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Quantization dimension for inhomogeneous bi-Lipschitz IFS

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QUANTIZATION DIMENSION FOR INHOMOGENEOUS BI-LIPSCHITZ IFS

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ABSTRACT. Let ν be a Borel probability measure on a d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 1$, with a compact support, and let $(p_0, p_1, p_2, \ldots, p_N)$ be a probability vector with $p_j > 0$ for $1 \leq j \leq N$. Let $\{S_j : 1 \leq j \leq N\}$ be a set of contractive mappings on $\mathbb{R}^d$. Then, a Borel probability measure μ on $\mathbb{R}^d$ such that $\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1} + p_0 \nu$ is called an inhomogeneous measure, also known as a condensation measure on $\mathbb{R}^d$. For a given $r \in (0, +\infty)$, the quantization dimension of order r, if it exists, denoted by $D_r(\mu)$, of a Borel probability measure μ on $\mathbb{R}^d$ represents the speed at which the nth quantization error of order r approaches to zero as the number of elements n in an optimal set of n-means for μ tends to infinity. In this paper, we investigate the quantization dimension for such a condensation measure.

1. Introduction

Quantization is a process of discretization, in other words, to represent a continuous or a large set of values by a set with smaller number of values. It has broad application in signal processing and data compression. For a detailed survey on the subject and a comprehensive list of references to the literature one is referred to [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 20, 21].

Given a Borel probability measure P on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error* of order r for P, is defined by

$$V_{n,r} := V_{n,r}(P) = \inf\{\int d(x,\alpha)^r dP(x) : \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \leq n\}, \tag{1}$$

where $d(x, \alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. The numbers

$$\underline{D}_r(P) := \liminf_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(P)} \text{ and } \overline{D}_r(P) := \limsup_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(P)} \tag{2}$$

are called the *lower* and the *upper quantization dimensions* of P of order r, respectively. If $\underline{D}_r(P) = \overline{D}_r(P)$, the common value is called the *quantization dimension* of P of order r and is denoted by $D_r(P)$. Quantization dimension measures the speed at which the specified measure of the error goes to zero as n tends to infinity. For any $\kappa > 0$, the numbers $\liminf_n n^{\frac{r}{\kappa}} V_{n,r}(P)$ and $\limsup_n n^{\frac{r}{\kappa}} V_{n,r}(P)$ are called the *κ-dimensional lower* and *upper quantization coefficients* for P, respectively. If the κ-dimensional lower and upper quantization coefficients are finite and positive, then κ equals the quantization dimension $D_r(P)$.

Let $\mathcal{F} = \{\mathbb{R}^d; S_1, S_2, \ldots, S_N\}$ be a family of contractive iterated function system (IFS) such that for all $x, y \in \mathbb{R}^d$, we have

$$s_i d(x, y) \leq d(S_i(x), S_i(y)) \leq c_i d(x, y),$$

where $0 < s_i \leq c_i < 1$ and $i \in \{1, 2, \ldots, N\}$. Let $\mathcal{G} = \{\mathbb{R}^d; g_1, g_2, \ldots, g_M\}$ be another IFS such that

$$d(g_i(x), g_i(y)) \leq r_i d(x, y),$$

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where $0 < r_i < 1$ and $i \in \{1, 2, \ldots, M\}$. Let K_ϕ and C be the attractor of the IFS $\mathcal{F}$ and $\mathcal{G}$, respectively, that is,

$$K_\emptyset = \bigcup_{i=1}^{N} S_i(K_\emptyset), \quad C = \bigcup_{i=1}^{M} g_i(C).$$

Let $(t_1, t_2, \ldots, t_M)$ be a probability vector. Then, there exists a unique invariant measure ν supported on C such that

$$\nu = \sum_{i=1}^{M} t_i \nu \circ g_i^{-1}.$$

Let $(p_0, p_1, p_2, \ldots, p_N)$ be a probability vector. Then, there exists a unique Borel probability measure μ and a compact set K such that

$$\mu = p_0 \nu + \sum_{i=1}^{N} p_i \mu \circ S_i^{-1}, \quad K = \bigcup_{i=1}^{N} S_i(K) \bigcup C.$$

We call $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ a *condensation system*. The measure μ is called the *attracting measure* or the *condensation measure* for $(S_{ii=1}^N, (p_i)_{i=0}^N, \nu)$, and the set K, which is the support of the measure μ, is called the *attractor* of the system (see [3, 14]). Such a measure is also termed as an *inhomogeneous measure*. We say that the condensation system satisfies the *strong separation condition* (SSC) if $S_1(K), S_2(K), \ldots, S_N(K)$ and C are pairwise disjoint. An IFS $\mathcal{F} = \{\mathbb{R}^d; S_1, S_2, \ldots, S_N\}$ satisfies the *open set condition* (OSC) if there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\cup_{j=1}^N S_j(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all $1 \leq i \neq j \leq N$. Furthermore, $\mathcal{F}$ satisfies the *strong open set condition* (SOSC) if U can be chosen such that $U \cap K_\emptyset \neq \emptyset$, where $\emptyset$ is the empty set and $K_\emptyset$ is the attractor of $\mathcal{F}$. Notice that in the case of an iterated function system consisting of a finite number of mappings on the Euclidean space, the open set condition implies the strong open set condition (see [16, 18]), and it is not true for an iterated function system with infinitely many mappings (see [19]).

