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Second-order parametrized-post-Newtonian Lagrangian

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A many-body Lagrangian to second post-Newtonian order using an extension of the parametrized-post-Newtonian (PPN) formalism is introduced and the properties of new parameters are explored. A parametrized gauge transformation is developed to permit comparison with theories of gravity in a variety of different coordinate systems. A procedure to impose Lorentz invariance on a general second-order post-Newtonian Lagrangian is developed. The Lagrangian is then constrained to possess Lorentz invariance and a "Lorentz-invariant" gauge is introduced. The constrained Lagrangian is found to be described by ten new second-order PPN parameters. When the Lagrangian is further constrained to describe theories of gravity for which test particles move along geodesics, one of the ten new parameters is given entirely in terms of first-order PPN parameters, leaving only nine PPN parameters to describe the second-order gravitational interaction. A "metric" gauge is introduced which allows the metric to be easily found from the Lagrangian and is shown to reduce to the gauge associated with the canonical formalism of Arnowitt, Deser, and Misner when the general-relativity values of the PPN parameters are used.

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I. INTRODUCTION

Experimental tests of general relativity have focused primarily on observations of first post-Newtonian (1PN) effects using the parametrized-post-Newtonian (PPN) formalism [1]. Originally developed as an expansion of the metric to 1PN order [2], this formalism was later applied to a 1PN expansion of a many-body Lagrangian [1], [3]. The robustness of the PPN formalism combined with improved accuracy in precision experiments has led to a need for a coherent second post-Newtonian (2PN) extension of the PPN framework to accompany experimental tests of relativity beyond the 1PN order [4], [5].

A second-order parametrized-post-Newtonian (2PPN) many-body Lagrangian is here developed using an extension of the procedures introduced by Nordtvedt in obtaining the PPN Lagrangian [3]. The 2PPN Lagrangian so obtained possesses a more complicated class of 2PPN parameters which can be dimensionless scalar functions of the interbody separation vectors. In addition, different gauges are required for different applications of the Lagrangian. If it is to be invariant under a Lorentz transformation (without an accompanying gauge change), the Lagrangian must be written in an *acceleration-dependent* form. The 2PPN metric can be obtained from the Lagrangian if it is written in a gauge which allows it to take the form of a line element when one of the bodies is taken to be a point mass. Such a Lagrangian must be an ordinary, *acceleration-independent* Lagrangian. A general gauge transformation is introduced during the development of the 2PPN many-body Lagrangian, while specific gauge choices are discussed in conjunction with imposing Lorentz invariance on the Lagrangian and obtaining the metric from the motion of a point mass.

II. THE 2PPN LAGRANGIAN

A many-body 2PPN Lagrangian for describing a general class of Lagrangian-based theories of gravity can be obtained by considering the end result of deriving the 2PN expansion for any specific theory of gravity [3]. In this procedure, one would begin with a specific field Lagrangian $\mathcal{L}$ which can be written

$$\mathcal{L}=\mathcal{L}_{\rm NG}+\mathcal{L}_{\rm G} \tag{1}$$

where $\mathcal{L}_{\rm G}$ is the part of the Lagrangian which contains only the gravitational field variables and no matter variables, while $\mathcal{L}_{\rm NG}$ contains only matter variables and the metric field [6]. Since the intent is to produce a many-body Lagrangian, the matter is considered to be concentrated into a finite number of bodies. For ease of development, the internal dynamics of these bodies is taken to be negligible. Thus, the matter variables are reduced to the time-dependent coordinate trajectories of each body ($\mathbf{r}_i$) and time-independent mass parameters which are related to the structure and content of each body. The metric field in $\mathcal{L}_{\rm NG}$ is then replaced by a functional of the trajectories (and includes the mass parameters) which leaves $\mathcal{L}_{\rm NG}$ as a functional of the trajectories alone. The field equations for the gravitational field variables in $\mathcal{L}_{\rm G}$ can then be solved to yield the field variables in terms of the trajectories and mass parameters. The end result is a many-body Lagrangian, L which incorporates the mass parameters and is a functional only of the trajectories. The action is then given by

$$I=\int_{\lambda_1}^{\lambda_2} L\left[\mathbf{r}_i(\lambda),\frac{d\mathbf{r}_i}{d\lambda},\ldots,\frac{d^n\mathbf{r}_i}{d\lambda^n}\right]d\lambda\ , \tag{2}$$

where λ is a parametrization of each trajectory. Although it is not necessary for λ to be the same parametrization for each body, it is usually taken to be a universal Newtonian-type time coordinate. This choice is made in the post-Newtonian expansion of L, and units are chosen so that $G=c=1$. The expansion variables are taken to be the potentials (which are of the form M/r) and the time derivatives of the trajectories $(d^n\mathbf{r}_i/dt^n)$. The degree of "smallness" of these variables is of order $(1/c^2)$ for the potentials and $(1/c)$ for each differentiation with respect to time. An expansion to 2PN order is carried out to order $(1/c^6)$. The class of theories for which this procedure is applicable is restricted to those theories for which there is no radiation present to the desired order of approximation. Since dipole radiation first appears at order $(1/c^5)$, theories which allow dipole radiation are excluded from this discussion. To obtain a 2PPN Lagrangian for describing the class of theories for which the above procedure is applicable, one simply writes down a general expansion incorporating all possible terms up to the appropriate order. The mass parameters in each term are incorporated into the expansion coefficients which are then taken as 2PPN parameters to be determined by appeal to observation or eliminated by judicious choice of gauge.

The expansion terms of the 2PPN Lagrangian can be conveniently grouped according to the mass dimension of the 2PPN parameters and the number of bodies involved in the specific interaction described by the 2PPN parameter. The simplest of these groups contains only one term which has mass dimension one and involves one body. It appears in the kinetic expansion

$$L=-\sum_i M_i(1-\tfrac{1}{2}v_i^2-\tfrac{1}{8}v_i^4)-M_i^{(6)}v_i^6+\cdots \; . \tag{3}$$

This term would be present even in the absence of any gravitational field. Since the evidence for the validity of special relativity is fairly strong, $M_i^{(6)}$ can be set equal to the inertial mass M_i without any significant loss of generality.

The next group of terms have mass dimension two [i.e., they have units of (mass)^2] and involve the interaction of two bodies. These were referred to as "linear field" terms in a previous paper [7]. They appear in the Lagrangian as

$$\begin{aligned}
&\sum_{i\neq j} r_{ij}[A_{ij}^1\ddot{\mathbf{a}}_i\cdot\mathbf{r}_{ij}+A_{ij}^2(\dot{\mathbf{a}}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})+A_{ij}^3(\dot{\mathbf{a}}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})+A_{ij}^4\dot{\mathbf{a}}_i\cdot\mathbf{v}_i\\
&\qquad+A_{ij}^5\dot{\mathbf{a}}_i\cdot\mathbf{v}_j+A_{ij}^6a_i^2+A_{ij}^7\mathbf{a}_i\cdot\mathbf{a}_j+A_{ij}^8(\mathbf{a}_i\cdot\hat{\mathbf{r}}_{ij})^2+A_{ij}^9(\mathbf{a}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{a}_j\cdot\hat{\mathbf{r}}_{ij})]\\
&+\sum_{i\neq j}\frac{1}{r_{ij}}\{(\mathbf{a}_i\cdot\mathbf{r}_{ij})[B_{ij}^1v_i^2+B_{ij}^2v_j^2+B_{ij}^3\mathbf{v}_i\cdot\mathbf{v}_j+B_{ij}^4(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2+B_{ij}^5(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2+B_{ij}^6(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})]\\
&\qquad+(\mathbf{a}_i\cdot\mathbf{v}_i)(B_{ij}^7\mathbf{v}_i\cdot\mathbf{r}_{ij}+B_{ij}^8\mathbf{v}_j\cdot\mathbf{r}_{ij})+(\mathbf{a}_i\cdot\mathbf{v}_j)(B_{ij}^9\mathbf{v}_i\cdot\mathbf{r}_{ij}+B_{ij}^{10}\mathbf{v}_j\cdot\mathbf{r}_{ij})\\
&\qquad+C_{ij}^1v_i^4+C_{ij}^2v_i^2(\mathbf{v}_i\cdot\mathbf{v}_j)+C_{ij}^3(\mathbf{v}_i\cdot\mathbf{v}_j)^2+C_{ij}^4v_i^2v_j^2+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})(C_{ij}^5v_i^2+C_{ij}^6\mathbf{v}_i\cdot\mathbf{v}_j)\\
&\qquad+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(C_{ij}^7v_i^2+C_{ij}^8v_j^2+C_{ij}^9\mathbf{v}_i\cdot\mathbf{v}_j)+C_{ij}^{10}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^4+C_{ij}^{11}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^3(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})+C_{ij}^{12}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2\}\; .
\end{aligned}\tag{4}$$

