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Hankel Partial Contraction, Contractive Completion, Moore-Penrose Inverse, Extremal Case

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HANKEL PARTIAL CONTRACTION, CONTRACTIVE COMPLETION, MOORE-PENROSE INVERSE, EXTREMAL CASE

A Thesis

by

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ABSTRACT

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In this article we find concrete necessary and sufficient conditions for the existence of contractive completions of Hankel partial contractions of size 4×4 extremal case. We will show a counter example for the projection of Hankel partial contractions of size 4×4 extremal case in $\mathbb{R}^3$ is a prisim.

DEDICATION

I would like to dedicate this paper to Dr. Yoon and the School of Mathematical and Statistical Sciences. Both have always believed in me since day one. I would also like to thank my thesis committee for supporting me through this process. I would also like to thanks my friends and family that have had to put up with me.

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CHAPTER I

INTRODUCTION

A *Hankel matrix* is a square matrix with constant skew-diagonals. A *Hankel partial contraction* is a Hankel matrix such that not all of its entries are determined, but in which every well-defined submatrix is a contraction. In this article, we study the problem of whether a Hankel partial contraction in which the upper left triangle is known can be completed to a contraction. Then we review some known solution sets and definitions as we build up to
Recall:

- Conjecture (Curto, Lee, JYoon (JMP 12): The solution set

$$\{(x,y,z) \in \mathbb{R}^3 : \text{the Problem 1 is soluble for } H_4\}$$

is a prism in $\mathbb{R}^3$.

- Result (Kim, Yoo, JYoon, (JKMS 15)): The solution set

$$\{(x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for } H_4\}$$

needs not a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2 \neq 1$

Problem

Based on Theorem 10, we try to find examples which show that the set

$$\left\{(x,y,z)\in\mathbb{R}^3 : \text{Problem 1 is soluble for } H_4\right\}$$

needs NOT a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2=1$.

Given real numbers $a_1,a_2,\cdots,a_n$, find real numbers $x_1,\cdots,x_{n-1}$ such that

$$H(a_1,a_2,\cdots,a_n;x_1,\cdots,x_{n-1})$$

is contractive.

1.1 Helkel Partial Contraction

Given real numbers $a_1,\cdots,a_n$, let

$$H\equiv H(a_1,a_2,\cdots,a_n;x_1,\cdots,x_{n-1}):=\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix} \tag{1.1}$$

be a Hankel matrix, where $x_1,\cdots,x_{n-1}$ are real numbers to be determined. We say that H is a *partial contraction* if all completely determined submatrices of H are contractions (in the sense that their operator norms are at most 1).

Let

$$
\begin{aligned}
H_n &\equiv H(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1}) \\
:&= \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix},
\end{aligned}
$$

where $x_1, \cdots, x_{n-1}$ are real numbers to be determined. Given real numbers $a_1, a_2, \cdots, a_n$, find the necessary and sufficient conditions on the given data to guarantee the existence of contractive Hankel completion.

Since $\|S\| \leq \|T\|$ if S is a submatrix of the matrix T, it follows that each submatrix of a contraction is again a contraction. Thus, a necessary condition for a partial matrix T to be contraction is that each submatrix must be a contraction. We call a partial matrix meeting this necessary condition a *partial contraction* (*well-posed condition*). We say that Problem 1 just given above is *well-posed* if

$$H_{\triangle}(a_1, a_2, \cdots, a_n)$$

is partially contractive, and that it is *soluble* if

$$H(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$$

is contractive for some $x_1, \cdots, x_{n-1}$. We also say that $H_{\triangle}(a_1, a_2, \cdots, a_n)$ is *extremal* if $a_1^2 + \cdots + a_n^2 = 1$.

1.2 Known Solution Sets

Known: The solution set

$$\left\{ (x,y) \in \mathbb{R}^2 : \text{Problem 1 is soluble for} H_3 = \begin{pmatrix} a & b & c \\ b & c & x \\ c & x & y \end{pmatrix} \right\}$$

is a rectangle in $\mathbb{R}^2$. The solution set

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for} H_3 \right\}$$

is a prism in $\mathbb{R}^3$. The solution set

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for} H_4 \right\}$$

needs not a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2 \neq 1$. The solution set

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for} H_4 \right\}$$

needs not a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2 = 1$. Norm of matrix defined on $\mathbb{R}^2$ For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{R}$ $\|A\| = \sup\left\{ \|Ax\| : x \in \mathbb{R}^2, \|x\| = 1 \right\}$, where $x = (x_1, x_2)$, $\|x\| = \sqrt{x_1^2 + x_2^2}$, $Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (ax_1 + bx_2, cx_1 + dx_2)$, and $\|Ax\| = \sqrt{(ax_1+bx_2)^2 + (cx_1+dx_2)^2}$

CHAPTER II

DEFINITIONS AND THEOREMS

2.1 Preliminaries

Definition:

(i): We say that A is contractive if $\|A\| \leq 1$

(ii): We say that A is positive semidefinite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^2$, where $\langle x, y\rangle :=$ $x_1y_1 + x_2y_2$, $x = (x_1, x_2)$, and $y = (y_1, y_2)$.

