrow X$ for compact or finite-dimensional paracompact free G -space X proved in [32] with the help of the cohomological index (alternative proof is given in [25]). We show that this fact follows directly from one of Dold’s result proved in his paper [10]. Besides we give homological versions of Dold’s results proved in [10].

In Sections 5 and 6 we use these notions to obtain G -generalizations of Theorem A.

Assume that G can act freely on \mathbb{S}^{d-1} . Then there is an obvious semi-free G -action on \mathbb{R}^d with the unique fixed point $0 \in \mathbb{R}^d$ (and free on $\mathbb{R}^d \setminus 0$). Note that the degree of any equivariant map $\mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$ equals 1 modulo $|G|$ (see [14]). Actually, it implies that for any continuous map $f : \mathbb{B}^d \rightarrow \mathbb{R}^d$ which is equivariant on the boundary the zero set $f^{-1}(0)$ is not empty. Theorem 5.5 extends this fact for a case when \mathbb{S}^{d-1} is embedded into a space X .

In particular, Theorems 5.1–5.3 for G -spaces imply the following result for manifolds (or pseudomanifolds):

Theorem 5.4. *Let M^n be a compact connected orientable manifold (or a pseudomanifold) with the connected boundary ∂M , and assume that G can act freely on ∂M . Consider a continuous mapping $f : M \rightarrow \mathbb{R}^n$ such that $f|_{\partial M} : \partial M \rightarrow \mathbb{R}^n$ is an equivariant map, where \mathbb{R}^n is considered as a semifree G -space with the unique fixed point in the origin $0 \in \mathbb{R}^n$. If $\text{ind}^G \partial M = n - 1$ then the zero set $Z_f = f^{-1}(0)$ is not empty.*

In Section 6 we discuss an alternative approach for G -versions of the Borsuk–Ulam theorem and prove also Bourgin–Yang type theorems for G -spaces.

Last section is devoted to G -versions of the Tucker lemma [28], which is known to be a discrete analog of the Borsuk–Ulam theorem.

Let T be some triangulation of the d -dimensional ball \mathbb{B}^d . We call T *antipodally symmetric on the boundary* if the set of simplices of T contained in the boundary $\partial \mathbb{B}^d = \mathbb{S}^{d-1}$ of the ball \mathbb{B}^d is an antipodally symmetric triangulation of \mathbb{S}^{d-1} , that is if $s \subset \mathbb{S}^{d-1}$ is a simplex of T , then $-s$ is also a simplex of T .

Theorem B. (Tucker’s lemma) *Let T be a triangulation of \mathbb{B}^d that is antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be a labeling of the vertices of T that satisfies $L(-v) = -L(v)$ for every vertex v on the boundary. Then there exists an edge in T that is **complementary**, i.e. its two vertices are labeled by opposite numbers.*

Consider also the following version of Tucker’s lemma:

Theorem C. *Let T be a centrally symmetric triangulation of the sphere \mathbb{S}^d . Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

be an equivariant labeling, i.e. $L(-v) = -L(v)$. Then there exists a complementary edge.

It is well known, see [17], that these theorems are equivalent to the Borsuk–Ulam theorem.

Let X be a finite simplicial complex, $V(X)$ denote its vertex set and C be a finite set that we call the set of colors. Recall that a C -labeling (coloring) of $V(X)$ is a map $V(X) \rightarrow C$. In case X is a finite G -simplicial complex and C is a G -set we say that a C -labeling is *equivariant* if the map $V(X) \rightarrow C$ is equivariant. When $C = G \times \{1, \dots, n\}$ we call a C -labeling a (G, n) -labeling. Thus, a (G, n) -labeling prescribes to each vertex some pair (g, k) , $g \in G$, $k \in \{1, \dots, n\}$. In what follows we consider $G \times \{1, \dots, n\}$ with the following G -action: $h \cdot (g, k) = (hg, k)$.

An edge of X is called *complementary* if labels of its vertices are (g_1, k_1) and (g_2, k_2) with $g_1 \neq g_2$ and $k_1 = k_2$.

If $G = \mathbb{Z}_2 \cong C_2 = \{1, -1\}$ is the cyclic group of order 2 then a (G, n) -labeling is a Tucker labeling. Indeed, it follows from the obvious bijection $(\pm 1, k) \leftrightarrow \pm k$ between sets $\{1, -1\} \times \{1, \dots, n\}$ and $\{+1, -1, +2, -2, \dots, +n, -n\}$.

The main result of Section 7 is the following extension of Tucker’s lemma:

Theorem 7.1. *Let X be a simplicial complex with a free simplicial G -action. Then $\text{t-ind}^G X \geq n$ if and only if for any equivariant (G, n) -labeling of the vertex set of an arbitrary equivariant triangulation of X there exists a complementary edge.*

We consider also Tucker type lemmas for bounded spaces. In particular, Theorem 7.2 yields the following theorem for manifolds:

Theorem 7.3. *Let M^n be a connected compact orientable PL-manifold such that its boundary ∂M is homeomorphic to the sphere \mathbb{S}^{n-1} . Let T be a triangulation of M . Suppose that there exists a free simplicial action of a finite group G on ∂T . Then for any (G, n) -labeling of $V(T)$ that is an equivariant on ∂T there exists a complementary edge.*

In what follows we assume that all spaces in consideration are paracompact.

2 Topological index

Consider a group G as a discrete free G -space. Let $J^m(G) = G * \dots * G$ be the join of m -copies of G with the diagonal action of G .

Definition 2.1. *Let X be a free G -space. Topological index $\text{t-ind}^G X$ equals minimal n such that there exists an equivariant map $X \rightarrow J^{n+1}(G)$. If no such n exists, then $\text{t-ind}^G X = \infty$.*

Correctness of this definition follows from the fact (discussed below) that there exists no equivariant map $J^{n+1}(G) \rightarrow J^n(G)$.

Remark 2.1. 1) We can take $E_G = G * \dots * G * \dots = J^\infty(G)$ as a total space of the universal G -bundle $E_G \rightarrow B_G$.

2) If $G = \mathbb{Z}_2$ then $J^{m+1}(\mathbb{Z}_2)$ is equivariantly homeomorphic to \mathbb{S}^m , since $SY = Y * \mathbb{Z}_2$, where SY is the suspension, and

$$\mathbb{S}^m = S\mathbb{S}^{m-1} = \mathbb{S}^{m-1} * \mathbb{Z}_2 = \mathbb{S}^{m-2} * \mathbb{Z}_2 * \mathbb{Z}_2 = \dots = J^{m+1}(\mathbb{Z}_2).$$

3) For a cyclic group $G = \mathbb{Z}_q$, $q > 2$, we can take in the definition of index the following sequence of test spaces: $G, \mathbb{S}^1, \mathbb{S}^1 * G, \mathbb{S}^3, \mathbb{S}^3 * G, \mathbb{S}^5, \mathbb{S}^5 * G, \dots$, where each odd dimensional sphere is considered with some free action of $G = \mathbb{Z}_q$. For example we can use the following free action of \mathbb{Z}_q on spheres. Let \mathbb{S}^{2n-1} be considered as a standard unit sphere in \mathbb{C}^n . Then the generator of G acts as multiplication of coordinates of n -tuples by $e^{2\pi i/q}$.

The main property of the topological index:

$$\text{If } X \rightarrow Y \text{ is equivariant then } \text{t-ind}^G X \leq \text{t-ind}^G Y.$$

It is not hard to see that if X is either compact, or paracompact and finite-dimensional, then $\text{t-ind}^G X < \infty$, in the second case $\text{t-ind}^G X \leq \dim X$. For the proof one can use nerves of G -invariant coverings of a G -space and the fact that $J^{n+1}(G)$ is n -universal, i.e. any G -CW-complex of dimension not exceeding n can be mapped equivariantly to $J^{n+1}(G)$. If G can act freely on a sphere of some dimension N then this sphere is N -universal.

