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Abstract

Balanced sets appeared in the 1960s in cooperative game theory as a part of nonempty core conditions. In this paper we present a classification of balanced families containing only 2–element subsets. We also discuss generalizations of the classical Sperner and Tucker lemmas using balanced sets.

1 Introduction

Balanced sets as families of weighted subsets of a finite set first appeared in Bondareva [1] and Shapley [6] papers.

Denote by $[d]$ the set $\{1, 2, ..., d\}$. Let Φ be a family of subsets $\{S_1, ..., S_m\}$ of $[d]$. Following Shapley [7], this family is called *balanced* if there is a set of non–negative weights $\{w_1, ..., w_m\}$ such that

$$\sum_{k=1}^{m} w_k \eta_k = (1, ..., 1),$$

where η_k is the characteristic (indicator) vector of S_k in $[d]$. A balanced family Φ is called *minimal* if there are no proper balanced subfamilies in Φ.

Now we consider families that contain only 2–element subsets. We say that a family F is of *odd size* if it contains an odd number of subsets.

Let $I = \{i_1, ..., i_n\}$, where $n \geq 3$. We say that a family F of subsets from I is *cyclic with respect to* I if $F = \{(i_1, i_2), (i_2, i_3), ..., (i_n, i_1)\}$. If $n = 2$, then there is only one 2–subset of I. In this case we call $F = \{(i_1, i_2)\}$ *isolated.*

The proof of the following theorem is given in Section 2.

Theorem 1. *Let Φ be a minimal balanced family of 2–subsets in $[d]$. Then $[d]$ is the disjoint union of subsets $I_1, ..., I_k$ and $\Phi = \{\Phi_\ell\}_{\ell=1,...,k}$, where Φ_ℓ is either cyclic of odd size with respect to I_ℓ or it is isolated.*

These decompositions correspond to partitions of d into parts 2, 3, 5,... The generating function of this partition is

$$\frac{1}{1+x} \prod_{i=0}^{\infty} \frac{1}{1-x^{2i+1}}.$$

It is easy to show the connection with the odd partitions. Denote by $b(d)$ the number of our partitions. Let $q(d)$ be the number of odd partitions. Then

$$b(d) = q(d) - q(d-1) + ... + (-1)^d q(0).$$

We can apply theorem 5 for cooperative games with 2-players coalitions. From the Bondareva – Shapley theorem it follows that we can check the non–empty core condition only on the minimal balanced sets. Since we know a classification of these sets we can simplify the condition for the non-emptiness of the core.

Here we give a general geometrical definition of balanced sets.

Let $V = \{v_1, v_2, \ldots, v_m\}$ be a set of points in $\mathbb{R}^d$. A subset $\{v_{i_1}, \ldots, v_{i_k}\}$ is called *balanced* if c_V lies in the convex hull $\mathrm{conv}(v_{i_1}, \ldots, v_{i_k})$, where c_V is the center of mass of V. The corresponding set of indices $I = \{i_1, \ldots, i_k\}$ is also called balanced.

A balanced subset of V is *minimal* if and only if it does not contain a subset that is balanced. Denote the family of minimal balanced subsets as $\mathrm{BS}(V)$.

Sperner's lemma on colorings of triangulations vertices and its extension to coverings Knaster –Kuratowski – Mazurkiewicz (KKM) lemma are discrete analogs of the Brower's fixed point theorem. KKM may be extended to KKMS theorem, see [7, 8]. All these theorems have many applications, particularly in game theory and mathematical economics.

Tucker, Ky Fan, and Shashkin's lemmas are discrete versions of the Borsuk–Ulam theorem. They also have various applications.

In Section 3, we consider Theorems A and B, which are generalizations of discrete versions of fixed point theorems that rely on balanced sets of V. If for some V we know its $\mathrm{BS}(V)$, then that gives explicit versions of these theorems.