The parameters in this group are symmetric under interchange of their indices. Each can be thought of as defining a separate gravitational interaction and therefore incorporating a numerical coupling constant and the associated gravitational mass parameters of each body involved in the interaction. However, it is not necessary that the parameter be a simple product of the coupling constant and mass parameters [3]. It is possible to determine the value of the coupling constant and the mass parameters in the limit that one or both of the bodies in the interaction are taken to be test bodies. In the test-body limit, the mass of the body approaches zero in such a way that all the mass parameters $M(X)$ for the body reduce to the inertial mass M, so that

$$\lim_{i\to\mathrm{tb}}\frac{M(X)_i}{M_i}=1\; . \tag{5}$$

Thus, a particular two-body parameter in this group (X_{ij}) reduces to the coupling constant X in the limit that both bodies are test bodies:

$$\lim_{i,j\to\mathrm{tb}}\frac{X_{ij}}{M_iM_j}=X\; . \tag{6}$$

The gravitational mass parameter $M_j(X)$ associated with the interaction described by X_{ij} is found in the limit that only one of the bodies is a test body:

$$\lim_{i\to\mathrm{tb}}\frac{X_{ij}}{M_i}=XM_j(X)\; . \tag{7}$$

Terms which have mass dimension three, but only involve the interaction of two bodies are the simplest of the nonlinear interactions at 2PN order. They appear in the Lagrangian as

$$\sum_{i\neq j}\frac{1}{r_{ij}^2}[a_{ij}^1(\mathbf{a}_i\cdot\mathbf{r}_{ij})-a_{ji}^1(\mathbf{a}_j\cdot\mathbf{r}_{ij})+a_{ij}^2v_i^2+a_{ji}^2v_j^2+a_{ij}^3\mathbf{v}_i\cdot\mathbf{v}_j+a_{ij}^4(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2+a_{ji}^4(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2+a_{ij}^5(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})]\; . \tag{8}$$

With the exception of a_{ij}^3 and a_{ij}^5, these parameters are not symmetric under interchange of the indices i and j. All parameters a_{ij}^n describe interactions which are of order $M_i^2 M_j$ as well as those of order $M_i M_j^2$. When one of the bodies is taken to be a test body, its second-order contribution to the parameters a_{ij}^n is reduced to zero, leaving a mass parameter of mass dimension two:

$$\lim_{i \to \mathrm{tb}} \frac{a_{ij}^n}{M_i} = \lambda_j^n \; , \tag{9}$$

$$\lim_{j \to \mathrm{tb}} \frac{a_{ij}^n}{M_j} = \lambda_i'^n \; . \tag{10}$$

For interactions involving more than two bodies, the parameters must also incorporate possible dimensionless scalar functions involving the interbody separation vectors of the bodies involved in the interaction. The simplest class of these types of parameters are of mass dimension three and involve the interaction of three bodies. These terms appear in the Lagangian as

$$\sum_{i \neq j \neq k} \frac{1}{r_{ij} r_{ik}} (A_{ijk} \mathbf{a}_i \cdot \mathbf{r}_{ij} + B_{ijk}^1 v_i^2 + B_{ijk}^2 \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbb{C}_{ijk}^1 \cdot \mathbf{v}_i + \mathbf{v}_i \cdot \mathbb{C}_{ijk}^2 \cdot \mathbf{v}_j) \tag{11}$$

where the summation is to be taken over three distinct bodies, so that $j \neq i$ and $k \neq i$ or j. (This summation convention will be followed throughout.) The parameters A_{ijk} and B_{ijk}^n, each define a separate three-body interaction and incorporate an associated coupling constant and mass parameter. In addition, they are dimensionless scalar functions of the interbody separation vectors $\mathbf{r}_{ij}$ and $\mathbf{r}_{ik}$. In the limit that one or more of the three bodies is taken to be a test body, the body's position will still be of importance and so reference to the test body does not entirely drop out of the parameter (although its mass does). The notation of an underlined subscript is introduced to distinguish between compact celestial bodies and test bodies in these parameters. Thus, for a given parameter X_{ijk},

$$\lim_{i \to \mathrm{tb}} \frac{X_{ijk}}{M_i} = X_{\underline{i}jk} \tag{12}$$

and the mass dimension of X_{ijk} is reduced by one for each underlined subscript. The parameters $\mathbb{C}_{ijk}^n$ are matrices built up from outer products of unit interbody separation vectors in the following fashion:

$$\mathbb{C}_{ijk} = C_{ijk}^1 \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij} + C_{ijk}^2 \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik} + C_{ijk}^3 \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ik} + C_{ijk}^4 \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ij} \tag{13}$$

where the subparameters C_{ijk}^n obey the same rules as A_{ijk} and B_{ijk}^n. Since $\mathbb{C}_{ijk}^1$ is a symmetric matrix and is also symmetric under interchange of its last two indices, it can be described using only two subparameters. Having no special symmetries, $\mathbb{C}_{ijk}^2$ is described by four subparameters.

The last terms needed at 2PN order are of mass dimension four. They appear in the Lagrangian as

$$\sum_{i \neq j} \frac{\Psi_{ij} + \Theta_{ij}}{r_{ij}^3} + \sum_{i \neq j \neq k} \frac{\Omega_{ijk}}{r_{ij} r_{ik} r_{jk}} + \sum_{i \neq j \neq k \neq l} \frac{\Pi_{ijkl}}{r_{ij} r_{ik} r_{il}} \; . \tag{14}$$

The two-body parameters Ψ_{ij} and Θ_{ij} are symmetric under an interchange of their indices. The parameter Ψ_{ij} is of order $M_i^3 M_j$ and $M_i M_j^3$, while Θ_{ij} is of order $M_i^2 M_j^2$. In the limit that body i becomes a test body, Ψ_{ij} defines a third-order mass parameter for body j:

$$\lim_{i \to \mathrm{tb}} \frac{\Psi_{ij}}{M_i} = \psi_j \; . \tag{15}$$

In this same limit, Θ_{ij} goes to zero.

The three-body parameter Ω_{ijk} is symmetric under interchange of any two indices and can be a dimensionless scalar function of the interbody separation vectors. Its mass dependence is of order $M_i^2 M_j M_k$. The underlined subscript notation is used to denote a body taken to the test-body limit. The parameter Π_{ijkl} is also a dimensionless scalar function of the interbody separation vectors, and describes an interaction involving four bodies. It is symmetric in its last three indices. Again, an underlined subscript is used to denote test bodies.

The 2PPN Lagrangian given above in (3), (4), (8), (11), and (14) contains more parameters than are actually needed to describe the class of gravitational theories at 2PN order, since a number of the parameters can be eliminated by a suitable gauge choice. At the 1PN level the Lagrangian can be written in a single standard gauge which is valid for all applications, but this is not the case for the 2PPN Lagrangian. Therefore, a general gauge transformation is needed which alters the Lagrangian at the 2PN level, but not at 1PN order. This transformation consists of a generalized contact transformation of the trajectories,

$$\mathbf{r}_i' = \mathbf{r}_i + \delta \mathbf{r}_i \; , \tag{16}$$

plus the addition to the Lagrangian of a total time derivative of a function Q which depends on the trajectories and their time derivatives. If $\delta \mathbf{r}_i$ is entirely of order $(1/c^4)$ and Q is entirely of order $(1/c^5)$, then this transformation alters the Lagrangian by

$$\delta L = \sum_i M_i \mathbf{v}_i \cdot \frac{d \mathbf{r}_i}{dt} - \frac{1}{2} \sum_{i \neq j} \frac{\Gamma_{ij}}{r_{ij}^3} \mathbf{r}_{ij} \cdot (\delta \mathbf{r}_i - \delta \mathbf{r}_j) + \frac{dQ}{dt} \; , \tag{17}$$

where δL is entirely of order $(1/c^6)$. The parameter Γ_{ij} comes from the 1PPN Lagrangian given by Nordtvedt [3]:

$$L_{1\mathrm{PPN}} = - \sum_i M_i (1 - \tfrac{1}{2} v_i^2 - \tfrac{1}{8} v_i^4) + \frac{1}{2} \sum_{i \neq j} \frac{1}{r_{ij}} (\Gamma_{ij} \{ 1 - \tfrac{1}{2} [v_{ij}^2 + \mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{r}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{r}}_{ij})] \} + (1+\gamma) M_i M_j v_{ij}^2) + (\tfrac{1}{2} - \beta) \sum_{j,k \neq i} \frac{\Gamma_{ijk}}{r_{ij} r_{ik}} \; , \tag{18}$$

where $v_{ij}^2=(\mathbf{v}_i-\mathbf{v}_j)^2$.

The general contact transformation of the appropriate order is

$$M_i\delta\mathbf{r}_i=\sum_{j\neq i}\frac{1}{r_{ij}}\{\mathbf{r}_{ij}[\Delta_{ij}^1 v_i^2+\Delta_{ij}^2 v_j^2+\Delta_{ij}^3\mathbf{v}_i\cdot\mathbf{v}_j+\Delta_{ij}^4(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2+\Delta_{ij}^5(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2+\Delta_{ij}^6(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})+\Delta_{ij}^7\mathbf{a}_i\cdot\mathbf{r}_{ij}+\Delta_{ij}^8\mathbf{a}_j\cdot\mathbf{r}_{ij}]$$

$$+\mathbf{v}_i(\Delta_{ij}^9\mathbf{v}_i\cdot\mathbf{r}_{ij}+\Delta_{ij}^{10}\mathbf{v}_j\cdot\mathbf{r}_{ij})+\mathbf{v}_j(\Delta_{ij}^{11}\mathbf{v}_i\cdot\mathbf{r}_{ij}+\Delta_{ij}^{12}\mathbf{v}_j\cdot\mathbf{r}_{ij})\}+\sum_{j\neq i}\frac{\Delta_{ij}^{13}}{r_{ij}^2}\mathbf{r}_{ij}+\sum_{j\neq k\neq i}\frac{\Delta_{ijk}^{14}}{r_{ij}r_{ik}}\mathbf{r}_{ij}\ , \tag{19}$$

where the Δ^n parametrize the transformation. These parameters are similar to the 2PPN parameters which appear in the Lagrangian so that Δ_{ijk}^{14} possesses the same properties as the parameters in (11), Δ_{ij}^{13} is similar to the parameters in (8), and the rest are analogous with the parameters in (4). The appropriate function Q is given by