The equality $\text{t-ind}^G J^{n+1}(G) = n$ provides correctness of the definition of topological index. This fact can be proved using (co)homological index (see next section), which is a lower bound for topological index. Proofs that don't use cohomological indexes can be found in [17]. One more or less elementary proof is sketched below.

We need to show that there exists no equivariant map $J^{N+1}(G) \rightarrow J^N(G)$ for any N . Let us call continuous equivariant map of G -spaces also as a G -map. Obviously a G -map $J^2(G) = G * G \rightarrow J^1(G) = G$ cannot exist, since $G * G$ is connected while G is not. Thus we can assume that $N > 1$.

First let $G = \mathbb{Z}_p$ and suppose that a G -map $J^{N+1}(G) \rightarrow J^N(G)$ exists for some $N > 1$. If $p = 2$, then this map is just a map of spheres and the contradiction follows from the Borsuk–Ulam theorem. If p is odd then taking join with G we obtain also a G -map $J^{N+2}(G) \rightarrow$

$J^{N+1}(G)$. One of $N+1$ and $N+2$ is even, say equals $2m$, hence we have a G -map $J^{2m}(G) \rightarrow J^{2m-1}(G)$. Taking the composition of this map with the embedding $J^{2m-1}(G) \subset J^{2m}(G)$ we obtain an equivariant nullhomotopic map $J^{2m}(G) \rightarrow J^{2m}(G)$. Now there exist equivariant maps $\mathbb{S}^{2m-1} \rightarrow J^{2m}(G)$ and $J^{2m}(G) \rightarrow \mathbb{S}^{2m-1}$. Taking composition of these three maps we obtain an equivariant nullhomotopic map $\mathbb{S}^{2m-1} \rightarrow \mathbb{S}^{2m-1}$, a contradiction, since the degree of an equivariant map of spheres equals 1 modulo order of G by Krasnosel'skii theorem (see e.g. [13, 14]). Also at this point one can apply Dold's arguments [10] (based only on the notion of fixed point index) showing that an equivariant selfmap of a sphere cannot be nullhomotopic.

In general case let G_0 be any nontrivial subgroup of G . Consider $J^n(G)$ as a G_0 -space. Then it is easy to see that $J^n(G)$ like $J^n(G_0)$ is a $(n-1)$ -universal G_0 -space. Hence there exist G_0 -maps $J^n(G_0) \rightarrow J^n(G)$ and $J^n(G) \rightarrow J^n(G_0)$. Therefore the existence of a G -map $J^{N+1}(G) \rightarrow J^N(G)$ implies the existence of a G_0 -map $J^{N+1}(G_0) \rightarrow J^N(G_0)$. To finish the proof it remains to take $G_0 \cong \mathbb{Z}_p$ where p is any prime divisor of the order of G .

Remark 2.2. For $G = \mathbb{Z}_2$ this index was introduced by Yang [35] under the name B -index (Yang also introduced homological index which is discussed below). For finite groups topological index was introduced by M. Krasnosel'skii and in general case (for topological groups) by Albert Schwarz under the name genus (more precise genus is by 1 greater than topological index). In fact Schwarz [26] introduced and studied more general notion of genus of a fiber space which generalize the notions of the Lusternik–Shnirelman category and of Krasnosel'skii genus of a covering (it is valid for a continuous surjective map).

On his web page Schwarz [27] writes: "The same notion was rediscovered (under another name) 25 years later by S. Smale who used to estimate topological complexity of algorithms".

Nowadays this notion is usually called sectional category.

3 Cohomological index

Consider first the case of an action of the group \mathbb{Z}_p of prime order p (the case $p = 2$ was considered in [23]). Using Smith's sequences we can define for a free \mathbb{Z}_p -space its cohomological index $\text{ind}_p X \in \{0, 1, 2, \dots; \infty\}$ possessing the following properties (see [32] for details):

1. If there exists an equivariant map $X \rightarrow Y$ of free \mathbb{Z}_p -spaces then $\text{ind}_p(X) \leq \text{ind}_p(Y)$.
2. If $X = A \cup B$ are open invariant subspaces, then

$$\text{ind}_p(X) \leq \text{ind}_p(A) + \text{ind}_p(B) + 1.$$

3. Tautness: If Y is a closed invariant subspace of X , then there exists an open invariant neighborhood of Y such that $\text{ind}_p(Y) = \text{ind}_p(U)$.

4. $\text{ind}_p(X) > 0$ if X is connected.
5. Let X be either compact, or paracompact and finite dimensional. Then $\text{ind}_p(X) < \infty$.
6. Assume that X is connected and $H^i(X; \mathbb{Z}_p) = 0$ for $0 < i < N$. Then $\text{ind}_p(X) \geq N$.

7. Assume that X is finite dimensional and $H^i(X; \mathbb{Z}_p) = 0$ for $i > d$. Then $\text{ind}_p(X) \leq d$.

8. If there exists an equivariant map $f : X \rightarrow Y$ and $\text{ind}_p(X) = \text{ind}_p(Y) = k < \infty$ then $0 \neq f^* : H^k(Y; \mathbb{Z}_p) \rightarrow H^k(X; \mathbb{Z}_p)$.

Here Čech cohomology groups are used.

For $G = \mathbb{Z}_2$ this index was introduced by Yang [35]. In [36] Yang actually used this index for $G = \mathbb{Z}_3$ without naming it. Conner & Floyd in [7] introduced for any finite group G and a commutative ring with unit L a cohomological index for which they used notation $\text{co-ind}_L(\cdot)$. At the same time A.S. Schwarz [26] introduced homological genus. It can be shown that homological genus equals $\text{co-ind}_{\mathbb{Z}}(\cdot) + 1$, and $\text{co-ind}_{\mathbb{Z}_p}(\cdot)$ for $G = \mathbb{Z}_p$ coincides with $\text{ind}_p(\cdot)$.

In what follows the property 8 will serve as our main tool.

For example, from properties 1 and 8 we immediately obtain that $\text{ind}_p(\cdot)$ is stable, i.e. $\text{ind}_p X * \mathbb{Z}_p = \text{ind}_p X + 1$ (see [32, Corollary 3.1]), so if $\text{ind}_p X$ is finite then there exists no equivariant map $X * \mathbb{Z}_p \rightarrow X$.

Now we recall the definition of $\text{ind}_p(\cdot)$. Denote by $\pi : X \rightarrow X/\mathbb{Z}_p$ the projection. Then there are two Smith sequences (see e.g. [5]):

$$\dots \rightarrow H_\rho^k(X) \longrightarrow H^k(X) \xrightarrow{\pi^!} H^k(X/\mathbb{Z}_p) \xrightarrow{\delta_1} H_\rho^{k+1}(X) \rightarrow \dots$$

and

$$\dots \rightarrow H^k(X/\mathbb{Z}_p) \xrightarrow{\pi^*} H^k(X) \longrightarrow H_\rho^k(X) \xrightarrow{\delta_2} H^{k+1}(X/\mathbb{Z}_p) \rightarrow \dots$$

Here coefficients \mathbb{Z}_p are omitted, groups $H_\rho^*(X)$ are called special Smith cohomology groups, and $\pi^!$ is called the transfer (see [5]).

Let us define $s_{2d} : H^0(X/\mathbb{Z}_p) \rightarrow H^{2d}(X/\mathbb{Z}_p)$ and $s_{2d+1} : H^0(X/\mathbb{Z}_p) \rightarrow H_\rho^{2d+1}(X)$ as $s_{2d+1} = \delta_1 s_{2d}$ and $s_{2d+2} = \delta_2 s_{2d+1}$ where $s_0 = \text{id}$, and put $u_n(X) = s_n(1)$, $1 \in H^0(X/\mathbb{Z}_p)$. Then $\text{ind}_p X$ equals maximal n such that $u_n(X) \neq 0$.