In this paper, we state and prove the following theorem which is the main result of the paper.

Theorem 1.1. *Let $r \in (0, \infty)$, and let μ be the condensation measure associated with the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$. Assume that the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition. Then,*

$$\max\{k_r, \underline{D}_r(\nu)\} \leq \underline{D}_r(\mu),$$

where k_r is uniquely determined by $\sum_{i=1}^N (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1$. Furthermore, if the IFS $\mathcal{G}$ satisfies the strong open set condition, then

$$\max\{k_r, \underline{D}_r(\nu)\} \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq \max\{l_r, d_r\},$$

where l_r and d_r are given by $\sum_{i=1}^N (p_i c_i^r)^{\frac{l_r}{r+l_r}} = 1$ and $\sum_{i=1}^M (t_i r_i^r)^{\frac{d_r}{r+d_r}} = 1$, respectively.

Remark 1.2. In the above theorem, if we consider the mappings S_i and g_i as similarity mappings, i.e.,

$$d(S_i(x), S_i(y)) = s_i d(x, y), \quad d(g_j(x), g_j(y)) = r_j d(x, y)$$

for all $i \in \{1, 2, \cdots, N\}$ and $j \in \{1, 2, \ldots, M\}$, then the quantization dimension of the condensation measure μ exists, and

$$D_r(\mu) = \max\{k_r, D_r(\nu)\},$$

where k_r and $D_r(\nu)$ are given by $\sum_{i=1}^N (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1$ and $\sum_{i=1}^M (t_i r_i^r)^{\frac{D_r(\nu)}{r+D_r(\nu)}} = 1$, respectively.

Remark 1.3. In [24], Zhu estimated the quantization dimension of the condensation measure μ corresponding to the condensation system $(\{f_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ with an inhomogeneous open set condition, where each f_i is a similarity map and the measure ν is an invariant measure corresponding to some self-similar IFS satisfying the open set condition. For determining the upper and the lower bound, the author used the separation condition on the condensation system as well as the separation condition on the self-similar IFS associated with the measure ν. On the other hand, in Theorem 1.1, we note that for determining the lower bound, we only take separation condition on the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ and the measure ν is any probability measure, and in obtaining the upper bound, we consider the the condensation system with strong separation condition and the measure ν corresponding to some IFS with strong open set condition.

Remark 1.4. In [22], Zhu determined the quantization dimension of the condensation measure μ corresponding to the condensation system $(\{f_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ with the strong separation condition, where f_i's are similarity mappings and the measure ν is a self similar measure corresponding to the IFS $\{\mathbb{R}^d; f_1, f_2, \ldots, f_N\}$ with a probability vector $(t_1, t_2, \ldots, t_N)$. By taking the same conditions as in [22], Roychowdhury [17] estimated the quantization dimension of the condensation measure μ and related the upper bound of the quantization dimension with the temperature function of the thermodynamic formalism that arises in the multifractal analysis of μ. However, in Theorem 1.1, we consider more general condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ with the strong separation condition by taking the maps S_i's as bi-Lipschitz and the measure ν as the invariant measure corresponding to some IFS. Thus, Theorem 1.1 provides the quantization dimension of the condensation measure in a more general setting.

2. Preliminaries

Define $I := \{1, 2, \ldots, N\}$ and let $I^n = \{w = w_1 w_2 w_3 \cdots w_n : w_i \in I \ \forall \ i \in \{1, 2, \ldots, n\}\}$. Let $I^* = \cup_{n \in \mathbb{N} \cup \{0\}} I^n$, where I^0 denotes the set consisting of the empty sequence $\emptyset$. Thus, I^* denotes the set of all finite sequences of symbols belonging to I including the empty sequence $\emptyset$. Notice that if $\omega \in I^*$, then $\omega \in I^n$ for some $n \in \mathbb{N} \cup \{0\}$. Such an n is called the length of the word ω and is denoted by $|\omega|$. Notice that the length of the empty sequence is zero. Let Ω denote the set of all infinite sequences of symbols from the set I. For $w = w_1 w_2 \cdots w_n \in I^n$ with $n \geq 1$, we define

$$S_w = S_{w_1} \circ S_{w_2} \circ \cdots S_{w_n}, \ p_w = \prod_{i=1}^n p_{w_i}, \ s_w = \prod_{i=1}^n s_{w_i}, \ c_w = \prod_{i=1}^n c_{w_i} \text{ and } w^- = w_1 w_2 \cdots w_{n-1}.$$