$$Q=\sum_{i\neq j}\frac{1}{r_{ij}}\{(\mathbf{v}_i\cdot\mathbf{r}_{ij})[q_{ij}^1 v_i^2+q_{ij}^2 v_j^2+q_{ij}^3\mathbf{v}_i\cdot\mathbf{v}_j+q_{ij}^4(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2+q_{ij}^5(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2]$$

$$+r_{ij}^2(q_{ij}^6\mathbf{a}_i\cdot\mathbf{v}_i+q_{ij}^7\mathbf{a}_i\cdot\mathbf{v}_j+q_{ij}^8\dot{\mathbf{a}}_i\cdot\mathbf{r}_{ij})+(\mathbf{a}_j\cdot\mathbf{r}_{ij})(q_{ij}^9\mathbf{v}_i\cdot\mathbf{r}_{ij}+q_{ij}^{10}\mathbf{v}_j\cdot\mathbf{r}_{ij})\}$$

$$+\sum_{i\neq j}\frac{q_{ij}^{11}}{r_{ij}^2}\mathbf{v}_i\cdot\mathbf{r}_{ij}+\sum_{i\neq j\neq k}\frac{q_{ijk}^{12}}{r_{ij}r_{ik}}\mathbf{v}_i\cdot\mathbf{r}_{ij}\ , \tag{20}$$

where the q^n parametrize the function Q. It is possible to eliminate up to 26 of the 2PPN parameters in the Lagrangian by correctly choosing Δ^n and q^n, leaving 24 parameters to describe the general class of Lagrangian-based theories of gravity at the 2PN order.

III. LORENTZ INVARIANCE

The 2PPN Lagrangian given above describes a class of gravitational theories which includes preferred-frame theories. A preferred-frame theory may define *a priori* a universal rest frame so that the motion of any system with respect to that frame will produce measurable effects in the gravitational motion of bodies in the system. It would then be possible to determine the absolute motion of the system by performing local measurements on the system itself. Such theories are said to violate Lorentz invariance. If a theory is Lorentz invariant, it may still possess preferred frames which are determined by the motion or location of proximate matter which is not itself part of the system described by the Lagrangian and thus violate a generalized Lorentz invariance described by Nordtvedt [3], or the strong equivalence principle. It has been shown in a previous paper [7] that the number of free 2PPN parameters in the two-body interaction terms (the "linear-field" Lagrangian) can be significantly reduced by requiring the 2PPN Lagrangian to be Lorentz invariant. In that paper, Lorentz invariance was imposed on the Lagrangian through the use of gedanken experiments on special systems of bodies.

A more formal procedure using "post-Galilean" transformations on the 1PPN Lagrangian was described in [1]. In this procedure, one considers two observers who are located far from a system of bodies whose motion is to be described by the Lagrangian. One of the observers is at rest with respect to the center of mass of the system while the other is moving at a velocity $\mathbf{w}$ with respect to the system. If the gravitational interaction is Lorentz invariant, the trajectories calculated by one observer and then transformed by a Lorentz boost in the direction of $\mathbf{w}$ should be identical to the trajectories calculated by the other observer. In the correct gauge, this can be guaranteed if the Lagrangian itself is invariant under a Lorentz boost, modulo a total time derivative:

$$L(\mathbf{r},t)=L(\mathbf{r}',t')\frac{dt'}{dt}+\frac{d\chi}{dt} \tag{21}$$

where $\mathbf{r}'$ and t' are the Lorentz-transformed coordinates.

When this procedure is extended to the 2PPN Lagrangian, two complications arise. First, the correct gauge is as yet unknown and so a very general gauge must be used to allow the procedure itself to select the proper gauge. Second, the trajectories of the bodies can no longer be parametrized by a simple Newtonian-type time coordinate for both observers. If a universal time is used to parametrize all the trajectories in the primed coordinate system, then the initial conditions will all be given at a single initial time t_1'. However, in the unprimed coordinates, a different initial time will be needed for each body in the system. In addition, the parametrization of any individual trajectory will vary as the trajectory itself is varied to minimize the action. This difficulty can be accommodated by splitting the many-body Lagrangian into many one-body Lagrangians L_i. In each L_i, all terms in the many-body Lagrangian which contain a mass parameter of body i are retained while the rest are discarded. The time coordinate of body i (t_i) is then used to parametrize the trajectories of all the bodies in the system. When the "post-Galilean" transformations are applied to the coordinates, the time coordinate for each body is changed, and thus the parametrization in each one-body Lagrangian is different.

To second post-Galilean order, the Lorentz transformations are

$$\mathbf{r}'=\mathbf{r}+\mathbf{w}t+\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{r})\mathbf{w}+\tfrac{1}{2}w^2\mathbf{w}t+\tfrac{3}{8}w^2(\mathbf{w}\cdot\mathbf{r})\mathbf{w}+\tfrac{3}{8}w^4\mathbf{w}t\ , \tag{22}$$

$$t'=t+\mathbf{w}\cdot\mathbf{r}+\tfrac{1}{2}w^2t+\tfrac{1}{2}w^2(\mathbf{w}\cdot\mathbf{r})+\tfrac{3}{8}w^4t+\tfrac{3}{8}w^4(\mathbf{w}\cdot\mathbf{r})+\tfrac{5}{16}w^6t\ . \tag{23}$$

For notational convenience, the primed coordinate system is taken to be one in which t' is used as the parametrization for the trajectories of all bodies—thus, $t_i'=t_j'$. In this system, the equations of motion for body i found from L_i are identical to those found from the many-body Lagrangian, and the boundary conditions (which are given along hypersurfaces of constant t') are the same for all bodies. For a system of N bodies, there are N trajectories $\mathbf{r}_i'$, each parametrized by a separate time coordinate t_i'. These are transformed by (22) and (23) to yield

$$\mathbf{r}_i'(t_i')=\mathbf{r}_i(t_i)+\mathbf{w}t_i+\tfrac{1}{2}[\mathbf{w}\cdot\mathbf{r}_i(t_i)]\mathbf{w}+\tfrac{1}{2}w^2\mathbf{w}t_i + \tfrac{3}{8}w^2[\mathbf{w}\cdot\mathbf{r}_i(t_i)]\mathbf{w}+\tfrac{3}{8}w^4\mathbf{w}t_i \;, \tag{24}$$

$$t_i'=t_i+\mathbf{w}\cdot\mathbf{r}_i(t_i)+\tfrac{1}{2}w^2t_i+\tfrac{1}{2}w^2[\mathbf{w}\cdot\mathbf{r}_i(t_i)] + \tfrac{3}{8}w^4t_i+\tfrac{3}{8}w^4[\mathbf{w}\cdot\mathbf{r}_i(t_i)]+\tfrac{5}{16}w^6t_i \;. \tag{25}$$

Since $t_i'=t_j'$,

$$t_j=t_i+\mathbf{w}\cdot\mathbf{r}_i(t_i)-\mathbf{w}\cdot\mathbf{r}_j(t_j) \;. \tag{26}$$

The action for each body is then found by integrating along the path given by each body's trajectory:

$$I_i=\int_{t_{i1}}^{t_{i2}} L_i(\mathbf{r}_i(t_i),\mathbf{r}_j(t_i))dt_i \;. \tag{27}$$

Because (24) and (25) give $\mathbf{r}_j'$ and t_j' in terms of $\mathbf{r}_j(t_j)$ and t_j, they must be rewritten with a Taylor expansion of $\mathbf{r}_j(t_j)$ about t_i using (26):

$$\mathbf{r}_j(t_j)=\mathbf{r}_j+(\mathbf{w}\cdot\mathbf{r}_{ij})\mathbf{v}_j-(\mathbf{w}\cdot\mathbf{r}_{ij})(\mathbf{w}\cdot\mathbf{v}_j)\mathbf{v}_j + \tfrac{1}{2}(\mathbf{w}\cdot\mathbf{r}_{ij})^2\mathbf{a}_j \tag{28}$$

where $\mathbf{r}_{ij}=\mathbf{r}_i(t_i)-\mathbf{r}_j(t_i)$ and all time derivatives are taken with respect to t_i. Unless otherwise noted, all trajectories and their derivatives are assumed to be parametrized by t_i. The condition for Lorentz invariance (21), involves making the following substitutions in the Lagrangian:

$$\mathbf{r}_i'(t_i')=\mathbf{r}_i+\mathbf{w}t_i+\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{r}_i)\mathbf{w}+\tfrac{1}{2}w^2\mathbf{w}t_i+\tfrac{3}{8}w^2(\mathbf{w}\cdot\mathbf{r}_i)\mathbf{w}+\tfrac{3}{8}w^4\mathbf{w}t_i \;, \tag{29}$$

$$\mathbf{r}_j'(t_j')=\mathbf{r}_j+\mathbf{w}t_i+(\mathbf{w}\cdot\mathbf{r}_{ij})\mathbf{v}_j+(\mathbf{w}\cdot\mathbf{r}_{ij})\mathbf{w}+\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{r}_j)\mathbf{w}+\tfrac{1}{2}w^2\mathbf{w}t_i-(\mathbf{w}\cdot\mathbf{r}_{ij})(\mathbf{w}\cdot\mathbf{v}_j)\mathbf{v}_j-\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{r}_{ij})(\mathbf{w}\cdot\mathbf{v}_j)\mathbf{w}+\tfrac{1}{2}w^2(\mathbf{w}\cdot\mathbf{r}_{ij})\mathbf{w} + \tfrac{3}{8}w^2(\mathbf{w}\cdot\mathbf{r}_j)\mathbf{w}+\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{r}_{ij})^2\mathbf{a}_j+\tfrac{3}{8}w^4\mathbf{w}t_i \;, \tag{30}$$