The following proposition is a partial converse to Property 8 (see also [31, Proposition 3.3]).

Proposition 3.1. *Let X and Y be free \mathbb{Z}_p -spaces and $f : X \rightarrow Y$ an equivariant map. Assume that k is odd and*

- a) $\text{ind}_p(Y) = k$,
- b) $\dim X = k$,
- c) $H^k(X; \mathbb{Z}_p) = H^k(Y; \mathbb{Z}_p) = \mathbb{Z}_p$,
- d) $f^* : H^k(Y; \mathbb{Z}_p) \rightarrow H^k(X; \mathbb{Z}_p)$ is an isomorphism.

Then $\text{ind}_p(X) = k$.

Proof. Put $k = 2n + 1$.

An equivariant map $f : X \rightarrow Y$ between free \mathbb{Z}_p -spaces induces a map of factor spaces $X/\mathbb{Z}_p \rightarrow Y/\mathbb{Z}_p$ and we have two commutative diagrams (for p odd) since Smith's sequences are functorial. Consider one of these diagrams:

$$\begin{array}{ccccccc}
H^k(X/\mathbb{Z}_p) & \xrightarrow{\pi^*} & H^k(X) & \longrightarrow & H_\rho^k(X) & \xrightarrow{\delta_2} & H^{k+1}(X/\mathbb{Z}_p) \\
\uparrow & & \uparrow f^* & & \uparrow & & \uparrow \\
H^k(Y/\mathbb{Z}_p) & \longrightarrow & H^k(Y) & \longrightarrow & H_\rho^k(Y) & \xrightarrow{\delta_2} & H^{k+1}(Y/\mathbb{Z}_p)
\end{array}$$

Since $u_k(Y) \neq 0$ and $\delta_2 u_k(Y) = 0$, there is a nontrivial element $\alpha \in H^k(Y) = \mathbb{Z}_p$ which is mapped onto $u_k(Y)$. Now $u_k(Y)$ is mapped to $u_k(X)$ and from assumption d) it follows that $0 \neq f^* \alpha \in H^k(X) = \mathbb{Z}_p$ is mapped onto $u_k(X)$. Now we argue by contradiction. If $u_k(X) = 0$ then $H^k(X) \rightarrow H_\rho^k(X)$ is trivial. Since $H^{k+1}(X/\mathbb{Z}_p) = 0$, we obtain $H_\rho^k(X) = 0$. We have also $H_\rho^{k+1}(X) = 0$, since $\dim X = k$. From Smith's sequence

$$H_\rho^k(X) \longrightarrow H^k(X) \xrightarrow{\pi^!} H^k(X/\mathbb{Z}_p) \xrightarrow{\delta_1} H_\rho^{k+1}(X)$$

we see that $\pi^!$ is an isomorphism and $H^k(X/\mathbb{Z}_p) = \mathbb{Z}_p$. From the first row of the above diagram it follows that $\pi^* : H^k(X/\mathbb{Z}_p) \rightarrow H^k(X)$ is also an isomorphism, so $\pi^! \circ \pi^*$ is an isomorphism, but this contradicts with the fact that $\pi^! \circ \pi^*$ is the multiplication by p , i.e. zero homomorphism. \square

Note that if X is a free \mathbb{Z}_p -space where p is an odd prime and $\dim X = 2n + 1$, then there exists an equivariant map $f : X \rightarrow \mathbb{S}^{2n+1}$.

Corollary 3.1. *Let X be a free \mathbb{Z}_p -space where p is an odd prime. Assume that $\dim X = 2n + 1$ and $H^{2n+1}(X; \mathbb{Z}_p) = \mathbb{Z}_p$, and denote by $f : X \rightarrow \mathbb{S}^{2n+1}$ an equivariant map. Then $\text{ind}_p(X) = 2n + 1$ if and only if $f^* : H^{2n+1}(\mathbb{S}^{2n+1}; \mathbb{Z}_p) \rightarrow H^{2n+1}(X; \mathbb{Z}_p)$ is an isomorphism.*

In what follows we will use cohomological index with integer coefficients. This index is defined via homological genus introduced by Albert Schwarz in [26].

Definition 3.1. *Let X be a free G -space. We define $\text{ind}^G X$, the integer cohomological index of X , as its Schwarz's homological genus minus 1.*

Remark 3.1. 1) Using notation of Conner and Floyd [7] we have $\text{ind}^G(\cdot) = \text{co-ind}_{\mathbb{Z}}(\cdot)$.

2) $\text{ind}^G(\cdot)$ is the largest cohomological index. In particular for $G = \mathbb{Z}_p$ we have $\text{ind}_p(\cdot) \leq \text{ind}^G(\cdot)$. Also for any G we have $\text{ind}^G(\cdot) \leq \text{t-ind}^G(\cdot)$.

This cohomological index possesses similar properties:

1. If there exists an equivariant map $X \rightarrow Y$ then $\text{ind}^G(X) \leq \text{ind}^G(Y)$.
2. If $X = A \cup B$ are open invariant subspaces, then

$$\text{ind}^G(X) \leq \text{ind}^G(A) + \text{ind}^G(B) + 1.$$

3. Tautness: If Y is a closed invariant subspace of X , then there exists an open invariant neighborhood of Y such that $\text{ind}^G(Y) = \text{ind}^G(U)$.
4. $\text{ind}^G(X) > 0$ if X is connected.
5. If X is either compact, or paracompact and finite dimensional then $\text{ind}^G(X) < \infty$.
6. Assume that X is connected and $H^i(X; \mathbb{Z}) = 0$ for $0 < i < N$. Then $\text{ind}^G(X) \geq N$.
7. Assume that X is finite dimensional and $H^i(X; \mathbb{Z}) = 0$ for $i > d$. Then $\text{ind}^G(X) \leq d$.
8. If there exists an equivariant map $f : X \rightarrow Y$ and $\text{ind}^G(X) = \text{ind}^G(Y) = k < \infty$ then $0 \neq f^* : H^k(Y; \mathbb{Z}) \rightarrow H^k(X; \mathbb{Z})$.

4 Dold theorems and generalizations

Note that from properties 1 and 8 we immediately obtain that $\text{ind}_p(\cdot)$ is stable, i.e. $\text{ind}_p X * \mathbb{Z}_p = \text{ind}_p X + 1$ (see [32, Corollary 3.1]), so if $\text{ind}_p X$ is finite then there exists no equivariant map $X * \mathbb{Z}_p \rightarrow X$. As a direct consequence we have the following assertion:

Proposition 4.1. *Let H be any topological group which has a nontrivial finite subgroup and X be either compact or paracompact and finite dimensional space with a free action of H . Then there exists no equivariant map $X * H \rightarrow X$.*

An independent, alternative proof of this result is given by Passer in [25]. One of his arguments is used below in a more simple proof of this proposition. Also we show below that proposition 4.1 follows directly from the paper of Dold [10].

Proposition 4.1 gives the partial solution to the following conjecture of Baum, Dąbrowski and Hajac:

Conjecture 4.1 ([3], Conjecture 2.2). *Let X be a compact Hausdorff space with a continuous free action of a nontrivial compact Hausdorff group G . Then, for the diagonal action of G on the join $X * G$, there does not exist an equivariant continuous map $f : X * G \rightarrow X$.*

In [6], Chirvasitu and Passer proposed a possible approach to the open part of Conjecture 4.1 (and its analogue for compact group actions on C^* -algebras) using the ideas of [25] and [10]. The case of certain compact *quantum* group actions on C^* -algebras was considered by Dąbrowski, Hajac, and Neshveyev in [9].

Let us deduce proposition 4.1 directly from Dold's [10] result and give one more simple proof.