For the empty sequence $\emptyset$, we write

$$S_\emptyset = Id_{\mathbb{R}^d}, \, p_\emptyset = 1, \ \text{ and } s_\emptyset = c_\emptyset = 1,$$

where $Id_{\mathbb{R}^d}$ is the identity mapping on $\mathbb{R}^d$. For $n \geq 1$, by iterating, we obtain

$$K = \Big(\bigcup_{w \in I^n} S_w(K) \Big) \cup \Big(\bigcup_{m=0}^{n-1} \bigcup_{w \in I^m} S_w(C) \Big), \text{ and } \mu = \sum_{w \in I^n} p_w \mu \circ S_w^{-1} + p_0 \sum_{m=0}^{n-1} \sum_{w \in I^m} p_w \nu \circ S_w^{-1}.$$

For any $w \in I^*$ such that $|w| \geq n$, we denote $w|_n = w_1 w_2 \cdots w_n$. We define an ordering on I^* as follows

$$w \prec \sigma \text{ if and only if } |w| \leq |\sigma|, \ \sigma|_{|w|} = w,$$

where $\sigma, w \in I^*$. For any $\sigma, w \in I^*$, we say that σ and w are incomparable if neither $\sigma \prec w$ nor $w \prec \sigma$. On the other hand, if $\omega \prec \sigma$, we say that σ is an extension of ω. For $w \in I^*$, we define

$$\Xi_w(m) = \{\sigma \in I^{|w|+m} : w \prec \sigma\} \quad \text{and} \quad \Xi_w^* = \cup_{m \in \mathbb{N}} \Xi_w(m),$$

where $m \geq 1$.

Define $J := \{1, 2, \ldots, M\}$. We define J^* and J^n in a similar way as we define I^* and I^n.

We call $\Gamma \subset I^*$ a finite maximal antichain if Γ is a finite set such that every sequence in Ω is an extension of some element in Γ, but no element of Γ is an extension of another element in Γ. Let Γ be a finite maximal antichain. Set

$$l_\Gamma = \min\{|\omega| : \omega \in \Gamma\} \quad \text{and} \quad M_\Gamma = \max\{|\omega| : \omega \in \Gamma\}.$$

For $w \in I^{l_\Gamma}$, we define

$$\Delta_\Gamma(w) = \{\sigma \in I^* : w \leq \sigma,\ \Xi^*_\sigma \cap \Gamma \neq \emptyset\}, \quad \Delta^*_\Gamma = \cup_{w\in I^{l_\Gamma}} \Delta_\Gamma(w).$$

Lemma 2.1. *(see [24]) Let Γ be a finite maximal antichain. Then,*

$$K = \Big(\bigcup_{w\in\Gamma} S_w(K)\Big) \cup \Big(\bigcup_{w\in\Delta^*_\Gamma} S_w(C)\Big) \cup \Big(\bigcup_{n=0}^{l_\Gamma-1} \bigcup_{w\in I^n} S_w(C)\Big).$$

For a given Borel probability measure P on $\mathbb{R}^d$, a positive real number r and a natural number n, let $V_{n,r}$ denote the nth quantization error of order r as defined in equation (1). If the infimum in (1) is attained for some $A \subset \mathbb{R}^d$, then A is called an *n-optimal set* for the measure P of order r. The existence of n-optimal set is guaranteed if P has a compact support (see [6]). The lower quantization dimension $\underline{D}_r$ and the upper quantization dimension $\overline{D}_r$ are as defined in equation (2). We recall the following important proposition which will be needed to prove our main results.

Proposition 2.2. *(see [6])*

(1) *If $0 \leq t_1 < \overline{D}_r < t_2$, then*

$$\limsup_{n\to\infty} nV_{n,r}^{\frac{t_1}{r}} = \infty \text{ and } \lim_{n\to\infty} nV_{n,r}^{\frac{t_2}{r}} = 0.$$

(2) *If $0 \leq t_1 < \underline{D}_r < t_2$, then*

$$\liminf_{n\to\infty} nV_{n,r}^{\frac{t_2}{r}} = 0 \text{ and } \lim_{n\to\infty} nV_{n,r}^{\frac{t_1}{r}} = \infty.$$

3. Main results

To prove Theorem 1.1, which gives the main results of the paper, we state and prove some lemmas and propositions.