$$\mathbf{v}_i'(t_i')=\mathbf{v}_i+\mathbf{w}-(\mathbf{w}\cdot\mathbf{v}_i)\mathbf{v}_i-\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{v}_i)\mathbf{w}-\tfrac{1}{2}w^2\mathbf{v}_i \;, \tag{31}$$

$$\mathbf{v}_j'(t_j')=\mathbf{v}_j+\mathbf{w}-(\mathbf{w}\cdot\mathbf{v}_j)\mathbf{v}_j-\tfrac{1}{2}(\mathbf{w}\cdot\mathbf{v}_j)\mathbf{w}-\tfrac{1}{2}w^2\mathbf{v}_j+(\mathbf{w}\cdot\mathbf{r}_{ij})\mathbf{a}_j \;, \tag{32}$$

$$\frac{dt_i'}{dt_i}=1+\mathbf{w}\cdot\mathbf{v}_i+\tfrac{1}{2}w^2+\tfrac{1}{2}w^2(\mathbf{w}\cdot\mathbf{v}_i)+\tfrac{3}{8}w^4+\tfrac{3}{8}w^4(\mathbf{w}\cdot\mathbf{v}_i)+\tfrac{5}{16}w^6 \;. \tag{33}$$

Since no time derivative of the trajectory higher than the acceleration appears in the transformations above, none need be present in the Lagrangian when it is written in a Lorentz-invariant gauge. Thus a gauge choice is made to yield an acceleration-dependent 2PPN Lagrangian for body i:

$$L_i=-M_i(1-\tfrac{1}{2}v_i^2-\tfrac{1}{8}v_i^4-\tfrac{1}{16}v_i^6)+L_1+L_2+L_3+L_4 \tag{34}$$

with

$$\begin{aligned} L_1=\sum_{j\neq i}\frac{1}{r_{ij}}\{&\Gamma_{ij}+[(1+\gamma)M_iM_j-\tfrac{1}{2}\Gamma_{ij}](v_i^2+v_j^2)-2[(1+\gamma)M_iM_j-\tfrac{1}{4}\Gamma_{ij}]\mathbf{v}_i\cdot\mathbf{v}_j-\tfrac{1}{2}\Gamma_{ij}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})\\ &+a_{ij}^1(v_i^4+v_j^4)+a_{ij}^2(v_i^2+v_j^2)\mathbf{v}_i\cdot\mathbf{v}_j+a_{ij}^3(\mathbf{v}_i\cdot\mathbf{v}_j)^2+a_{ij}^4v_i^2v_j^2\\ &+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})[b_{ij}^1(v_i^2+v_j^2)+b_{ij}^2\mathbf{v}_i\cdot\mathbf{v}_j]+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(b_{ij}^3v_i^2+b_{ij}^4v_j^2+b_{ij}^5\mathbf{v}_i\cdot\mathbf{v}_j)\\ &+(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2(b_{ij}^4v_i^2+b_{ij}^3v_j^2+b_{ij}^5\mathbf{v}_i\cdot\mathbf{v}_j)+c_{ij}^1[(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^4+(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^4]+c_{ij}^2(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2\\ &+c_{ij}^3[(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^3(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^3]+\alpha_{ij}^1[(\mathbf{a}_i\cdot\mathbf{r}_{ij})v_i^2-(\mathbf{a}_j\cdot\mathbf{r}_{ij})v_j^2]+\alpha_{ij}^2[(\mathbf{a}_i\cdot\mathbf{r}_{ij})v_j^2-(\mathbf{a}_j\cdot\mathbf{r}_{ij})v_i^2]\\ &+\alpha_{ij}^3(\mathbf{a}_i\cdot\mathbf{r}_{ij}-\mathbf{a}_j\cdot\mathbf{r}_{ij})\mathbf{v}_i\cdot\mathbf{v}_j+\alpha_{ij}^4[(\mathbf{a}_i\cdot\mathbf{v}_i)(\mathbf{v}_i\cdot\mathbf{r}_{ij})-(\mathbf{a}_j\cdot\mathbf{v}_j)(\mathbf{v}_j\cdot\mathbf{r}_{ij})]+\alpha_{ij}^5[(\mathbf{a}_i\cdot\mathbf{v}_i)(\mathbf{v}_j\cdot\mathbf{r}_{ij})-(\mathbf{a}_j\cdot\mathbf{v}_j)(\mathbf{v}_i\cdot\mathbf{r}_{ij})]\\ &+\alpha_{ij}^6[(\mathbf{a}_i\cdot\mathbf{v}_j)(\mathbf{v}_j\cdot\mathbf{r}_{ij})-(\mathbf{a}_j\cdot\mathbf{v}_i)(\mathbf{v}_i\cdot\mathbf{r}_{ij})]+\alpha_{ij}^7[(\mathbf{a}_i\cdot\mathbf{v}_j)(\mathbf{v}_i\cdot\mathbf{r}_{ij})-(\mathbf{a}_j\cdot\mathbf{v}_i)(\mathbf{v}_j\cdot\mathbf{r}_{ij})]\\ &+\alpha_{ij}^8[(\mathbf{a}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2-(\mathbf{a}_j\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2]+\alpha_{ij}^9[(\mathbf{a}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2-(\mathbf{a}_j\cdot\mathbf{r}_{ij})(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2]\\ &+\alpha_{ij}^{10}(\mathbf{a}_i\cdot\mathbf{r}_{ij}-\mathbf{a}_j\cdot\mathbf{r}_{ij})(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})\}\;, \end{aligned} \tag{35}$$

$$L_2=\sum_{j\neq i}\frac{1}{r_{ij}^2}[(\tfrac{1}{2}-\beta)(\Gamma_{ijj}+\Gamma_{jii})+\zeta_{ij}^1v_i^2+\zeta_{ji}^1v_j^2+\zeta_{ij}^2\mathbf{v}_i\cdot\mathbf{v}_j+\zeta_{ij}^3(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2+\zeta_{ji}^3(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2+\zeta_{ij}^4(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})] \tag{36}$$

$$L_3=\sum_{j\neq k\neq i}\frac{1}{r_{ij}r_{ik}}\left[\left(\frac{1}{2}-\beta\right)\left(\Gamma_{ijk}+2\frac{r_{ik}}{r_{jk}}\Gamma_{jik}\right)+a^1_{ijk}v_i^2+2\frac{r_{ik}}{r_{jk}}a^1_{jik}v_j^2+2\left(a^2_{ijk}+\frac{r_{ik}}{r_{jk}}a^2_{jik}\right)\mathbf{v}_i\cdot\mathbf{v}_j+\left(\frac{r_{ik}}{r_{jk}}a^2_{jki}+\frac{r_{ij}}{r_{jk}}a^2_{kji}\right)\mathbf{v}_j\cdot\mathbf{v}_k\right.$$
$$\left.+\mathbf{v}_i\cdot\mathbb{A}_{ijk}\cdot\mathbf{v}_i+2\mathbf{v}_j\cdot\left(\frac{r_{ik}}{r_{jk}}\mathbb{A}_{jik}\right)\cdot\mathbf{v}_j+2\mathbf{v}_i\cdot\left(\mathbb{B}_{ijk}+\frac{r_{ik}}{r_{jk}}\mathbb{B}^T_{jik}\right)\cdot\mathbf{v}_j+\mathbf{v}_j\cdot\left(\frac{r_{ik}}{r_{jk}}\mathbb{B}_{jki}+\frac{r_{ij}}{r_{jk}}\mathbb{B}^T_{kji}\right)\cdot\mathbf{v}_k\right]\;; \quad (37)$$

and

$$L_4=2\sum_{j\neq i}\frac{1}{r_{ij}^3}(\Psi_{ij}+\Theta_{ij})+3\sum_{j\neq k\neq i}\frac{\Omega_{ijk}}{r_{ij}r_{ik}r_{jk}}+\sum_{j\neq k\neq l\neq i}\frac{\Pi_{ijkl}}{r_{ij}r_{ik}r_{il}}+3\sum_{j\neq k\neq l\neq i}\frac{\Pi_{jikl}}{r_{ij}r_{jk}r_{jl}}\;, \quad (38)$$

where the summations are over all bodies except body i. The Lorentz invariance condition (21) can be imposed on the pieces of the Lagrangian (35)–(37) separately. The last piece of the Lagrangian (38) cannot contribute to any violations of Lorentz invariance at the 2PN order, so it is not included in the imposition of (21). There is a corresponding χ for each piece:

$$\chi_1=\sum_{j\neq i}\frac{1}{r_{ij}}[(\mathbf{w}\cdot\mathbf{r}_{ij})(\chi^1_{ij}v_i^2+\chi^2_{ij}v_j^2+\chi^3_{ij}\mathbf{v}_i\cdot\mathbf{v}_j+\chi^4_{ij}\mathbf{w}\cdot\mathbf{v}_i+\chi^5_{ij}\mathbf{w}\cdot\mathbf{v}_j+\chi^6_{ij}w^2)$$
$$+(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\chi^7_{ij}\mathbf{w}\cdot\mathbf{v}_i+\chi^8_{ij}\mathbf{w}\cdot\mathbf{v}_j+\chi^9_{ij}w^2)+(\mathbf{v}_j\cdot\mathbf{r}_{ij})(\chi^{10}_{ij}\mathbf{w}\cdot\mathbf{v}_i+\chi^{11}_{ij}\mathbf{w}\cdot\mathbf{v}_j+\chi^{12}_{ij}w^2)]\;, \quad (39)$$

$$\chi_2=\sum_{j\neq i}\frac{\chi^{13}_{ij}}{r_{ij}^2}(\mathbf{w}\cdot\mathbf{r}_{ij})\;, \quad (40)$$

and

$$\chi_3=\sum_{j\neq k\neq i}\frac{\chi^{14}_{ijk}}{r_{ij}r_{ik}}(\mathbf{w}\cdot\mathbf{r}_{ij})\;. \quad (41)$$