Dold in the proof of his last theorem in paper [10] showed for finite group $G \neq \{1\}$ that

If there exists an equivariant nullhomotopic map $X \rightarrow X$ of a free G -space X to itself then for every free G -space Y such that Y/G is a finite cell complex there exists an equivariant map $Y \rightarrow X$.

Proposition 4.1 is a consequence of this Dold's result. In particular we can take $Y = J^N(G)$ with any N and obtain an equivariant map $J^N(G) \rightarrow X$. Thus if there exists an equivariant nullhomotopic selfmap $X \rightarrow X$ then $\text{t-ind}^G X = \infty$. Also it follows that an equivariant map $Y \rightarrow X$ exists for any free G -space Y such that $\text{t-ind}^G Y < \infty$ because we can take the composition of maps $Y \rightarrow J^n(G)$ and $J^n(G) \rightarrow X$ where $\text{t-ind}^G Y = n - 1$.

Now if we assume equivariant map $F : X * G \rightarrow X$ then its composition with the natural embedding $X \subset X * G$ gives us the equivariant nullhomotopic map $X \rightarrow X$, so $\text{t-ind}^G X = \infty$. Thus in case $\text{t-ind}^G X < \infty$ a G -map $X * G \rightarrow X$ does not exist. In particular no such a G -map exists for compact X (with no restriction on dimension) and for finite dimensional paracompact X since in these cases $\text{t-ind}^G X < \infty$. \square

Note that for existence of a G -map $X * G \rightarrow X$ some restrictions on X are needed since there is an obvious equivariant homeomorphism $X * G \approx X$ for $X = J^\infty(G)$. Note also that if a G -map $X * G \rightarrow X$ exists then we can prove the equality $\text{t-ind}^G X = \infty$ by the following simple argument used in [25]¹. Taking the join with G we obtain an equivariant map $X * G * G \rightarrow X * G$, and hence a map $X * G * G \rightarrow X$. Iterating this procedure we obtain for any n an equivariant map $X * J^n(G) \rightarrow X$ (this argument was used in [25]). Since $J^n(G)$ is a G -subspace of $X * J^n(G)$ we obtain an equivariant map $J^n(G) \rightarrow X$ for any n , and therefore $\text{t-ind}^G X = \infty$.

The following assertion clarifies the situation.

Lemma 4.1. *Let X and Y be free G -spaces. There exists an equivariant map $F : X * G \rightarrow Y$ if and only if there exists a nullhomotopic equivariant map $f : X \rightarrow Y$.*

Proof. Given $F : X * G \rightarrow Y$ we can define the equivariant nullhomotopic map $f : X \rightarrow Y$ as a composition of the natural embedding $X \subset X * G$ with F , i.e. $f = F|_X$.

Now let $f : X \rightarrow Y$ be an equivariant nullhomotopic map. Elements of $X * G$ are written as $[x, t, h]$, where $x \in X$, $t \in [0, 1]$, $h \in G$, and $[x, 0, h] = [x, 0, e]$ and $[x, 1, h] = [x', 1, h]$ for any $x, x' \in X$ and $h \in G$. Then G acts on $X * G$ as $g[x, t, h] = [gx, t, gh]$, $g \in G$, and there is an equivariant inclusion of X into $X * G$ given as $x \mapsto [x, 0, e]$. Denote by f_t a homotopy between the equivariant map $f = f_0$ and a constant map f_1 such that $f_1(X) = \{y\}$, where $y \in X$ is some point. Define $F : X * G \rightarrow Y$ by the formula $F([x, t, h]) := hf_t(h^{-1}x)$.

We have $F([x, 0, h]) = hf_0(h^{-1}x) = hf(h^{-1}x) = f(x)$ and $F([x, 1, h]) = hf_1(h^{-1}x) = hy$, so F is correctly defined. The following calculation

$$F(g[x, t, h]) = F([gx, t, gh]) = ghf_t((gh)^{-1}gx) = ghf_t(h^{-1}x) = gF([x, t, h])$$

shows that F is equivariant. \square

Remark 4.1. Gottlieb [12] proved that the order of G divides the Lefschetz number of an equivariant selfmap of a finitely dominated manifold with a free G -action. As a corollary [12,

¹The authors thanks Benjamin Passer for his useful comments on the first version of this paper. The discussion with him led to our better understanding of the problem.

Corollary 4] he obtained that no equivariant nullhomotopic selfmap of a finitely dominated manifold with a free G -action exists when $G \neq \{1\}$. Dold [10] deduced his more general result from the partial case that there exists no equivariant nullhomotopic selfmap of a sphere with a free G -action, and therefore if there exists an equivariant map of spheres $\mathbb{S}^n \rightarrow \mathbb{S}^N$ with free G -actions then $n \leq N$. Dold's argument (calculation of fixed point indices of a map of factor spaces) for a selfmap of a sphere is just the same as Gottlieb's for a selfmap of a compact manifold. As was mentioned above this result for spheres follows also from earlier stronger theorem of Krasnosel'skii [13, 14] who proved that the degree of an equivariant map of a sphere to itself is 1 modulo the order of G .

Dold proved (see Remark and Theorem on page 68 in [10]) the following:

If a map $f : X \rightarrow Y$ commutes with some free actions of a nontrivial finite group G on X and Y then

$$\dim Y \geq 1 + \text{Connectivity}(X).$$

If $\dim Y = 1 + \text{Connectivity}(X) < \infty$ then f is not nullhomotopic (assuming Y paracompact).

If X is a finite-dimensional paracompact space and $f : X \rightarrow X$ is a nullhomotopic map then f does not commute with any free G -action on X for any finite group $G \neq \{1\}$.

Here are homological versions of these results.

Proposition 4.2. *Let H be a subgroup of G of prime order p and denote by $n = \text{ind}_p X$ the cohomological index of a space X with respect to H . If $0 < n < \infty$, then $H^n(X; \mathbb{Z}_p) \neq 0$ and the induced endomorphism $f^* : H^n(X; \mathbb{Z}_p) \rightarrow H^n(X; \mathbb{Z}_p)$ is nontrivial.*

Proposition 4.3. *Let X and Y be free G -spaces and p is a prime divisor of the order of G . Assume that $\tilde{H}^i(X; \mathbb{Z}_p) = 0$ for $i \leq n$ and that $f : X \rightarrow Y$ is an equivariant map. Then $\text{t-ind}^G Y \geq n + 1$, in particular $\dim Y \geq n + 1$. If $\text{t-ind}^G Y = n + 1$, then $f^* : H^{n+1}(Y; \mathbb{Z}_p) \rightarrow H^{n+1}(X; \mathbb{Z}_p)$ is a nontrivial homomorphism.*

Proof. The problem reduces to the case $G = \mathbb{Z}_p$. Then the first assertion follows from properties 1 and 6 of the index $\text{ind}_p(\cdot)$ and the fact that $\text{ind}_p Y \leq \text{t-ind}^G Y \leq \dim Y$, where Y is a free finite-dimensional \mathbb{Z}_p -space. In particular it follows that $\text{ind}_p J^{n+1}(G) = \text{t-ind}^G J^{n+1}(G) = n$.

For the proof of the second assertion note that there exists an equivariant map $h : Y \rightarrow J^{n+2}(G)$, and from property 8 it follows that the composition $h \circ f$ induces a nontrivial homomorphism of $(n + 1)$ -dimensional cohomology groups (with \mathbb{Z}_p -coefficients). Therefore $f^* \neq 0$ in dimension $n + 1$.

Actually for the proof the first assertion it is easier to use more simple index $\text{in}_p(\cdot)$ which equals weak homological genus (introduced in [26]) minus 1.