Lemma 3.1. *Let $r \in (0,\infty)$, and let μ be the condensation measure associated with the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$. Then,*

$$\underline{D}_r(\nu) \leq \underline{D}_r(\mu) \quad \text{and} \quad \overline{D}_r(\nu) \leq \overline{D}_r(\mu).$$

Proof. Let A be an n-optimal set for the Borel probability measure μ of order r. Then, we have

$$V_{n,r}(\mu) = \int d(x,A)^r d\mu(x).$$

Since $\mu = p_0\nu + \sum_{i=1}^N p_i\mu \circ S_i^{-1}$, we obtain $\mu \geq p_0\nu$. Then,

$$V_{n,r}(\mu) \geq p_0 \int d(x,A)^r d\nu(x) \geq p_0 V_{n,r}(\nu).$$

In the above, taking logarithm, and then dividing both side by $\log V_{n,r}(\mu)\log V_{n,r}(\nu)$, we get

$$\frac{1}{\log V_{n,r}(\mu)} \leq \frac{-\log p_0}{\log V_{n,r}(\mu)\log V_{n,r}(\nu)} + \frac{1}{\log V_{n,r}(\nu)},$$

which implies that

$$\Big(1 - \frac{\log p_0}{\log V_{n,r}(\mu)}\Big)\frac{r\log n}{-\log V_{n,r}(\nu)} \leq \frac{r\log n}{-\log V_{n,r}(\mu)}.$$

By [6, Lemma 6.1], we have $\lim_{n\to\infty} V_{n,r}(\mu) \to 0$. Hence, by taking liminf and limsup on both sides of the above inequality, we get our required result. □

In the following lemma, we only assume that the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition without assuming any condition on the probability measure ν.

Proposition 3.2. *Assume that the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition. Let μ be the inhomogeneous measure associated with $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$. Then for any $r > 0$, we have*

$$\liminf_{n\to\infty} nV_{n,r}^{\frac{k_r}{r}}(\mu) > 0,$$

where k_r is uniquely determined by $\sum_{i=1}^N (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1$.

Proof. Let $r \in (0,\infty)$. We define a map $\phi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = \sum_{i=1}^N (p_i s_i^r)^t.$$

The map ϕ is strictly decreasing and continuous. Moreover, $\phi(0) = N > 1$ and $\phi(1) = \sum_{i=1}^N (p_i s_i^r) < \sum_1^N p_i = 1$. Thus, there exists a unique $\tilde{t_r} \in (0,1)$ such that $\sum_{i=1}^N (p_i s_i^r)^{\tilde{t_r}} = 1$. By the uniqueness of $\tilde{t_r}$, there is a unique $k_r \in (0,\infty)$ such that

$$\sum_{i=1}^N (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1.$$

Since the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition, $S_1(K), S_2(K), \dots, S_N(K)$ and C are pairwise disjoint sets. Write

$$\delta = \min\{d(S_i(K), S_j(K)) : 1 \le i \neq j \le N\} > 0.$$

Let A_n be an n-optimal set for the Borel probability measure μ of order r. Set $\delta_n = \sup_{x\in K} d(x, A_n)$. Then, $\lim_{n\to\infty} \delta_n = 0$ (see [6, Lemma 6.1, Lemma 13.8]). This implies that there exists $n_0 \in \mathbb{N}$ such that $\delta_n \le \frac{\delta}{3}$ for all $n \ge n_0$. Set $A_{n_i} = \{a \in A_n : d(a, S_i(K)) \le \delta_n\}$. It is easy to see that $A_{n_i} \neq \emptyset$ and $A_{n_i} \cap A_{n_j} = \emptyset$ for $1 \le i \neq j \le N$. Let $k_i(n) = \text{card}(A_{n_i})$. Thus, we have $\sum_{i=1}^N k_i(n) \le n$. Also, $k_i(n) \le n-1$ for each $1 \le i \le N$. Since A_n is an n-optimal set for the Borel probability measure μ of order r, we have

$$\begin{aligned}
V_{n,r}(\mu) &= \int d(x, A_n)^r d\mu(x) \\
&\ge \sum_{i=1}^N \int_{S_i(K)} d(x, A_n)^r d\mu(x) \\
&= \sum_{i=1}^N \int_{S_i(K)} d(x, A_{n_i})^r d\mu(x) \\
&\ge \sum_{i=1}^N p_i \int_{S_i(K)} d(x, A_{n_i})^r d(\mu \circ S_i^{-1})(x) \\
&\ge \sum_{i=1}^N p_i s_i^r \int_K d(x, S_i^{-1}(A_{n_i}))^r d(\mu)(x) \\
&\ge \sum_{i=1}^N p_i s_i^r V_{k_i(n),r}(\mu).
\end{aligned}$$