When condition (21) is imposed on L_1, all but six of the 2PPN parameters in L_1 can be expressed in terms of the 1PPN parameters Γ_{ij} and γ. The remaining free parameters appear in the Lorentz-invariant Lagrangian as

$$\sum_{j\neq i}\frac{1}{r_{ij}}[a^1_{ij}v^4_{ij}+b^1_{ij}v^2_{ij}(\mathbf{v}_{ij}\cdot\hat{\mathbf{r}}_{ij})^2+c^1_{ij}(\mathbf{v}_{ij}\cdot\hat{\mathbf{r}}_{ij})^4+\alpha^1_{ij}v^2_{ij}(\mathbf{a}_{ij}\cdot\mathbf{r}_{ij})+\alpha^4_{ij}(\mathbf{a}_{ij}\cdot\mathbf{v}_{ij})(\mathbf{v}_{ij}\cdot\mathbf{r}_{ij})+\alpha^8_{ij}(\mathbf{a}_{ij}\cdot r_{ij})(\mathbf{v}_{ij}\cdot\hat{\mathbf{r}}_{ij})^2] \quad (42)$$

and do not contribute to any universal preferred frame effects. Five of these parameters can be eliminated by the gauge transformations (19) and (20). In a gauge which eliminates all but a^1_{ij}, the following constraints are imposed on the other 2PPN parameters:

$$a^2_{ij}=-4a^1_{ij}+(1+\gamma)M_iM_j-\tfrac{1}{2}\Gamma_{ij}\;, \quad (43a)$$

$$a^3_{ij}=4a^1_{ij}-2(1+\gamma)M_iM_j+\tfrac{3}{8}\Gamma_{ij}\;, \quad (43b)$$

$$a^4_{ij}=2a^1_{ij}+\tfrac{1}{8}\Gamma_{ij}\;, \quad (43c)$$

$$b^2_{ij}=(1+\gamma)M_iM_j-\tfrac{1}{2}\Gamma_{ij}\;, \quad (43d)$$

$$b^4_{ij}=-\tfrac{1}{2}(1+\gamma)M_iM_j+\tfrac{1}{8}\Gamma_{ij}\;, \quad (43e)$$

$$c^2_{ij}=\tfrac{3}{8}\Gamma_{ij}\;, \quad (43f)$$

$$\alpha^2_{ij}=\tfrac{1}{2}(1+\gamma)M_iM_j-\tfrac{1}{8}\Gamma_{ij}\;, \quad (43g)$$

$$\alpha^6_{ij}=-(1+\gamma)M_iM_j+\tfrac{1}{4}\Gamma_{ij}\;, \quad (43h)$$

$$\alpha^9_{ij}=-\tfrac{1}{8}\Gamma_{ij}\;, \quad (43i)$$

with all the remaining parameters except a^1_{ij} equal to zero.

Although the imposition of Lorentz invariance on L_1 does not affect the terms in L_2 and L_3, the selection of a Lorentz-invariant gauge to eliminate the unwanted parameters in (42) will contribute terms to those pieces. However, it can be assumed that the gauge choice has been made previous to the imposition of (21) and these new terms are absorbed into the parameters in L_2 and L_3. The constraints on the 2PPN parameters in L_2 are

$$\xi^2_{ij}=-\tfrac{1}{2}(\tfrac{1}{2}-\beta)(\Gamma_{ijj}+\Gamma_{jii})-\xi^1_{ij}-\xi^1_{ji}\;, \quad (44a)$$

$$\xi^4_{ij}=-(\tfrac{1}{2}-\beta)(\Gamma_{ijj}+\Gamma_{jii})-\xi^3_{ij}-\xi^3_{ji}\;, \quad (44b)$$

$$\xi^3_{ij}-\xi^3_{ji}=2\xi^1_{ji}-2\xi^1_{ij}\;. \quad (44c)$$

The undetermined parameters appear in the Lorentz invariant Lagrangian as

$$\sum_{j\neq i}\frac{1}{r_{ij}^2}\{(\xi^1_{ij}\mathbf{v}_i-\xi^1_{ji}\mathbf{v}_j)\cdot\mathbf{v}_{ij}$$
$$+[\xi^3_{ij}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})-\xi^3_{ji}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})](\mathbf{v}_{ij}\cdot\hat{\mathbf{r}}_{ij})\}\;. \quad (45)$$

A particular gauge can be chosen so that (45) becomes

$$\sum_{j\neq i}\frac{\xi_{ij}}{r_{ij}^2}v^2_{ij} \quad (46)$$

where

$$\xi_{ij}=\xi^1_{ij}+\tfrac{1}{2}\xi^3_{ij}\;. \quad (47)$$

By virtue of (44c), ξ_{ij} is symmetric under interchange of its indices.

Requiring L_3 to be Lorentz invariant yields the two constraints

$$a^1_{ijk} = -\frac{1}{2}\left[a^2_{ijk} + a^2_{ikj} + (\tfrac{1}{2}-\beta)\Gamma_{ijk} + \frac{r_{ik}}{r_{jk}}a^2_{jik} + \frac{r_{ij}}{r_{jk}}a^2_{kij}\right], \quad (48a)$$

$$\mathbb{A}_{ijk} = -\frac{1}{2}\left[\mathbb{B}^T_{ijk} + \mathbb{B}^T_{ikj} + \frac{r_{ik}}{r_{jk}}[\mathbb{B}_{jik} + (\tfrac{1}{2}-\beta)\Gamma_{jik}\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{ij}] + \frac{r_{ij}}{r_{jk}}[\mathbb{B}_{kij} + (\tfrac{1}{2}-\beta)\Gamma_{kij}\hat{\mathbf{r}}_{ik}\hat{\mathbf{r}}_{ik}]\right]. \quad (48b)$$

Since $\mathbb{A}_{ijk}$ is a symmetric matrix, there is an additional constraint on $\mathbb{B}_{ijk}$ given by

$$\mathbb{B}_{ijk} - \mathbb{B}^T_{ijk} + \mathbb{B}_{ikj} - \mathbb{B}^T_{ikj} = \frac{r_{ik}}{r_{jk}}(\mathbb{B}_{jik} - \mathbb{B}^T_{jik}) + \frac{r_{ij}}{r_{jk}}(\mathbb{B}_{kij} - \mathbb{B}^T_{kij}). \quad (48c)$$

The undetermined terms appear as

$$-\sum_{j\neq k\neq i}\frac{1}{r_{ij}r_{ik}}(a_{ijk}\mathbf{v}_i\cdot\mathbf{v}_{ij} + \mathbf{v}_i\cdot\mathbb{C}_{ijk}\cdot\mathbf{v}_{ij}) \quad (49)$$

where

$$a_{ijk} = a^2_{ijk} + \frac{r_{ik}}{r_{jk}}a^2_{jik}, \quad (50a)$$

$$\mathbb{C}_{ijk} = \mathbb{B}_{ijk} + \frac{r_{ik}}{r_{jk}}\mathbb{B}^T_{jik}. \quad (50b)$$

One of the parameters can be eliminated by choice of gauge leaving

$$-\sum_{j\neq k\neq i}\frac{1}{r_{ij}r_{ik}}\mathbf{v}_i\cdot\mathbb{C}_{ijk}\cdot\mathbf{v}_{ij}. \quad (51)$$

The remaining parameter $\mathbb{C}_{ijk}$ is not entirely free, since it must still satisfy (48c). If it is expanded out into its subparameters as in (13), it can be seen that (48.c) requires

$$C^3_{ijk} - C^3_{ikj} = C^4_{ijk} - C^4_{ikj}. \quad (52)$$

After the imposition of Lorentz invariance, the resulting 2PPN Lagrangian is completely described by the ten parameters a^1_{ij}, ξ_{ij}, Ψ_{ij}, Θ_{ij}, C^1_{ijk}, C^2_{ijk}, C^3_{ijk}, C^4_{ijk}, Ω_{ijk}, and Π_{ijkl} where the last six of these parameters can be general functions of the interbody separation vectors. The corresponding 2PPN many-body Lagrangian is