To define $\text{in}_p(X)$ for a paracompact free \mathbb{Z}_p -space X consider an equivariant map $X \rightarrow J^\infty(\mathbb{Z}_p) = E_{\mathbb{Z}_p}$ and the map of factor-spaces $\mu : X/\mathbb{Z}_p \rightarrow B_{\mathbb{Z}_p}$. Recall that $H^i(B_{\mathbb{Z}_p}; \mathbb{Z}_p) = \mathbb{Z}_p$. Say that $\text{in}_p(X) \geq n$ if $\mu^* \neq 0$ in dimension n . It is easy to see that this assumption is

equivalent to the assumption that $\mu^* : H^i(B_{\mathbb{Z}_p}; \mathbb{Z}_p) \rightarrow H^i(X/\mathbb{Z}_p; \mathbb{Z}_p)$ is a monomorphism for $i \leq n$. To prove proposition 4.3 we need only to show that $\text{index in}_p(\cdot)$ satisfies properties 1, 5 and 6. The most complicated property 6 follows easily from the consideration of the spectral sequence of a covering $X \rightarrow X/\mathbb{Z}_p$ (from the spectral sequence of a bundle $X \times_{\mathbb{Z}_p} E_{\mathbb{Z}_p} \rightarrow X/\mathbb{Z}_p$ with fiber X). \square

Remark 4.2. 1) Results like the first statement of proposition 4.3 (generalizations can be obtained using [26, Theorem 17]) belong to A.S. Schwarz, since they follow trivially from [26, Theorem 17] and its corollaries 1 and 2 and properties of homological and weak homological genus introduced in [26].

2) $\text{in}_p(\cdot)$ possesses all other properties except 2 and 8, and it can be shown that $\text{in}_p(\cdot) \leq \text{ind}_p(\cdot)$ with the equality for $p = 2$, see [32].

3) The definition of $\text{in}_p(\cdot)$ and its property 6 was rediscovered many times, see e.g. [16], [29, 30]. In [15] Liulevicius actually used this index without naming it.

4) It is easy to deduce from proposition 4.3 that $\text{in}_p \mathbb{S}^n = \text{ind}_p \mathbb{S}^n = \text{t-ind}^G \mathbb{S}^n = n$ and $\text{in}_p J^n(G) = \text{ind}_p J^n(G) = \text{t-ind}^G J^n(G) = n - 1$ where p is a prime divisor of the order of G and cohomological indices are taken in respect with any subgroup of G of prime order p .

5 Borsuk–Ulam type theorems for bounded spaces

Definition 5.1. We say that $h : X_0 \rightarrow X$ is n -cohomological trivial (n -c.t. map) over R if $h^* : H^n(X; R) \rightarrow H^n(X_0; R)$ is the trivial homomorphism of cohomology groups with coefficients in R in dimension n . In the case when h is an embedding we call X_0 an n -c.t.-subspace of X over R .

Example 5.1. Let X be a compact connected $(n+1)$ -dimensional manifold with the connected boundary $\partial X = X_0$. Then X_0 is an n -c.t.-subspace of X over \mathbb{Z}_2 , and if moreover X is orientable then X_0 is an n -c.t.-subspace of X over R for any R .

Let a space X_0 be a subspace of X . Denote by $i : X_0 \rightarrow X$ the inclusion. Suppose X_0 admits a free action of a finite group G . (Actually, we do not assume that X is a G -space.)

These assumptions on X and X_0 will be used in what follows.

Theorem 5.1. Let Y be a G -space, Y_0 its invariant closed subspace such that the action on $Y \setminus Y_0$ is free, and $f : X \rightarrow Y$ a continuous map. Assume that

- 1) $n = \text{ind}^G X_0 = \text{ind}^G (Y \setminus Y_0)$,
 - 2) X_0 is an n -c.t.-subspace of X over \mathbb{Z} ,
 - 3) $f|_{X_0} : X_0 \rightarrow Y$ is equivariant,
- then $f^{-1}(Y_0) \neq \emptyset$.

Proof. We argue by contradiction. If $f^{-1}(Y_0) = \emptyset$ then f maps X into $Y \setminus Y_0$ and $f|_{X_0} : X_0 \rightarrow Y \setminus Y_0$ is equivariant. Since $f|_{X_0} = f \circ i$ and i^* is trivial in dimension n , we obtain that $(f|_{X_0})^* : H^n(Y \setminus Y_0; \mathbb{Z}) \rightarrow H^n(X_0; \mathbb{Z})$ is trivial, a contradiction with property 8 of index. \square

Note that if $\text{ind}^G(Y \setminus Y_0) < n$ then by property 1 of index there exists no equivariant map from X to $Y \setminus Y_0$, hence $(f|_X)^{-1}(Y_0) \neq \emptyset$ (in this case we don't need the assumption 2).

The theorem follows also from the following result.

Theorem 5.2. *Let X_0 be a free G -space, $i : X_0 \subset X$. Let K be a free G -space and $f : X \rightarrow K$ is a map equivariant on X_0 . Assume that $\text{ind}^G X_0 = d$ and that X_0 is d - \mathbb{Z} -c.t.-subspace of X . Then $\text{ind}^G K \geq d + 1$.*

If in addition K is a connected closed orientable topological $(d + 1)$ -dimensional manifold or a pseudomanifold then for any $y \in K$ at least one of the sets $f^{-1}(gy)$ for some $g \in G$ depending on y is nonempty.

Proof. The map $f \circ i : X \rightarrow K$ is equivariant, so $\text{ind}^G K \geq \text{ind}^G X = d$. Since $(f \circ i)^* = i^* \circ f^* = 0$ in dimension d , it follows from property 8 of index that $\text{ind}^G K \neq d$. Therefore $\text{ind}^G K \geq d + 1$.

When K is a manifold we argue by contradiction. Let $y \in K$ be a point such that $f^{-1}(Gy) = \emptyset$ where Gy is the orbit of the point y . Then f maps X to $K \setminus Gy$ and $f \circ i : X_0 \rightarrow K \setminus Gy$ is equivariant. Applying the first statement we obtain that $\text{ind}^G(K \setminus Gy) \geq d + 1$. On the other hand $K \setminus Gy$ is an open manifold, hence $H^j(K \setminus Gy; \mathbb{Z}) = 0$ for $j \geq d + 1$, and from property 7 of index we obtain $\text{ind}^G(K \setminus Gy) < d + 1$. (Also $H^{d+1}(K \setminus Gy; \mathbb{Z}) = 0$ contradicts with $\text{ind}^G(K \setminus Gy) = d + 1$ by property 8.) \square

Definition 5.2. *Let Y be a G -space. A point $y \in Y$ is a fixed point of the action if $gy = y \ \forall g \in G$. Denote the set of fixed points by Y^G . We say that the action of G on Y is semifree if $Y \setminus Y^G \neq \emptyset$ and $Y^G \neq \emptyset$ and G acts freely on $Y \setminus Y^G$.*

Assume that Y is a semifree G -space and $f : X \rightarrow Y$ a continuous map. In this case directly from theorem 5.1 we obtain:

Theorem 5.3. *Let Y is a semifree G -space, $f : X \rightarrow Y$ a continuous map. Assume that*

- 1) $n = \text{ind}^G X_0 = \text{ind}^G(Y \setminus Y^G)$,
- 2) X_0 is n -c.t.-subspace of X over \mathbb{Z} .
- 3) $f|_{X_0} : X \rightarrow Y$ is equivariant.

Then $f^{-1}(Y^G) \neq \emptyset$.

We can apply this result in the case when $X = M$ is a manifold and $X_0 = \partial M$ is its boundary.