Define $M_0 = \min\{n^{\frac{r}{k_r}} V_{n,r}(\mu) : n < n_0\}$. Clearly, $M_0 > 0$ and $M_0 \le n^{\frac{r}{k_r}} V_{n,r}(\mu)$ for all $n < n_0$. Now our claim is that $M_0 \le n^{\frac{r}{k_r}} V_{n,r}(\mu)$ for all $n \in \mathbb{N}$. We prove our claim by induction on $n \in \mathbb{N}$. Let us assume that $M_0 \le \eta^{\frac{r}{k_r}} V_{\eta,r}(\mu)$ for all $\eta < n$ and $n \ge n_0$. Thus, we have

$$V_{n,r}(\mu) \ge \sum_{i=1}^N p_i s_i^r V_{k_i(n),r}(\mu) \ge M_0 \sum_{i=1}^N p_i s_i^r k_i(n)^{\frac{-r}{k_r}}.$$

Hence, by the generalized Hölder's inequality, we get

$$\sum_{i=1}^{N} p_i s_i^r k_i(n)^{\frac{-r}{k_r}} \geq \Big(\sum_{i=1}^{N} (p_i s_i^r)^{\frac{k_r}{r+k_r}}\Big)^{1+\frac{r}{k_r}} \cdot \Big(\sum_{i=1}^{N} k_i(n)\Big)^{\frac{-r}{k_r}}.$$

Using the facts that $\sum_{i=1}^N (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1$ and $\sum_{i=1}^N k_i(n) \leq n$, we get $V_{n,r}(\mu) \geq M_0 n^{\frac{-r}{k_r}}$, that is, $M_0 \leq n^{\frac{r}{k_r}} V_{n,r}(\mu)$. Thus, by induction, we obtain that $M_0 \leq n^{\frac{r}{k_r}} V_{n,r}(\mu)$ holds for all $n \in \mathbb{N}$. This implies that

$$\liminf_{n\to\infty} n V_{n,r}^{\frac{k_r}{r}}(\mu) \geq M_0^{\frac{k_r}{r}} > 0.$$

Thus, the proof is done. □

Motivated by [24], we fix some notations in the following paragraph.

First, we define a finite maximal antichain on I^* as follows

$$\Gamma_n^r = \{w \in I^* : p_w c_w^r < \xi_r^n \leq p_{w^-} c_{w^-}^r\}$$

where $n \geq 1$ and $\xi_r = \min\{\min_{1\leq j\leq M} t_j r_j^r, \min_{1\leq j\leq N} p_j c_j^r\}$. Let $|\Gamma_n^r|$ denote the cardinality of the set Γ_n^r. We write

$$\Phi_n^r = \bigcup_{m=0}^{l_{\Gamma_n^r}-1} I^m \cup \Delta_{\Gamma_n^r}^*$$

for all $n \geq 1$. By the definition of Φ_n^r, one can easily see that for any $w \in \Phi_n^r$, we have $p_w c_w^r \geq \xi_r^n$. Thus, for any $w \in \Phi_n^r$, we define a finite maximal antichain on J^* as follows

$$\Gamma_n^r(w) = \{\sigma \in J^* : p_w c_w^r t_\sigma r_\sigma^r < \xi_r^n \leq p_w c_w^r t_{\sigma^-} r_{\sigma^-}^r\}.$$

Let $|\Gamma_n^r(w)|$ be the cardinality of $\Gamma_n^r(w)$ for each $w \in \Phi_n^r$. We define

$$\phi_{n,r} := |\Gamma_n^r| + \sum_{w\in\Phi_n^r} |\Gamma_n^r(w)|. \tag{3}$$

Lemma 3.3. *For each $n \geq 1$, we have*

$$K = \Big(\bigcup_{w\in\Gamma_n^r} S_w(K)\Big) \cup \Big(\bigcup_{w\in\Phi_n^r} \bigcup_{\sigma\in\Gamma_n^r(w)} S_w(g_\sigma(C))\Big).$$

Proof. By Lemma 2.1, we have

$$K = \Big(\bigcup_{w\in\Gamma_n^r} S_w(K)\Big) \cup \Big(\bigcup_{w\in\Delta_{\Gamma_n^r}^*} S_w(C)\Big) \cup \Big(\bigcup_{m=0}^{l_{\Gamma_n^r}-1} \bigcup_{w\in I^m} S_w(C)\Big).$$

Since for each $w \in \Phi_n^r$, $\Gamma_n^r(w)$ is a finite maximal antichain in J^*, we obtian

$$C = \bigcup_{\sigma\in\Gamma_n^r(w)} g_\sigma(C).$$

Using the definition of Φ_n^r and the above relations, we get

$$K = \Big(\bigcup_{w\in\Gamma_n^r} S_w(K)\Big) \cup \Big(\bigcup_{w\in\Phi_n^r} \bigcup_{\sigma\in\Gamma_n^r(w)} S_w(g_\sigma(C))\Big).$$