$$\begin{aligned}
L = &-\sum_i M_i(1 - \tfrac{1}{2}v_i^2 - \tfrac{1}{8}v_i^4 - \tfrac{1}{16}v_i^6) + \tfrac{1}{2}\sum_{i\neq j}\frac{\Gamma_{ij}}{r_{ij}}[1 - v_i^2 + \tfrac{1}{2}\mathbf{v}_i\cdot\mathbf{v}_j - \tfrac{1}{2}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij}) - v_i^2\mathbf{v}_i\cdot\mathbf{v}_j + \tfrac{3}{4}(\mathbf{v}_i\cdot\mathbf{v}_j)^2 \\
&\quad + \tfrac{1}{8}v_i^2v_j^2 - \tfrac{1}{2}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})\mathbf{v}_i\cdot\mathbf{v}_j + \tfrac{1}{4}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2v_j^2 \\
&\quad + \tfrac{3}{8}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2 - \tfrac{1}{4}(\mathbf{a}_i\cdot\mathbf{r}_{ij})v_j^2 - \tfrac{1}{2}(\mathbf{a}_i\cdot\mathbf{v}_j)(\mathbf{v}_j\cdot\mathbf{r}_{ij}) - \tfrac{1}{4}(\mathbf{a}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2] \\
&+ (1+\gamma)\sum_{i\neq j}\frac{M_iM_j}{r_{ij}}[v_i^2 - \mathbf{v}_i\cdot\mathbf{v}_j + v_i^2\mathbf{v}_i\cdot\mathbf{v}_j - 2(\mathbf{v}_i\cdot\mathbf{v}_j)^2 - \tfrac{1}{2}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2v_j^2 \\
&\quad + \tfrac{1}{2}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})\mathbf{v}_i\cdot\mathbf{v}_j + \tfrac{1}{2}(\mathbf{a}_i\cdot\mathbf{r}_{ij})v_j^2 - (\mathbf{a}_i\cdot\mathbf{v}_j)(\mathbf{v}_j\cdot\mathbf{r}_{ij})] \\
&+ (\tfrac{1}{2}-\beta)\sum_{i\neq j}\frac{\Gamma_{ijj}}{r_{ij}^2}[1 - \tfrac{1}{2}\mathbf{v}_i\cdot\mathbf{v}_j - (\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})] + (\tfrac{1}{2}-\beta)\sum_{i\neq j\neq k}\frac{\Gamma_{ijk}}{r_{ij}r_{ik}}[1 - \tfrac{1}{2}v_i^2 - (\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2] \\
&+ \tfrac{1}{2}\sum_{i\neq j}\frac{a^1_{ij}}{r_{ij}}v_{ij}^4 + \tfrac{1}{2}\sum_{i\neq j}\frac{\xi_{ij}}{r_{ij}^2}v_{ij}^2 - \sum_{i\neq j\neq k}\frac{1}{r_{ij}r_{ik}}\mathbf{v}_i\cdot\mathbb{C}_{ijk}\cdot\mathbf{v}_{ij} + \sum_{i\neq j}\frac{\Psi_{ij}+\Theta_{ij}}{r_{ij}^3} + \sum_{i\neq j\neq k}\frac{\Omega_{ijk}}{r_{ij}r_{ik}r_{jk}} + \sum_{i\neq j\neq k\neq l}\frac{\Pi_{ijkl}}{r_{ij}r_{ik}r_{il}}.
\end{aligned} \quad (53)$$

IV. THE 2PPN METRIC

The Lagrangian given in (53) is written in a special gauge which allows its Lorentz invariance to be easily seen. In this gauge, however, it is not obvious that test bodies will move along geodesics. To find the 2PPN metric, it is necessary to use a different "metric" gauge which will allow the Lagrangian for a test body to be written as a line element. If the metric is written as an expansion about the Minkowski metric, $\eta_{\mu\nu}$, with signature $(+---)$, so that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (54)$$

then, in the metric gauge [3],

$$\lim_{i\to \mathrm{tb}}\frac{L_i}{M_i} = -\left[g_{\mu\nu}\frac{dx_i^\mu}{dt_i}\frac{dx_i^\nu}{dt_i}\right]^{1/2} = -\sqrt{1 - v_i^2 + h_{00} + 2\mathbf{h}\cdot\mathbf{v}_i + \mathbf{v}_i\cdot\mathbb{h}\cdot\mathbf{v}_i} \quad (55)$$

where h_{oa} and h_{ab} are the components of $\mathbf{h}$ and $\mathbb{h}$, respectively.

The metric gauge is found by applying the general gauge transformation (19) and (20) to the Lorentz invariant Lagrangian (53) and then requiring that the gauge parameters be such that (55) is satisfied. As in the imposition of Lorentz invariance, this procedure can be applied separately to the parts of the Lagrangian (35)–(38).

The gauge transformation which takes L_1 from the Lorentz invariant gauge to the metric gauge is

$$\delta\mathbf{r}_i=\frac{1}{M_i}\sum_{j\neq i}[\tfrac{1}{2}(1+\gamma)M_iM_j-\tfrac{3}{8}\Gamma_{ij}]v_j^2\mathbf{r}_{ij}-\tfrac{1}{8}\Gamma_{ij}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2\mathbf{r}_{ij}+\tfrac{1}{2}\Gamma_{ij}(\mathbf{v}_j\cdot\mathbf{r}_{ij})\mathbf{v}_i-[(1+\gamma)M_iM_j-\tfrac{1}{4}\Gamma_{ij}](\mathbf{v}_j\cdot\mathbf{r}_{ij})\mathbf{v}_j\} \tag{56}$$

$$Q=\tfrac{1}{2}\sum_{i\neq j}\frac{1}{r_{ij}}\{-\tfrac{1}{2}[(1+\gamma)M_iM_j-\tfrac{5}{4}\Gamma_{ij}][(\mathbf{v}_i\cdot\mathbf{r}_{ij})v_j^2-(\mathbf{v}_j\cdot\mathbf{r}_{ij})v_i^2]$$
$$-[(1+\gamma)M_iM_j-\tfrac{1}{4}\Gamma_{ij}](\mathbf{v}_{ij}\cdot\mathbf{r}_{ij})\mathbf{v}_i\cdot\mathbf{v}_j-\tfrac{1}{8}\Gamma_{ij}[(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(\mathbf{v}_j\cdot\mathbf{r}_{ij})-(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2]\}\ . \tag{57}$$

This transformation is not sufficient to guarantee that (55) is satisfied, and so an additional constraint is placed upon the remaining 2PPN parameter in L_1:

$$a_{ij}^1=\tfrac{1}{2}(1+\gamma)M_iM_j-\tfrac{1}{8}\Gamma_{ij}\ . \tag{58}$$

Thus, there are no 2PPN parameters left in L_1 after the imposition of Lorentz invariance and the requirement that a gauge exist so that (55) is valid.

The transformation to the metric gauge (56) and (57) introduces additional terms to the remaining pieces of the Lagrangian. The contribution to L_2 is

$$\delta L_2=-\tfrac{1}{2}\sum_{i\neq j}\frac{\Gamma_{ij}}{r_{ij}^2}\left[\left[(1+\gamma)M_i-\frac{3}{4}\frac{\Gamma_{ij}}{M_j}\right]v_i^2-2\left[(1+\gamma)M_i-\frac{1}{8}\frac{\Gamma_{ij}}{M_j}\right](v_i\cdot\hat{\mathbf{r}}_{ij})^2+\frac{\Gamma_{ij}}{M_j}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})\right] \tag{59}$$

which can easily be put into the form of the line element, so it is unnecessary to apply any further gauge transformation to L_2. Although (56) and (57) do not introduce any new terms to L_3 which cannot be put into the form of the line element, the metric which results does not approach $\eta_{\mu\nu}$ at points far removed from the system of bodies. This problem can be resolved by introducing an additional gauge transformation at the level of L_3:

$$\delta\mathbf{r}_i=\tfrac{1}{4}\sum_{j\neq k\neq i}\frac{1}{r_{ij}r_{jk}}\frac{\Gamma_{ij}\Gamma_{jk}}{M_iM_j}\mathbf{r}_{ij}\ . \tag{60}$$

The transformation to the metric gauge produces the following addition to L_3:

$$\delta L_3=\sum_{i\neq j\neq k}\frac{\Gamma_{jk}}{r_{jk}^2}\left\{\frac{1}{2}\left[(1+\gamma)M_i-\frac{3}{4}\frac{\Gamma_{ij}}{M_j}\right](\hat{\mathbf{r}}_{ij}\cdot\hat{\mathbf{r}}_{jk})v_i^2-\mathbf{v}_i\cdot\left[\frac{1}{8}\frac{\Gamma_{ij}}{M_j}(\hat{\mathbf{r}}_{ij}\cdot\hat{\mathbf{r}}_{jk})\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{ij}+\left[(1+\gamma)M_i-\frac{1}{4}\frac{\Gamma_{ij}}{M_j}\right]\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{jk}\right]\cdot\mathbf{v}_i\right.$$
$$\left.+\frac{1}{4}\mathbf{v}_i\cdot\left[\frac{\Gamma_{ij}}{M_j}\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{jk}-\frac{\Gamma_{ik}}{M_k}\hat{\mathbf{r}}_{ik}\hat{\mathbf{r}}_{jk}\right]\mathbf{v}_j\right\}$$
$$+\frac{1}{4}\sum_{i\neq j\neq k}\frac{1}{r_{ij}r_{ik}}\frac{\Gamma_{ij}\Gamma_{ik}}{M_i}[v_j^2-\mathbf{v}_i\cdot\mathbf{v}_j-(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})]\ , \tag{61}$$

and alters L_4 by

$$\delta L_4=-\frac{1}{4}\sum_{i\neq j\neq k}\frac{1}{r_{ij}r_{ik}r_{jk}}\left[\frac{\Gamma_{ij}^2\Gamma_{ik}}{M_iM_j}\frac{r_{jk}}{r_{ij}}+\frac{\Gamma_{ij}\Gamma_{ik}\Gamma_{jk}}{M_iM_k}\frac{r_{ik}}{r_{ij}}\hat{\mathbf{r}}_{ij}\cdot\hat{\mathbf{r}}_{ik}\right]+\frac{1}{4}\sum_{i\neq j\neq k\neq 1}\frac{1}{r_{ij}r_{jk}r_{kl}}\frac{\Gamma_{ij}\Gamma_{jk}\Gamma_{kl}}{M_jM_k}\frac{r_{jk}}{r_{ij}}\hat{\mathbf{r}}_{ij}\cdot\hat{\mathbf{r}}_{jk}\ . \tag{62}$$