Theorem 5.4. *Let M^n be a compact connected orientable manifold (or a pseudomanifold) with the connected boundary ∂M , and assume that G can act freely on ∂M . Consider a continuous mapping $f : M \rightarrow \mathbb{R}^n$ such that $f|_{\partial M} : \partial M \rightarrow \mathbb{R}^n$ is an equivariant map, where \mathbb{R}^n is considered as a semifree G -space with the unique fixed point $0 \in \mathbb{R}^n$, the origin. If $\text{ind}^G \partial M = n - 1$ then the zero set $Z_f = f^{-1}(0)$ is not empty.*

Here we consider any semifree action of G on \mathbb{R}^n with the unique fixed point $0 \in \mathbb{R}^n$, the origin. Such an action exists since we assume that G can act freely on \mathbb{S}^{n-1} . For example we can take the action which is obtained by linearity from the G -action on \mathbb{S}^{n-1} .

As a partial case of the previous assertion we obtain:

Corollary 5.1. *Let M^n be a compact connected orientable manifold (or a pseudomanifold) with the boundary ∂M which is homeomorphic to the sphere \mathbb{S}^{n-1} , and assume that G can act freely on $\partial M \approx \mathbb{S}^{n-1}$. Consider a continuous mapping $f : M \rightarrow \mathbb{R}^n$ such that $f|_{\partial M} : \partial M \rightarrow \mathbb{R}^n$ is an equivariant map, where \mathbb{R}^n is considered as a semifree G -space with the unique fixed point $0 \in \mathbb{R}^n$, the origin. Then the zero set $Z_f = f^{-1}(0)$ is not empty.*

This follows also from

Proposition 5.1. *If there is an embedding $i : \mathbb{S}^{d-1} \rightarrow X$ such that*

$$\text{Im } i^* \cap \{k \in \mathbb{Z} \mid k \equiv 1 \pmod{|G|}\} = \emptyset,$$

where $i^* : H^{d-1}(X; \mathbb{Z}) \rightarrow H^{d-1}(\mathbb{S}^{d-1}; \mathbb{Z})$, and $f : X \rightarrow \mathbb{R}^d$ a continuous map such that $f|_{\mathbb{S}^{d-1}} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ is equivariant, then $0 \in f(X)$.

Actually a more general assertion holds

Theorem 5.5. *Assume that there is a map $j : \mathbb{S}^{d-1} \rightarrow X$ such that*

$$\text{Im } j^* \cap \{k \in \mathbb{Z} \mid k \equiv 1 \pmod{|G|}\} = \emptyset,$$

where $j^* : H^{d-1}(X; \mathbb{Z}) \rightarrow H^{d-1}(\mathbb{S}^{d-1}; \mathbb{Z})$ is induced by j , and let $f : X \rightarrow \mathbb{R}^d$ be a continuous map such that $f \circ j : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ is equivariant. Then $0 \in f(X)$.

Proof. We argue by a contradiction. If $0 \notin f(X)$, then $f \circ j : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d \setminus 0$ is an equivariant map, hence its degree equals 1 modulo $|G|$ (see e.g. [14]), but this contradicts with the assumption $\text{Im } j^* \cap \{k \in \mathbb{Z} \mid k \equiv 1 \pmod{|G|}\} = \emptyset$. \square

6 Bourgin–Yang type theorems

Definition 6.1. *Let X be a space and X_0 its subspace which is a G -space. A camomile C is a G -space for which there is an embedding $X \subset C$ such that $C = GX$, induced embedding $X_0 \subset C$ is equivariant, the action of G on $C \setminus X_0$ is free, and $C \setminus X_0 = \bigcup_{g \in G} g(X \setminus X_0)$.*

Example 6.1. If X is a cone over X_0 , i.e. $X = X_0 * \text{pt}$, then $C = X_0 * G$.

Let Y be a G -space and Y_0 its invariant subspace such that the G -action on $Y \setminus Y_0$ is free. From the definition of camomile we easily obtain the following assertion.

Theorem 6.1. *There exists $f : X \rightarrow Y$ equivariant on X_0 and such that $f^{-1}(Y_0) = \emptyset$ if and only if there exists an equivariant map $C \rightarrow Y \setminus Y_0$ where C is the camomile associated with the embedding $X_0 \subset X$ of the G -space X_0 into X .*

Theorem 6.2. *Let X_0 be an n -c.t.-subspace of X over \mathbb{Z} such that $\text{ind}^G X_0 = n$. Then $\text{ind}^G C = n + 1$.*

Proof. Since the inclusion $X_0 \subset C$ is equivariant, we have from property 1 that $\text{ind}^G C \geq n$ and from property 8 obtain that $\text{ind}^G C \geq n + 1$. By property 3 there exists an invariant neighborhood of X_0 in C of index n . A complement of this neighborhood is a G -space that can be mapped equivariantly to G , so its index equals zero. Hence from property 2 we obtain that $\text{ind}^G C \leq n + 1$. \square

Now we show how to construct a camomile in the case when X is a finite-dimensional compact space and X_0 its closed subspace (so X_0 is a compactum also).

By Mostow theorem [18] we can equivariantly embed X_0 into finite-dimensional Euclidean G -space V . By Tietze lemma we can extend this embedding to the map $\varphi : X \rightarrow V$. If $\dim X = k$ then using Nöbeling–Pontryagin theorem (see e.g. [2]) we can embed X into the unit sphere $S^{2k+1} \subset \mathbb{R}^{2k+2}$. Denote this embedding by $\psi : X \rightarrow \mathbb{R}^{2k+2}$. Define a real-valued function $h : X \rightarrow \mathbb{R}$ as $h(x) = \rho(x, X_0)$, the distance between a point x and X_0 . This function takes zero values on X_0 and is positive on $X \setminus X_0$. Define $\eta : X \rightarrow \mathbb{R}^{2k+2}$ as $\eta(x) = h(x)\psi(x)$. Then $\zeta : X \rightarrow V \oplus \mathbb{R}^{2k+2}$, $\zeta(x) = (\varphi(x), \eta(x))$, is an embedding. We will consider $V \oplus \mathbb{R}^{2k+2}$ as a Euclidean G -space (G acts trivially on \mathbb{R}^{2k+2}). Then ζ is the embedding which is equivariant on X_0 . Finally put $W = V \oplus \mathbb{R}^{2k+2} \oplus \mathbb{R}[G]$ where $\mathbb{R}[G]$ is the group ring considered as Euclidean space of dimension $|G|$, the order of G . The group acts on $\mathbb{R}[G]$ by left multiplication and it is convenient to denote basis vectors as elements of the group G , so $\mathbb{R}[G] = \bigoplus_{g \in G} \mathbb{R} \cdot g$. Now we define an embedding $\mu : X \rightarrow V \oplus \mathbb{R}^{2k+2} \oplus \mathbb{R} \cdot e \subset W$ where $e \in G$ is the unit of G by the formula $(\varphi(x), h(x)\psi(x), h(x) \cdot e)$. Then $\mu : X \rightarrow W$ is equivariant on X_0 and $C = G\mu(X)$ is a camomile.

In fact the same construction of the camomile is valid for finite-dimensional separable metric space X and closed subspace X_0 (with G -action).

Camomile is convenient for proving results of Bourgin–Yang type.

Theorem 6.3. *Assume that Y is a G -space, Y_0 its invariant closed subspace such that the action on $Y \setminus Y_0$ is free, and $f : X \rightarrow Y$ a continuous map. If*

- 1) $n = \text{ind}^G X_0 > \text{ind}^G (Y \setminus Y_0)$,
- 2) X_0 is an n -c.t.-subspace of X over \mathbb{Z} ,
- 3) $f|_{X_0} : X_0 \rightarrow Y$ is equivariant,

then $\dim f^{-1}(Y_0) \geq n - \text{ind}^G (Y \setminus Y_0)$.