2 Proof of Theorem 1

Let Φ be a family that contains only 2–element subsets from $[d]$. This family can be described geometrically with an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. To every 2-element subset (i, j) we assign a point in $\mathbb{R}^d$:

$$e_{ij} := \frac{1}{2}(e_i + e_j) \ \ (1 \leqslant i < j \leqslant d).$$

This set of points we denote by V_d. Obviously, all these $\binom{d}{2}$ points in V_d lie in a hyperplane Π_d that defined by an equation $x_1 + \ldots + x_d = 1$.

(Note that the points V_d are midpoints of edges of a $(d-1)$-dimensional simplex in Π_d with the vertex set $e_1, ..., e_d$. A polytope $P_d := \mathrm{conv}(V_d)$ plays an important role in discrete geometry, graph theory and coding theory. In particular, V_d is an example of a 2–distance set in $\mathbb{R}^{d-1}$. There are many cases when the number of points in a maximal 2–distance set is at most $|V_d| = \binom{d}{2}$.)

It is easy to see that there is one–to–one correspondence between balanced subsets of V_d and balanced families $\Phi = \{S_1, ..., S_m\}$, where all S_i are 2-subsets of $[d]$.

Denote by c_d the center of mass of V_d. Then $c_d = (\frac{1}{d}, \ldots, \frac{1}{d}) \in \Pi_d$. The main goal of this section is to describe the set of all minimal balanced sets $S \subset V_d$.

Let K_d be a complete graph on d vertices $\{a_1, ..., a_d\}$. We will identify the vertices $\{a_1, ..., a_d\}$ with the vectors $\{e_1, ..., e_d\}$ and with their indices $\{1, 2, ..., d\}$.

Let S be a subset of V_d. Define a graph $G(S)$ as a subgraph of K_d by the rule: (i, j) *is an edge of* $G(S)$ *iff* $e_{ij} \in S$.

Denote by $n(S)$ the number of vertices of $G(S)$. Note that the edges of this graph correspond to the elements of S. It is clear that

$$S \subseteq V_{n(S)} \subset \Pi_{n(S)} \subset \mathbb{R}^{n(S)} \subseteq \mathbb{R}^d$$

and we have the following statement.

Lemma 2. *Let S be a subset of V_d. Suppose that $S = S_1 \cup S_2$, where $S_1 \cap S_2 = \emptyset$ and $G(S)$ is the disjoint union of graphs $G(S_1)$ and $G(S_2)$. Then*

$$S_i \subset \mathbb{R}^{n(S_i)},\ i = 1, 2; \quad \mathbb{R}^{n(S)} = \mathbb{R}^{n(S_1)} \oplus \mathbb{R}^{n(S_2)}$$

This lemma yields that if $G(S_1), ..., G(S_k)$, $k > 1$, are connected components of $G(S)$ then the sets S_i lie in mutually orthogonal subspaces of $\mathbb{R}^{n(S)}$.

Lemma 3. *Let $S \subset V_d$ be a balanced set. Suppose $G(S_1), ..., G(S_k)$ are connected components of $G(S)$. Then for all $i = 1, \ldots, k$ the set S_i is also balanced for $V = V_{n(S_i)}$.*

The next lemma follows directly from Shapley's definition:

Lemma 4. *Let S be a minimal balanced set, $|S| > 1$. Suppose that the graph $G(S)$ is connected. Then $G(S)$ has no vertices of degree 1.*

In Section 1 we defined cyclic and isolated families. For $S \subset V_n$ these definitions mean: S is cyclic if $G(S)$ is a polygon with n–vertices and S is isolated if $n = 2$ and so S contains only one vertex.