The Lagrangian can be used to find the metric at any point $\wp$ by placing a "virtual" test body at $\wp$. When written in the metric gauge, the resulting Lagrangian for the test body will be of the form

$$\lim_{i\to\text{tb}}\frac{L_i}{M_i}=-\sqrt{1-v_i^2}+U_i \tag{63}$$

so that

$$1-v_i^2-2U_i\sqrt{1-v_i^2}+(U_i)^2=g_{00}+2\mathbf{g}\cdot\mathbf{v}_i+\mathbf{v}_i\cdot\mathbf{g}\cdot\mathbf{v}_i\ . \tag{64}$$

By matching coefficients, the components of the metric can easily be determined. Since body i is a virtual test body, its mass does not enter into U_i and its location is identified with the coordinates of the point $\wp$. Thus, the subscript is dropped from $\mathbf{r}_i$ in favor of the generic vector $\mathbf{r}$. In the description of the metric, $\mathbf{r}_j$ is the vector joining $\wp$ and the location of body j. The functional dependence of $\mathbb{C}_{ijk}$ is given by

$$\mathbb{C}_{-jk}=\mathbb{C}(\mathbf{r}_j,\mathbf{r}_k,\mathbf{r}_{jk}) , \tag{65a}$$

$$\mathbb{C}_{j\_k}=\mathbb{C}(-\mathbf{r}_j,\mathbf{r}_{jk},\mathbf{r}_k) , \tag{65b}$$

$$\mathbb{C}_{jk-}=\mathbb{C}(\mathbf{r}_{jk},-r_j,-r_k) , \tag{65c}$$

and the parameters Ω_{ijk} and Π_{ijkl} are similarly defined.

The metric is found to be

$$\begin{aligned}
g_{00}=1&-2\sum_j \frac{1}{r_j}[M_j(G)(1-\tfrac{1}{2}v_j^2-\tfrac{1}{8}v_j^4)+(1+\gamma)M_j(v_j^2+\tfrac{1}{2}v_j^4)]\\
&-2\sum_j \frac{1}{r_j^2}\{(\tfrac{1}{2}-\beta)\lambda_{jj}+\lambda_j(\xi)v_j^2-\tfrac{1}{2}M_j(G)^2[1-\tfrac{7}{4}v_j^2+\tfrac{1}{4}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2]-(1+\gamma)M_jM_j(G)(\tfrac{3}{2}v_j^2-(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2)\}\\
&-2\sum_{j\neq k}\frac{1}{r_jr_k}\{(\tfrac{1}{2}-\beta)\lambda_{jk}[1-(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2]-\tfrac{1}{2}M_j(G)M_k(G)[1-v_j^2+\tfrac{1}{2}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2]-(1+\gamma)M_jM_k(G)v_j^2\}\\
&-2\sum_{j\neq k}\frac{1}{r_jr_{jk}}\{(1-2\beta)\lambda'_{jk}[1-\tfrac{1}{2}v_j^2-\tfrac{1}{2}(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})^2]-\mathbf{v}_j\cdot(\mathbb{C}_{j_k}+\mathbb{C}_{jk_})\cdot\mathbf{v}_j+\mathbf{v}_j\cdot\mathbb{C}_{jk_}\cdot\mathbf{v}_k\\
&\qquad-\tfrac{1}{4}\Gamma_{jk}\frac{M_j(G)}{M_j}[\mathbf{v}_j\cdot\mathbf{v}_k+(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})^2-(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{jk})(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})]\}\\
&+2\sum_{j\neq k}\frac{1}{r_j^2}\left[\Gamma_{jk}\frac{M_j(G)}{M_j}\{(\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_{jk})[\tfrac{3}{8}v_k^2+\tfrac{1}{8}(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})^2]-\tfrac{1}{4}[(\mathbf{v}_k\cdot\hat{\mathbf{r}}_j)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})+(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})]\}\right.\\
&\qquad-\tfrac{1}{8}M_j(G)M_k(G)[3(\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_k)v_k^2+(\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_k)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_k)^2-2(\mathbf{v}_k\cdot\hat{\mathbf{r}}_j)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_k)-2(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_k)]\\
&\qquad\left.-(1+\gamma)M_j(G)M_k\{\tfrac{1}{2}[(\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_{jk})+(\hat{\mathbf{r}}_k\cdot\hat{\mathbf{r}}_{jk})]v_k^2-(\mathbf{v}_k\cdot\hat{\mathbf{r}}_j)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_{jk})+(\mathbf{v}_k\cdot\hat{\mathbf{r}}_j)(\mathbf{v}_k\cdot\hat{\mathbf{r}}_k)\}\right]\\
&-\sum_j\frac{1}{r_j^3}[4\psi_j-(1-2\beta)M_j(G)\lambda_{jj}]\\
&-\sum_{j\neq k}\frac{1}{r_jr_kr_{jk}}\left[6\Omega_{-jk}-(1-2\beta)M_j(G)\left[\lambda_{jk}+2\frac{r_k}{r_{jk}}\lambda'_{jk}+\frac{r_{jk}}{r_k}\lambda_{kk}\right]-\tfrac{1}{2}\Gamma_{jk}\frac{M_j(G)^2}{M_j}\frac{r_k}{r_j}+\frac{\Gamma_{jk}^2}{M_k}\frac{M_j(G)}{M_j}\frac{r_k}{r_{jk}}\right.\\
&\qquad\left.+\frac{\Gamma_{jk}}{M_j}M_j(G)M_k(G)(\frac{r_j}{r_k}\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_k-\frac{r_j}{r_{jk}}\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_{jk})\right]\\
&-\sum_{j\neq k\neq l}\frac{1}{r_jr_kr_l}\left[2\Pi_{-jkl}+6\frac{r_kr_l}{r_{jk}r_{jl}}\Pi_{j_kl}-(1-2\beta)M_j(G)\left[\lambda_{kl}+2\frac{r_l}{r_{kl}}\lambda'_{kl}\right]\right.\\
&\qquad\left.+\frac{r_kr_l}{r_jr_{kl}}\left[\frac{\Gamma_{jk}\Gamma_{kl}}{M_k}\frac{M_j(G)}{M_j}\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_{jk}-M_j(G)\Gamma_{kl}\frac{M_k}{M_k}\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_k\right]\right] ,
\end{aligned}\tag{66}$$

$$\begin{aligned}
\mathbf{g}=&\sum_j\frac{1}{r_j}\{\tfrac{1}{2}M_j(G)[(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)\hat{\mathbf{r}}_j-\mathbf{v}_j]+2(1+\gamma)M_j\mathbf{v}_j\}(1+\tfrac{1}{2}v_j^2)\\
&+\sum_j\frac{1}{r_{ij}^2}\{\tfrac{1}{2}(\tfrac{1}{2}-\beta)\lambda_{jj}[(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)\hat{\mathbf{r}}_j+\mathbf{v}_j]+2\lambda_j(\xi)\mathbf{v}_j+\tfrac{1}{2}M_j(G)^2\mathbf{v}_j-2(1+\gamma)M_jM_j(G)\mathbf{v}_j\}\\
&-\sum_{j\neq k}\frac{1}{r_jr_k}\{\mathbf{v}_j\cdot\mathbb{C}^T_{-jk}-\tfrac{1}{4}M_j(G)M_k(G)[\mathbf{v}_j-(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)\hat{\mathbf{r}}_j]+(1+\gamma)M_jM_k(G)\mathbf{v}_j\}\\
&-\sum_{j\neq k}\frac{1}{r_jr_{jk}}\mathbf{v}_j\cdot\mathbb{C}_{j_k}+\frac{1}{4}\sum_{j\neq k}\frac{M_j(G)M_k(G)}{r_j^2}(\mathbf{v}_k\cdot\hat{\mathbf{r}}_k)\hat{\mathbf{r}}_j-\frac{1}{4}\sum_{j\neq k}\frac{\Gamma_{jk}}{r_{jk}^2}\frac{M_j(G)}{M_j}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{jk})\hat{\mathbf{r}}_j ,
\end{aligned}\tag{67}$$

$$g=-\mathbb{1}+2\sum_j \frac{1}{r_j}(M_j(G)\{[(1+\tfrac{3}{8}v_j^2-\tfrac{3}{8}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2]\mathbb{1}-\tfrac{1}{4}\mathbf{v}_j\mathbf{v}_j+\tfrac{1}{4}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)(\hat{\mathbf{r}}_j\mathbf{v}_j+\mathbf{v}_j\hat{\mathbf{r}}_j)-\tfrac{3}{8}[v_j^2+(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2]\hat{\mathbf{r}}_j\hat{\mathbf{r}}_j\}$$

$$-(1+\gamma)M_j\{[1+\tfrac{1}{2}v_j^2-\tfrac{1}{2}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)^2]\mathbb{1}+\tfrac{1}{2}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_j)(\hat{\mathbf{r}}_j\mathbf{v}_j+\mathbf{v}_j\hat{\mathbf{r}}_j)-\tfrac{1}{2}v_j^2\hat{\mathbf{r}}_j\hat{\mathbf{r}}_j\})$$

$$+2\sum_j \frac{1}{r_j^2}[\tfrac{1}{2}(\tfrac{1}{2}-\beta)\lambda_{jj}-\lambda_j(\xi)-\tfrac{1}{2}M_j(G)^2+(1+\gamma)M_jM_j(G)]\mathbb{1}$$