Proof. We have $\text{ind}^G C = n + 1$, where C is the camomile. Denote by $h : C \rightarrow Y$ the equivariant extension of f . Then $\text{ind}^G h^{-1}Y_0 \geq \text{ind}^G C - \text{ind}^G (Y \setminus P) - 1 = n - \text{ind}^G (Y \setminus Y_0)$, hence $\dim h^{-1}Y_0 \geq n - \text{ind}^G (Y \setminus Y_0)$. Since $h^{-1}Y_0 = \bigcup_{g \in G} g \cdot f^{-1}(Y_0)$ and $\dim g \cdot f^{-1}(Y_0) = \dim f^{-1}(Y_0)$ for any $g \in G$, we are done. \square

Since a free G -space is a free space with respect to any subgroup we have analogs of the above results in which $\text{ind}^G(\cdot)$ is replaced by $\text{ind}_p(\cdot)$ where $p = |H|$ is a prime and H is some subgroup of G . For example we have the following result:

Theorem 6.4. *Let Y be a G -space, Y_0 its invariant closed subspace such that the action on $Y \setminus Y_0$ is free, and $f : X \rightarrow Y$ a continuous map. Let $H = \mathbb{Z}_p$, p is a prime, be a subgroup of G . Assume that*

- 1) $n = \text{ind}_p X_0 \geq \text{ind}_p(Y \setminus Y_0)$,
- 2) X_0 is an n -c.t.-subspace of X over \mathbb{Z}_p ,
- 3) $f|_{X_0} : X \rightarrow Y$ is equivariant.

Then $f^{-1}(Y_0) \neq \emptyset$.

If $n = \text{ind}_p X > \text{ind}_p(Y \setminus Y_0)$ then $\dim f^{-1}(Y_0) \geq n - \text{ind}_p(Y \setminus Y_0)$.

7 Tucker type lemmas

7.1 Tucker type lemmas for G -spaces

Let X be a simplicial complex and C be a finite set. Recall that a C -labeling (coloring) of X is a map $V(X) \rightarrow C$ of the vertex set $V(X)$ to C . For $C = G \times \{1, \dots, n\}$ we say that we have a (G, n) -labeling. Thus a (G, n) -labeling prescribes to each vertex some pair (g, k) where $g \in G$ and $k \in \{1, \dots, n\}$.

Now we define equivariant labelings.

Definition 7.1. *Let X be a simplicial complex with a simplicial G -action, where G is a finite group, and C is a finite G -set. An equivariant C -labeling (coloring) of X is an equivariant map $V(X) \rightarrow C$ of the vertex set $V(X)$ to C . For $C = G \times \{1, \dots, n\}$, where G acts on the first factor by left multiplication and on the second factor the action is trivial, we call C -labeling as equivariant (G, n) -labeling.*

Definition 7.2. *An edge in X is called complementary if labels of its vertices belong to the same orbit in C . For (G, n) -labeling it means that vertices of a complementary edge have the form (g_1, k) and (g_2, k) , $g_1 \neq g_2$, for some $k \in \{1, \dots, n\}$.*

If $G = \mathbb{Z}_2 \cong C_2 = \{1, -1\}$ is the cyclic group of order 2 then a (G, n) -labeling is just a Tucker labeling since there is an obvious bijection $(\pm 1, k) \leftrightarrow \pm k$ between sets $\{1, -1\} \times \{1, \dots, n\}$ and $\{+1, -1, +2, -2, \dots, +n, -n\}$. Under this identification an equivariant (\mathbb{Z}_2, n) -labeling becomes an equivariant $\{\pm 1, \dots, \pm n\}$ -labeling and a complementary edge (for (\mathbb{Z}_2, n) -labeled complex X) is just a complementary edge in Tucker's sense (the sum of its labels equals zero).

Theorem 7.1. *$\text{t-ind}^G X \geq d$ if and only if for any equivariant (G, d) -labeling of the vertex set of an arbitrary equivariant triangulation of X there exists a complementary edge.*

Proof. Assume there is a (G, d) -labeling of the vertex set of an equivariant triangulation of X without complementary edges. Then such a labeling provides an equivariant map $X \rightarrow J^d(G)$. This contradicts with the assumption that $\text{t-ind}^G X \geq d$.

Now assume that for any equivariant (G, d) -labeling of the vertex set of an arbitrary equivariant triangulation of X there exists a complementary edge. Assume that $\text{t-ind}^G X < d$. Then there exists an equivariant continuous map $X \rightarrow J^d(G)$ and an equivariant simplicial approximation of this map which is a simplicial map of some triangulation of X . So there exists a (G, d) -labeling of $J^d(G)$ without complementary edges. Thus, the inverse image of this labeling is a (G, d) -labeling of X without complementary edges, a contradiction. \square

Remark 7.1. 1) From equivariance of labeling it follows that if there exists a complementary edge, then there is a whole orbit of complementary edges, i.e. there exist at least $|G|$ different complementary edges.

2) The first statement of theorem 7.1 holds for any numerical index $\text{Ind}(\cdot)$ which possesses the main property of index and dimension property, i.e.:

A) If there exists an equivariant map $X \rightarrow Y$ of G -spaces then $\text{Ind}(X) \leq \text{Ind}(Y)$.

B) $\text{Ind } J^{d+1}(G) = d$.

All indexes considered above satisfy these properties.

Thus the following assertion holds:

Let X be a simplicial complex with a free simplicial G -action such that $\text{Ind}(X) \leq d$. Then for any equivariant (G, d) -labeling of the vertex set of X there exists a complementary edge (actually there exist at least $|G|$ complementary edges).

7.2 Tucker type lemmas for bounded spaces

Consider the case of simplicial complex X and its subcomplex X_0 . We assume that G acts freely and simplicially on X_0 .

Theorem 7.2. *Assume that $\text{ind}^G X = n - 1$ and that X_0 is an $(n - 1)$ -c.t.-subspace of X over \mathbb{Z} . Then for any (G, n) -labeling of the vertex set of an arbitrary triangulation of X which is equivariant on X_0 there exists a complementary edge.*

Proof. We argue by contradiction. A (G, n) -labeling of the vertex set of a triangulation of X without complementary edges provides a map $\psi : X \rightarrow J^n(G)$, and this map is equivariant on X since our (G, n) -labeling is equivariant on X . Since i^* is trivial in dimension $n - 1$, where $i : X_0 \subset X$ is the inclusion, we see that $(\psi|_{X_0})^* : H^{n-1}(J^n(G); \mathbb{Z}) \rightarrow H^{n-1}(X; \mathbb{Z})$ is trivial, and we obtain a contradiction with property 8 of cohomological index, because $\text{ind}^G X = n - 1 = \text{ind}^G J^n(G)$. \square

As a partial case we obtain:

Theorem 7.3. *Let M^n be a connected compact orientable PL-manifold such that its boundary ∂M is homeomorphic to the sphere \mathbb{S}^{n-1} . Let T be a triangulation of M . Suppose that there exists a free simplicial action of a finite group G on ∂T . Then for any (G, n) -labeling of $V(T)$ that is an equivariant on ∂T there exists a complementary edge.*

7.3 Bourgin–Yang type results and Tucker type lemmas

Consider first the case $G = \mathbb{Z}_2$. By $\text{in}(\cdot)$ and $\text{t-ind}^G(\cdot)$, $G = \mathbb{Z}_2$, we denote Yang’s homological and topological indexes respectively, so $\text{in}(\mathbb{S}^n) = \text{t-ind}^G(\mathbb{S}^n) = n$.

Proposition 7.1. *Let $f : \mathbb{S}^n \rightarrow M$ be a continuous map to m -dimensional manifold M where $d := n - m > 0$. Assume that $K := \{x \in \mathbb{S}^n \mid f(x) = f(-x)\}$ is a triangulable space. Then for any equivariant triangulation of K and an equivariant labeling of its vertices by $\{+1, -1, +2, -2, \dots, +d, -d\}$ there exists a complementary edge.*

Proof. It follows from [24] (see also [29]) that $\text{in} K \geq d$, and by Theorem 7.1 we are done. \square

Note that if M is a PL -manifold and f is simplicial with respect to some equivariant triangulation of \mathbb{S}^n , then K is a triangulable space (but not necessarily a simplicial subspace).