Lemma 5. *Let S be a minimal balanced set of $V = V_n$ with $n > 2$. Suppose that $G(S)$ is connected and $n(S) = n$. Then n is odd and S is cyclic.*

Proof. Our proof relies on the following well–known theorem:

Carathéodory's theorem. *If a point x lies in the convex hull of a set P in $\mathbb{R}^m$, then x can be written as the convex combination of at most $m + 1$ points in P.*

Note that S is a subset of $(n - 1)$–dimensional Euclidean space. By the assumption the number of vertices of $G(S)$ is n and c_n lies in conv(S).

Caratheodory's theorem implies that c_n is the convex combination of $\ell \leq n$ vertices from S. From the minimality of S it follows that this subset of vertices coincides with S and $|S| = \ell$, hence $|S| \leq n$.

We see that the graph $G(S)$ is on n vertices and has at most n edges. Then from Lemma 4 it follows that this graph is an n–polygon.

Suppose $n = 2k$, $S = \{(1, 2)(2, 3), ..., (2k, 1)\}$, and $S' = \{(1, 2)(3, 4), ..., (2k - 1, 2k)\}$. Then $|S'| = k$. Since c_n is covered by the convex hull of S' and $S' \subset S$, we see that S is not minimal and n can not be even. □

Theorem 1 directly follows from lemmas 3 and 5.

3 Tucker, Fan, and Shashkin lemmas as corollaries of the balanced sets theorem

3.1 Discrete versions of fixed point theorems.

Let a space X is covered by m open (or closed) sets and $V = \{v_1, v_2, \dots, v_m\} \subset \mathbb{R}^d$. Following [4], we can construct a map $f_V : X \to \mathrm{conv}(V)$.

Let T be a triangulation of a manifold M. A vertex coloring $L : V(T) \to \{1, ..., m\}$ is a special case of covering. Then we can define a map $f_L : T \to \mathrm{conv}(V)$. Main results about these maps may be found in [4, Theorem 3.1], [4, Cor. 3.2], and [5, Th. 4.2]. Here we give corollaries of these theorems.

Theorem A. *Let $V := \{v_1, \dots, v_m\} \subset \mathbb{R}^d$. Let $F = \{F_1, \dots, F_m\}$ be a closed (or open) covering of n-dimensional disc. Suppose that F is such that f_V is not null–homotopic on the boundary. Then there is a minimal balanced set $I \in \mathrm{BS}(V)$ such that $\cap_{i \in I} F_i \neq \emptyset$.*

Theorem B. *Let $V := \{v_1, \dots, v_m\} \subset \mathbb{R}^d$. Let T be a triangulation of n-dimensional disc and $L : V(T) \to \{1, ..., m\}$ be a coloring of $V(T)$. Suppose that f_L is not null–homotopic on the boundary. Then there is a simplex s in T and $I \in \mathrm{BS}(V)$ such that vertices of s are colored with all colors from I.*

Note that the *"not null-homotopic"* on the boundary condition is true in the case of classical fixed point theorems. In particular, *Sperner's coloring*, *KKM's covering*, and *antipodal coloring* on the boundary are special cases of this condition.

Using theorems A and B we can obtain discrete versions of fixed point theorems.

Suppose $m = d + 1$, $n = d$. Let F be a covering of d–dimensional simplex $\Delta^d \subset \mathbb{R}^d$ with the vertex set V. Assume that F satisfies the boundary conditions of the KKM lemma. Then this covering is not null–homotopic on the boundary of Δ^d and the KKM lemma follows from Theorem A. Sperner's lemma can be easily deduced from the KKM lemma or from Theorem B.

Every Sperner coloring of a triangulation of Δ^d contains a cell whose vertices all have different colors.

Shapley's KKMS lemma [7, 8] also can be easily deduced from Theorem A, see [5, Cor. 4.2]. In this case the set of points V is the set of all centers of mass of k-vertex subsets of Δ^d, where $1 \leq k \leq d$.