□

Lemma 3.4. *Assume that the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition and the IFS $\mathcal{G}$ satisfies the strong open set condition. Let μ be the inhomogeneous measure associated with $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$. Then, we have*

$$V_{\phi_{n,r},r}(\mu) \leq C^*\Big(\sum_{w\in\Gamma_n^r} p_w c_w^r + \sum_{w\in\Phi_n^r} \sum_{\sigma\in\Gamma_n^r(w)} p_w c_w^r t_\sigma r_\sigma^r\Big),$$

where $C^* = \max\{(diam(K))^r, (diam(C))^r\}$ *and* $\phi_{n,r}$ *is given by equation* (3).

Proof. First, we define a set A such that $\text{card}(A) \leq \phi_{n,r}$. For each $w \in \Gamma_n^r$, we choose an element $a_w \in S_w(K)$ and for each $w \in \Phi_n^r$, we choose an element $a_{w,\sigma} \in S_w(g_\sigma(C))$ such that $\sigma \in \Gamma_n^r(w)$. We collect all such type of elements and form a set A. Clearly, $\text{card}(A) \leq \phi_{n,r}$. Thus, by Lemma 3.3, we have

$$\begin{aligned}
V_{\phi_{n,r},r}(\mu) &\leq \int_K d(x,A)^r d\mu(x) \\
&\leq \sum_{w\in\Gamma_n^r} \int_{S_w(K)} d(x,A)^r d\mu(x) + \sum_{w\in\Phi_n^r} \sum_{\sigma\in\Gamma_n^r(w)} \int_{S_w(g_\sigma(C))} d(x,A)^r d\mu(x) \\
&\leq \sum_{w\in\Gamma_n^r} \int_{S_w(K)} d(x,a_w)^r d\mu(x) + \sum_{w\in\Phi_n^r} \sum_{\sigma\in\Gamma_n^r(w)} \int_{S_w(g_\sigma(C))} d(x,a_{w,\sigma})^r d\mu(x) \\
&\leq \sum_{w\in\Gamma_n^r} (\text{diam}(S_w(K)))^r \mu(S_w(K)) + \sum_{w\in\Phi_n^r} \sum_{\sigma\in\Gamma_n^r(w)} (\text{diam}(S_w(g_\sigma(C))))^r \mu(S_w(g_\sigma(C))) \\
&\leq \sum_{w\in\Gamma_n^r} c_w^r (\text{diam}(K))^r \mu(S_w(K)) + \sum_{w\in\Phi_n^r} \sum_{\sigma\in\Gamma_n^r(w)} c_w^r r_\sigma^r (\text{diam}(C))^r \mu(S_w(g_\sigma(C))).
\end{aligned}$$

Writing $C^* = \max\{(\text{diam}(K))^r, (\text{diam}(C))^r\}$, we get

$$V_{\phi_{n,r},r}(\mu) \leq C^* \Bigg(\sum_{w\in\Gamma_n^r} c_w^r \mu(S_w(K)) + \sum_{w\in\Phi_n^r} \sum_{\sigma\in\Gamma_n^r(w)} c_w^r r_\sigma^r \mu(S_w(g_\sigma(C))) \Bigg) \tag{4}$$

Now, we determine the values of $\mu(S_w(K))$ and $\mu(S_w(g_\sigma(C))$ for $w \in \Gamma_n^r$ and $\sigma \in \Gamma_n^r(w)$. Let $w \in \Gamma_n^r$ and $|w| = n_1$. Since $\mu = \sum_{\tau\in I^{n_1}} p_\tau \mu \circ S_\tau^{-1} + p_0 \sum_{m=0}^{n_1-1} \sum_{\tau\in I^m} p_\tau \nu \circ S_\tau^{-1}$, and the condensation system satisfies the strong separation condition, we have

$$\begin{aligned}
\mu(S_w(K)) &= \sum_{\tau\in I^{n_1}} p_\tau \mu \circ S_\tau^{-1}(S_w(K)) + p_0 \sum_{m=0}^{n_1-1} \sum_{\tau\in I^m} p_\tau \nu \circ S_\tau^{-1}(S_w(K)) \\
&= p_w \mu(S_w^{-1}(S_w(K))) + p_0 \sum_{m=0}^{n_1-1} \sum_{\tau\in I^m} p_\tau \nu \circ S_\tau^{-1}(S_w(K)) \\
&= p_w \mu(K) + p_0 \sum_{m=0}^{n_1-1} \sum_{\substack{\tau\in I^m \\ \tau\leq w}} p_\tau \nu \circ S_\tau^{-1}(S_w(K)).
\end{aligned}$$