$$+2\sum_{j\neq k}\frac{1}{r_jr_k}[(\tfrac{1}{2}-\beta)\lambda_{jk}\mathbb{1}+\mathbb{C}_{-jk}+\tfrac{1}{2}M_j(G)M_k(G)\mathbb{1}-(1+\gamma)M_jM_k(G)\mathbb{1}]$$

$$+2\sum_{j\neq k}\frac{1}{r_{jk}^2}\left[\frac{1}{8}\Gamma_{jk}\frac{M_j(G)}{M_j}[(\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_{jk})(3\mathbb{1}+\hat{\mathbf{r}}_j\hat{\mathbf{r}}_j)-\hat{\mathbf{r}}_j\hat{\mathbf{r}}_{jk}-\hat{\mathbf{r}}_{jk}\hat{\mathbf{r}}_j]-\tfrac{1}{2}(1+\gamma)\Gamma_{jk}[(\hat{\mathbf{r}}_j\cdot\hat{\mathbf{r}}_{jk})\mathbb{1}-\hat{\mathbf{r}}_j\hat{\mathbf{r}}_{jk}-\hat{\mathbf{r}}_{jk}\hat{\mathbf{r}}_j]\right], \tag{68}$$

where $M_j(G)$, λ_{jk}, and λ'_{jk} are test-body limits of the 1PPN parameters given by Nordtvedt [3] and $\mathbb{1}$ is the identity matrix.

V. DISCUSSION

For most applications of a 2PPN Lagrangian, the form given in (34)–(38) is sufficiently general to incorporate all reasonable theories of gravity. If one wishes to restrict the class of gravitational theories to those which possess Lorentz invariance, then the Lagrangian given in either the Lorentz invariant gauge or the metric gauge should be used. The Lorentz invariant gauge chosen for (53) has the advantage of eliminating the need for an additional gauge change whenever a Lorentz boost is applied to the Lagrangian, thus allowing for greater ease in comparing data taken from different rest frames and exploring theories which do violate Lorentz invariance. The Lorentz-invariant gauge is actually a class of gauges with the remaining freedom apparent in the 2PPN parameters which are not constrained by the imposition of Lorentz invariance. With the additional constraint on the two-body parameter a_{ij}^1 obtained by requiring that test bodies move along geodesics, the Lorentz invariant gauge is restricted to a class of gauges described by

$$\delta\mathbf{r}_i=\frac{1}{M_i}\sum_{j\neq i}\frac{\Delta_{ij}^1}{r_{ij}^2}\mathbf{r}_{ij}+\frac{1}{M_i}\sum_{j\neq k\neq i}\frac{\Delta_{ijk}^2}{r_{ij}r_{ik}}\mathbf{r}_{ij}\ . \tag{69}$$

The metric gauge can be obtained from the Lorentz invariant gauge by the application of (56), (57), and (60), and so it also retains the freedom of (69).

If the general relativity values of the 1PPN parameters are used, then the Lagrangian written in the metric gauge is nearly identical to the Lagrangian written in the Arnowitt-Deser-Misner (ADM) gauge given by Damour and Schäfer [8]. If the gauge choice is not made to eliminate the three-body parameter a_{ijk}, then the 2PPN Lagrangian can be made to be identical with the ADM Lagrangian given by [8], and the general-relativity values of the 2PPN parameters can be obtained. In [8], one velocity-independent contribution to the ADM Lagrangian was not evaluated explicitly for more than two bodies. In a later paper [9], Schäfer was able to evaluate this contribution for the case of three bodies. These results were generalized to give a description for N bodies in which only the four-body interactions remain unknown. The term in question is given by (in units with $G=c=1$) [9]

$$U^{TT}=U_{(4)}^{TT}-\frac{1}{4}\sum_{i\neq j}\frac{1}{r_{ij}^3}M_i^2M_j^2-\frac{1}{64}\sum_{i\neq j\neq k}\frac{M_i^2M_jM_k}{r_{ij}r_{ik}r_{jk}}\left[18-24\left[\frac{r_{ij}}{r_{ik}}+\frac{r_{jk}}{r_{ik}}\right]-60\frac{r_{jk}^2}{r_{ik}^2}+60\frac{r_{jk}^2}{r_{ij}r_{ik}}+53\frac{r_{ij}r_{jk}}{r_{ik}^2}+6\frac{r_{ij}^2}{r_{ik}^2}-72\frac{r_{jk}^3}{r_{ij}r_{ik}^2}+35\frac{r_{jk}^4}{r_{ij}^2r_{ik}^2}\right], \tag{70}$$

where $U_{(4)}^{TT}$ is the piece of U^{TT} which corresponds to four-body interactions. Using (70) the 2PPN parameters are

$$\xi_{ij}=2(M_i^2M_j+M_iM_j^2)\ , \tag{71a}$$

$$a_{ijk}=-2+4\frac{r_{ij}}{r_{jk}}-2\frac{r_{ik}}{r_{jk}}-2\frac{r_{ij}}{R}+\frac{3}{2}\frac{r_{ik}}{R}-2\frac{r_{ij}r_{ik}}{Rr_{ijk}}\ , \tag{71b}$$

$$\mathbb{C}_{ijk}=-\frac{3}{2}\frac{r_{ik}}{R}\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{ij}+2\frac{r_{ij}}{R}\hat{\mathbf{r}}_{ik}\hat{\mathbf{r}}_{ik}+2\frac{r_{ij}r_{ik}}{Rr_{jk}}\hat{\mathbf{r}}_j\hat{\mathbf{r}}_{jk}-\frac{1}{2}\frac{r_{ij}r_{ik}}{R^2}(3\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{ij}-9\hat{\mathbf{r}}_{ik}\hat{\mathbf{r}}_{ij}+9\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{jk}+5\hat{\mathbf{r}}_{ik}\hat{\mathbf{r}}_{jk}-4\hat{\mathbf{r}}_{ik}\hat{\mathbf{r}}_{ik}+12\hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{ik}-4\hat{\mathbf{r}}_{jk}\hat{\mathbf{r}}_{jk}-12\hat{\mathbf{r}}_{jk}\hat{\mathbf{r}}_{ij}-16\hat{\mathbf{r}}_{jk}\hat{\mathbf{r}}_{ik})\ , \tag{71c}$$

$$\Theta_{ij}=\tfrac{1}{8}M_i^2M_j^2 \;, \tag{71d}$$

$$\Psi_{ij}=\tfrac{1}{8}(M_iM_j^3+M_i^3M_j) \;, \tag{71e}$$

$$\begin{aligned}\Omega'_{ijk}=M_i^2M_jM_k\Bigg[&\frac{3}{4}+\frac{5}{16}\left[\frac{r_{jk}}{r_{ij}}+\frac{r_{jk}}{r_{ik}}\right]+\frac{3}{16}\left[\frac{r_{ij}}{r_{ik}}+\frac{r_{ik}}{r_{ij}}\right]-\frac{15}{32}\left[\frac{r_{jk}}{r_{ij}}-\frac{r_{jk}}{r_{ik}}\right]^2\\&+\frac{53}{128}\left[\frac{r_{ik}r_{jk}}{r_{ij}^2}+\frac{r_{ij}r_{jk}}{r_{ik}^2}\right]+\frac{3}{64}\left[\frac{r_{ij}}{r_{ik}}-\frac{r_{ik}}{r_{ij}}\right]^2-\frac{9}{16}\left[\frac{r_{jk}^3}{r_{ij}^2r_{ik}}+\frac{r_{jk}^3}{r_{ij}r_{ik}^2}\right]\\&+\frac{35}{64}\frac{r_{jk}^4}{r_{ij}^2r_{ik}^2}-\frac{1}{8}\frac{r_{jk}}{r_{ij}}\hat{\mathbf{r}}_{ij}\cdot\hat{\mathbf{r}}_{jk}+\frac{1}{8}\frac{r_{jk}}{r_{ik}}\hat{\mathbf{r}}_{ik}\cdot\hat{\mathbf{r}}_{jk}\Bigg] \;,\end{aligned} \tag{71f}$$

$$\Pi_{ijkl}=\frac{1}{4}M_iM_jM_kM_l\left[1+\frac{1}{4}\frac{(r_{jk}+r_{jl})(r_{ij}r_{ik}+r_{ij}r_{il}+r_{ik}r_{il})}{r_{jk}r_{jl}r_{kl}}+\frac{1}{8}\frac{r_{ik}r_{kl}}{r_{ij}r_{il}}\hat{\mathbf{r}}_{ij}\cdot\hat{\mathbf{r}}_{ik}\right]-U^{TT}_{(4)} \;, \tag{71g}$$

where

$$\Omega_{ijk}=\frac{1}{3}(\Omega'_{ijk}+\Omega'_{jik}+\Omega'_{kji}) \tag{72}$$

and

$$R=r_{ij}+r_{ik}+r_{jk} \;. \tag{73}$$

By choosing $\Delta^1_{ij}=0$ and

$$\Delta^2_{ijk}=-a_{ijk} \tag{74}$$

in the transformation (69), then the ADM gauge can be put into the form of the metric gauge given in this paper.

In a recent paper [10], Damour and Schäfer have presented a gauge transformation which will put the N-body Lagrangian given in [8] into a harmonic gauge which can then be compared to the 2PPN Lagrangian in the "Lorentz invariant" gauge (53). It is hoped that when the 2PPN Lagrangian and its associated metric are put to use, a standard choice for Δ^1_{ij} and Δ^2_{ijk} will arise from practical considerations.

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