Let $V = V_d$ and we are in the conditions of Theorems A and B. *Then Theorem 1 yields a new result for colorings with $\binom{d}{2}$ colors.*

3.2 Antipodal balanced 2-subsets.

Let P be a centrally symmetric polytope in $\mathbb{R}^d$ with the vertex set $V = V(P)$. In other words, if $v \in V$, then $(-v) \in V$. We see that the center of mass $c_V = O$, where O is the origin of $\mathbb{R}^d$. It is clear that the *family of balanced 2-subsets of V is the set of all antipodal pairs $(v, -v)$, where $v \in V$.*

Suppose that T and L from Theorem B are both antipodally symmetric on the boundary. Then (see [3, 4, 5]) f_L is not null-homotopic on the boundary, hence we can use Theorem B.

1. Let $e_1, \dots, e_d$ be a standard orthonormal basis for $\mathbb{R}^d$. Let P be a regular cross polytope with the vertex set $V = \{\pm e_1, ..., \pm e_d\}$. Then we see that BS($V$) coincides with the set of all pairs of antipodal vertices. Then Tucker's lemma follows from Theorem B.

Let T be a triangulation of a d-dimensional disc such that T is antipodally symmetric on the boundary. Let $L : V(T) \to \{+1, -1, +2, -2, \dots, +d, -d\}$ be a coloring that is antipodal (i.e. $L(-v) = -L(v)$) for all vertices v on the boundary. Then there exists a complementary edge, i.e. $[u, v] \in T$ such that $L(u) = -L(v)$.

2. In [2, Th. 5.2] was constructed a convex polytope $P(n, d) \subset \mathbb{R}^d$ with $2n$ centrally symmetric vertices $V = \{\pm v_1, \dots, \pm v_n\}$ such that BS(V) consists of antipodal pairs $(v_i, -v_i)$, $i = 1, ..., n$ and d-simplices with vertices $(v_{k_0}, -v_{k_1}, \dots, (-1)^d v_{k_d})$ or $\{-v_{k_0}, v_{k_1}, \dots, (-1)^{d+1} v_{k_d})$, where $1 \leq k_0 < \dots < k_d \leq n$. Then Theorem B yields Ky Fan's lemma:

Let T be a triangulation of a d-dimensional disc that is antipodally symmetric on the boundary. Let $L : V(T) \to \{+1, -1, +2, -2, \dots, +n, -n\}$ be a coloring that is antipodal on the boundary. Suppose that there are no complementary edges in T. Then there are an odd number of alternating d-simplices, i.e. simplices that are colored by $(k_0, -k_1, k_2, \dots, (-1)^d k_d)$, where $1 \leq |k_0| < \dots < |k_d| \leq n$ and all k_i are of the same sign.

3. In [3] is considered an extended version of Shashkin's lemma.

Let T be a triangulation of a $(d-1)$-dimensional disc that is antipodally symmetric on the boundary. Let $L : V(T) \to \{+1, -1, +2, -2, \dots, +d, -d\}$ be a coloring that is antipodal on the boundary. Suppose that there are no complementary edges in T. Then for every set of colors $\Lambda = \{\ell_1, \dots, \ell_d\}$, where $|\ell_i| = i$ for $i = 1, \dots, d$, there are an odd number of cells in T that are labelled by Λ or $(-\Lambda)$.

The proofs in [3] do not rely on Theorem B. Here we show that Shashkin's lemma follows from this theorem.

Let Δ be a $(d-1)$–dimensional simplex in $\mathbb{R}^{d-1}$ with the center of mass at the origin O. Let $v_1, ..., v_d$ be the vertices of Δ. Suppose that V is the set of points $\{\pm v_1, ..., \pm v_d\}$. It is easy to prove that BS(V) consists of the set of all pairs $(v_i, -v_i)$ and the sets $\{v_1, ..., v_d\}$ and $\{-v_1, ..., -v_d\}$. Assign a color ℓ_i to a vertex v_i and $(-\ell_i)$ to $-v_i$. Then f_L is antipodal on the boundary and Shashkin's lemma follows from Theorem B.

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