Since $\tau \in I^m$ and $\tau \leq w$, we write $w = \tau w_\tau$ for some $w_\tau \in I^*$. Then, using $\mu(K) = 1$, we have

$$\begin{aligned}
\mu(S_w(K)) &= p_w + p_0 \sum_{m=0}^{n_1-1} \sum_{\substack{\tau\in I^m \\ \tau\leq w}} p_\tau \nu(S_\tau^{-1}(S_{\tau w_\tau}(K))) \\
&= p_w + p_0 \sum_{m=0}^{n_1-1} \sum_{\substack{\tau\in I^m \\ \tau\leq w}} p_\tau \nu(S_{w_\tau}(K) \cap C).
\end{aligned}$$

Since the condensation system satisfies the strong separation condition, we have $S_{w_\tau}(K) \cap C = \emptyset$. Thus, we get

$$\mu(S_w(K)) = p_w. \tag{5}$$

Since the IFS J satisfies the strong open set condition, we have $\nu(g_\sigma(C)) = t_\sigma$ for all $\sigma \in J^*$ (see [2, 15]).

$$\begin{aligned}
\mu(S_w(g_\sigma(C))) &= \sum_{\tau\in I^{n_1}} p_\tau \mu\circ S_\tau^{-1}(S_w(g_\sigma(C))) + p_0\sum_{m=0}^{n_1-1}\sum_{\tau\in I^m} p_\tau \nu\circ S_\tau^{-1}(S_w(g_\sigma(C)))\\
&= p_w\mu(S_w^{-1}(S_w(g_\sigma(C)))) + p_0\sum_{m=0}^{n_1-1}\sum_{\substack{\tau\in I^m\\ \tau\leq w}} p_\tau \nu\circ S_\tau^{-1}(S_w(g_\sigma(C)))\\
&= p_w\mu(g_\sigma(C)) + 0\\
&= p_w\Big(p_0\nu(g_\sigma(C)) + \sum_{i=1}^N p_i\mu\circ S_i^{-1}(g_\sigma(C))\Big).
\end{aligned}$$

Since the condensation system satisfies the strong separation condition, we obtain

$$\mu(S_w(g_\sigma(C))) = p_w p_0 \nu(g_\sigma(C)) = p_w p_0 t_\sigma \leq p_w t_\sigma. \tag{6}$$

By using (5) and (6) in the inequality (4), we get the required result. □

Proposition 3.5. *Assume that the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition and the IFS $\mathcal{G}$ satisfies the strong open set condition. Let μ be the inhomogeneous measure associated with $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$. Then,*

$$\limsup_{n\to\infty} n V_{n,r}^{\frac{t}{r}}(\mu) < \infty,$$

for all $t > \max\{l_r, d_r\}$, where l_r and d_r are given by $\sum_{i=1}^N (p_i c_i^r)^{\frac{l_r}{r+l_r}} = 1$ and $\sum_{i=1}^M (t_i r_i^r)^{\frac{d_r}{r+d_r}} = 1$, respectively.

Proof. By Lemma 3.4 and using the definition of Γ_n^r and $\Gamma_n^r(w)$, we have

$$\begin{aligned}
V_{\phi_{n,r},r}(\mu) &\leq C^*\Big(\sum_{w\in\Gamma_n^r} p_w c_w^r + \sum_{w\in\Phi_n^r}\sum_{\sigma\in\Gamma_n^r(w)} p_w c_w^r t_\sigma r_\sigma^r\Big)\\
&\leq C^*\xi_r^n\phi_{n,r}.
\end{aligned}$$

Now, we determine the bound of $\phi_{n,r}$. Suppose the values of l_r and d_r are uniquely determined by $\sum_{i=1}^N (p_i c_i^r)^{\frac{l_r}{r+l_r}} = 1$ and $\sum_{i=1}^M (t_i r_i^r)^{\frac{d_r}{r+d_r}} = 1$, respectively. Since Γ_n^r is a finite maximal antichain in I^* and for each $w \in \Phi_n^r$, $\Gamma_n^r(w)$ is a finite maximal antichain in J^*, we have

$$\sum_{w\in\Gamma_n^r} (p_w c_w^r)^{\frac{l_r}{r+l_r}} = 1 \quad \text{and} \quad \sum_{\sigma\in\Gamma_n^r(w)} (t_\sigma r_\sigma^r)^{\frac{d_r}{r+d_r}} = 1.$$

Let $t > \max\{l_r, d_r\}$. Then, we have

$$\sum_{w\in\Gamma_n^r} (p_w c_w^r)^{\frac{t}{r+t}} < 1 \quad \text{and} \quad \sum_{\sigma\in\Gamma_n^r(w)} (t_\sigma r_\sigma^r)^{\frac{t}{r+t}} < 1.$$