Similar result holds for $G = \mathbb{Z}_p \oplus \dots \oplus \mathbb{Z}_p$, elementary Abelian group, p is a prime.

Proposition 7.2. *Let $f : J^{n+1}(G) \rightarrow M$ be a continuous map to m -dimensional manifold M where $d := n - m(|G| - 1) > 0$ and $G = \mathbb{Z}_p \oplus \dots \oplus \mathbb{Z}_p$ is an elementary Abelian group, p is a prime. Assume that $K := \{x \in J^{n+1}(G) \mid f(x) = f(gx) \forall g \in G\}$ is a triangulable space. Then for any equivariant triangulation of K and an equivariant (G, d) -labeling of its vertices there exists a complementary edge.*

Proof. It follows from [32] that $\text{t-ind}^G K \geq n - m(|G| - 1)$. Hence the result follows from Theorem 7.1. \square

References

- [1] P. S. Aleksandrov, H. Hopf, *Topology*, AMS Chelsea Publishing Series (vol. 197), 1965.
- [2] A. V. Arkhangel'skii, V. V. Fedorchuk, *General Topology I: Basic Concepts and Constructions Dimension Theory*, Encyclopaedia of Mathematical Sciences, vol. 17, A. V. Arkhangel'skii, L. S. Pontryagin (eds.), 1990.
- [3] P. F. Baum, L. Dąbrowski, P. M. Hajac, Noncommutative Borsuk–Ulam-type conjectures, *Banach Center Publications*, **106** (2015), 9–18.
- [4] K. Borsuk, Drei Sätze über die n -dimensionale euklidische Sphäre, *Fund. Math.*, **20** (1933), 177–190.
- [5] Glen E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972
- [6] A. Chirvasitu, B. Passer, Compact Group Actions on Topological and Noncommutative Joins, *Proc. Amer. Math. Soc.* **146** (2018), 3217–3232.

- [7] P. E. Conner and E. E. Floyd, Fixed points free involutions and equivariant maps, *Bull. Amer. Mat. Soc.*, **60** (1960), 416–441.
- [8] P. E. Conner and E. E. Floyd, Differentiable periodic maps, Springer-Verlag, 1964.
- [9] L. Dąbrowski, P. M. Hajac, S. Neshveyev, Noncommutative Borsuk–Ulam-type conjectures revisited, *J. Noncommut. Geom.*, **14** (2020), 1305–1324
- [10] A. Dold, Simple proofs of some Borsuk-Ulam results, *Proc. Northwestern Homotopy Theory Conf. (H. R. Miller and S. B. Priddy, eds.)*, *Contemp. Math.*, **19** (1983), 65–69.
- [11] Ky Fan, A generalization of Tucker’s combinatorial lemma with topological applications. *Ann. of Math.*, **56** (1952), 431–437.
- [12] D. H. Gottlieb, The Lefschetz number and Borsuk–Ulam theorems, *Pacific J. Math.*, **103** No. 1, (1982), 29–37.
- [13] M. A. Krasnosel’skii, On computation of the rotation of a vector field on n -dimensional sphere, *Dokl. Acad. Nauk. SSSR (N.S.)*, **101** (1955), 401–404.
- [14] M. A. Krasnosel’skii, P. P. Zabreiko, Geometrical methods of nonlinear analysis, Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo, 1984.
- [15] A. Liulevicius, Borsuk-Ulam theorems for spherical space forms, *Proc. Northwestern Homotopy Theory Conf. (H. R. Miller and S. B. Priddy, eds.)*, *Contemp. Math.*, **19** (1983), 189–192.
- [16] E. L. Lusk, The mod p Smith index and a generalized Borsuk-Ulam theorem. *Michigan Math. J.*, **22** (1975), no. 2, 151–160.
- [17] J. Matoušek, Using the Borsuk-Ulam theorem, Springer-Verlag, Berlin, 2003.
- [18] G. D. Mostow, Equivariant embeddings in Euclidean space, *Annals of Math.* **65** (1957), 432–446.
- [19] O. R. Musin, Borsuk-Ulam type theorems for manifolds, *Proc. Amer. Math. Soc.* **140** (2012), 2551–2560.
- [20] O. R. Musin, Extensions of Sperner and Tucker’s lemma for manifolds, *J. of Combin. Theory Ser. A*, **132** (2015), 172–187.
- [21] O. R. Musin, Homotopy invariants of covers and KKM–type lemmas, *Algebraic & Geometric Topology*, **16:3** (2016), 1799–1812.
- [22] O. R. Musin, Generalizations of Tucker–Fan–Shashkin lemmas, *Arnold Math Journal*, **2:3** (2016), 299–308.

- [23] O. R. Musin and A. Yu. Volovikov, Borsuk–Ulam type spaces, *Mosc. Math. J.*, **15:4** (2015), 749–766.
- [24] M. Nakaoka, Equivariant point theorems, *Tsukuba Math Journal*, **2** (1977), 109–126.
- [25] B. Passer, Free Actions on C^* -algebra Suspensions and Joins by Finite Cyclic Groups, *Indiana University Mathematics Journal*, **67:1** (2018), 187–203.
- [26] A. S. Švarc (Schwarz), The genus of a fiber space, *Trudy Moskov. Mat. Obsc.*, **10** (1961), 217–272 and **11** (1962), 99–126; (Russian), English translation in *Amer. Math. Soc. Translat.*, II. Ser., **55** (1966), 49–140.
- [27] A. S. Schwarz, My life in science, <https://www.math.ucdavis.edu/~schwarz/bio.pdf>
- [28] A. W. Tucker, Some topological properties of the disk and sphere. In: Proc. of the First Canadian Math. Congress, Montreal, 285–309, 1945.
- [29] A. Yu. Volovikov, A generalization of the Borsuk–Ulam theorem, *Mat. Sb. (N.S.)*, 108 (150):2, 1979, 212–218 [in Russian], English translation: *Mathematics of the USSR-Sbornik*, 1980, 36:2, 195–202.
- [30] A. Yu. Volovikov, On the Bourgin–Yang theorem, *Uspekhi Mat. Nauk*, 1980, Volume 35, Issue 3(213), 159–162 [in Russian], English translation: *Russian Mathematical Surveys*, 1980, 35:3, 196–200.
- [31] A. Yu. Volovikov, On the index of G -spaces, *Sbornik: Mathematics*, **191**, 9 (2000), 1259–1277.
- [32] A. Yu. Volovikov, Coincidence points of maps of \mathbb{Z}_p^n -spaces, *Izvestiya: Mathematics*, **69**, 5 (2005), 913–962.
- [33] A. Yu. Volovikov, On cohomological index of free G -spaces, Algebraic Topology and Abelian Functions, Conference in honour of Victor Buchstaber on the occasion of his 70th birthday, Moscow, 18–22 June 2013, Abstracts, 91–92, <http://vmb70.mi.ras.ru/abstracts.pdf>
- [34] A. Yu. Volovikov, Cohomological index of free G -spaces, In preparation.
- [35] C. T. Yang, On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobo and Dyson, I, *Ann. of Math.*, **60** (1954), 262–282.
- [36] C. T. Yang, On maps from spheres to Euclidean spaces, *Amer. J. Math.*, **79** (1957), 725–732.

O. R. Musin, University of Texas Rio Grande Valley, Brownsville, TX, 78520
E-mail address: oleg.musin@utrgv.edu

A. Yu. Volovikov, Department of Higher Mathematics, MTU MIREA, Vernadskogo av.,
78, Moscow, 119454, Russia
E-mail address: a_volov@list.ru