By the definitions of $\phi_{n,r}$, Γ_n^r and $\Gamma_n^r(w)$ for each $w \in \Phi_n^r$, we obtain

$$\begin{aligned}
\phi_{n,r}\xi_r^{\frac{(n+1)t}{r+t}} &\leq \sum_{w\in\Gamma_n^r} (p_w c_w^r)^{\frac{t}{r+t}} + \sum_{w\in\Phi_n^r}\sum_{\sigma\in\Gamma_n^r(w)} (p_w c_w^r t_\sigma r_\sigma^r)^{\frac{t}{r+t}}\\
&< 1 + \sum_{w\in\Phi_n^r} (p_w c_w^r)^{\frac{t}{r+t}} \sum_{\sigma\in\Gamma_n^r(w)} (t_\sigma r_\sigma^r)^{\frac{t}{r+t}}\\
&< 1 + \sum_{w\in\Phi_n^r} (p_w c_w^r)^{\frac{t}{r+t}}.
\end{aligned}$$

Since $\Phi_n^r = \bigcup_{m=0}^{l_{\Gamma_n^r}-1} I^m \cup \Delta^*_{\Gamma_n^r}$ and $\Delta^*_{\Gamma_n^r} \subset \bigcup_{m=l_{\Gamma_n^r}}^{M_{\Gamma_n^r}} I^m$, we have

$$\begin{aligned}\phi_{n,r}\xi_r^{\frac{(n+1)t}{r+t}} &< 1 + \sum_{m=0}^{l_{\Gamma_n^r}-1}\sum_{w\in I^m}(p_w c_w^r)^{\frac{t}{r+t}} + \sum_{m=l_{\Gamma_n^r}}^{M_{\Gamma_n^r}}\sum_{w\in I^m}(p_w c_w^r)^{\frac{t}{r+t}} \\ &= 1 + \sum_{m=0}^{M_{\Gamma_n^r}}\sum_{w\in I^m}(p_w c_w^r)^{\frac{t}{r+t}} \\ &= 1 + \sum_{m=0}^{M_{\Gamma_n^r}}\left(\sum_{i=1}^{N}(p_i c_i^r)^{\frac{t}{r+t}}\right)^m \\ &< \frac{2}{1-\sum_{i=1}^N (p_i c_i^r)^{\frac{t}{r+t}}}.\end{aligned}$$

Thus, by the previous inequalities, we obtain

$$\begin{aligned}\phi_{n,r}V_{\phi_{n,r},r}^{\frac{t}{r}}(\mu) &\leq (C^*)^{\frac{t}{r}}\xi_r^{\frac{nt}{r}}\phi_{n,r}^{\frac{t+r}{r}} \\ &\leq (C^*)^{\frac{t}{r}}\xi_r^{\frac{-t}{r}}\left(\frac{2}{1-\sum_{i=1}^N (p_i c_i^r)^{\frac{t}{r+t}}}\right)^{\frac{t+r}{r}}.\end{aligned}$$

This implies that $\limsup_{n\to\infty}\phi_{n,r}V_{\phi_{n,r},r}^{\frac{t}{r}} < \infty$ for all $t > \max\{l_r, d_r\}$. By using [23, Theorem 2.4], we obtain

$$\limsup_{n\to\infty} nV_{n,r}^{\frac{t}{r}}(\mu) < \infty \quad \forall \quad t > \max\{l_r, d_r\}.$$

This completes the proof.

□

Proof of Theorem 1.1. Let μ be the condensation measure associated with the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$. Assume that the condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$ satisfies the strong separation condition. Then, by Lemma 3.1, Proposition 3.2 and (2) of Proposition 2.2, we obtain

$$\max\{k_r, \underline{D}_r(\nu)\} \leq \underline{D}_r(\mu),$$

where k_r is uniquely determined by $\sum_{i=1}^N (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1$. Furthermore, if the IFS $\mathcal{G}$ satisfies the strong open set condition, then, by Proposition 3.5 and (1) of Proposition 2.2, we get

$$\overline{D}_r(\mu) \leq \max\{l_r, d_r\},$$

where l_r and d_r are given by $\sum_{i=1}^N (p_i c_i^r)^{\frac{l_r}{r+l_r}} = 1$ and $\sum_{i=1}^M (t_i r_i^r)^{\frac{d_r}{r+d_r}} = 1$, respectively.

□

Remark 3.6. Let μ be the condensation measure for a condensation system $(\{S_i\}_{i=1}^N, (p_i)_{i=0}^N, \nu)$, where ν is any Borel probability measure supported on any compact set C. Then, by Lemma 3.1 and Proposition 3.2, the lower bound obtained in Theorem 1.1 is still valid in this more general setting. The assumption that the measure ν is the invariant measure corresponding to some IFS is needed only for the upper bound of the quantization dimension.

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