(iii): We say that A is positive $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{R}^2$.

(iv): An eigenvalue and eigenvector of a square matrix A are a scalar λ and a nonzero vector x so that $Ax = \lambda x$.

(v): A singular value and pair of singular vectors of a square matrix A are a nonnegative scalar σ and two nonzero vectors x and y so that $Ax = \sigma y$ and $A^* y = \overline{A}^t y = \sigma x$.

Note:

(i): We remark that there are several conditions that characterize positive matrices:

(a)$A \geq 0 \Longleftrightarrow A$ is self-adjoint ($A^* = A$) and all its eigenvalues are nonnegative.

(b) $A \geq 0 \Longleftrightarrow A = B^*B$ for some matrix B.

(ii): If A is self-adjoint, then the largest singular value $s_1(A) = \|A\|$.

Theorem 3:

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0 \text{ if and only if } a \geq 0 \text{ and } ac \geq b^2.$$

Proof of Theorem 3

Want: $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^2$.

$$\langle Ax,x\rangle \geq 0 \iff \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \geq 0$$

$$\iff \left\langle \begin{pmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \geq 0$$

$$\iff ax_1^2 + 2bx_1x_2 + cx_2^2 \geq 0$$

$$\iff a\left(x_1 + \tfrac{b}{a}x_2\right)^2 + \left(\tfrac{ac-b^2}{a}\right)x_2^2 \geq 0$$

$$\iff a \geq 0 \text{ and } ac \geq b^2.$$

For $a \in \mathbb{C}$,

$$|a| \leq 1 \iff |a|^2 \leq 1 \iff a\overline{a} \leq 1$$

$$\iff \begin{pmatrix} 1 & a \\ \overline{a} & 1 \end{pmatrix} \geq 0$$

$$\iff \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \geq 0.$$

Theorem 5 For any $n \times n$ matrix A,

$$\|A\| \leq 1 \iff \begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \geq 0 \iff I - A^*A \geq 0.$$

Proof of Theorem 5 $\|A\| \leq 1 \iff s_1 = \|A\| \leq 1$ (using Remark 2)

$$\iff s_1^2 \leq 1 \text{ and } s_2^2 \leq 1$$

(for readers' convenience, we consider the 2×2 case)

$$\iff \begin{pmatrix} 1 & s_1 \\ s_1 & 1 \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} 1 & s_2 \\ s_2 & 1 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} 1 & s_1 & 0 & 0 \\ s_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & s_2 \\ 0 & 0 & s_2 & 1 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & s_2 \\ s_1 & 0 & 1 & 0 \\ 0 & s_2 & 0 & 1 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I & S \\ S & I \end{pmatrix} \geq 0, \text{ where } S := \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \text{ and } s_1 \geq s_2$$

$$\Longleftrightarrow \begin{pmatrix} U & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} I & S \\ S & I \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & V \end{pmatrix} \geq 0, \quad \text{where } U \text{ and } V \text{ are unitary matrices}$$

$$\Longleftrightarrow \begin{pmatrix} I & USV \\ V^*S^*U^* & I \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \geq 0, \qquad \text{where } A = USV \text{ (singular value decomposition)}$$

$$\Longleftrightarrow I - A^*A \geq 0.$$

Theorem 6

Let $H_2 := H(a,b;x) = \begin{pmatrix} a & b \\ b & x \end{pmatrix} \in M_2(\mathbb{R})$.

If H_2 is well-posed, then H_2 admits a contractive completion.

Moreover, x is given by:

$$\begin{cases} x = -a, & \text{if } a^2 + b^2 = 1 \text{ and } b \neq 0 \\ -1 \leq x \leq 1, & \text{if } a^2 = 1 \text{ and } b = 0 \\ \frac{-1-a+b^2}{1+a} \leq x \leq \frac{1-a-b^2}{1-a}, & \text{if } a^2 + b^2 < 1. \end{cases}$$

Proof of Theorem 6

A straightforward calculation shows that

$$I - H_2 H_2^* = \begin{pmatrix} 1-a^2-b^2 & -b(a+x) \\ -b(a+x) & 1-b^2-x^2 \end{pmatrix}.$$

Hence, there exist x s.t. $\begin{pmatrix} 1-a^2-b^2 & -b(a+x) \\ -b(a+x) & 1-b^2-x^2 \end{pmatrix} \geq 0$ whenever $a^2+b^2 \leq 1$.

In fact, x is given by

$$\begin{cases} x = -a, & \text{if } a^2+b^2 = 1 \text{ and } b \neq 0 \\ -1 \leq x \leq 1, & \text{if } a^2 = 1 \text{ and } b = 0 \\ \frac{-1-a+b^2}{1+a} \leq x \leq \frac{1-a-b^2}{1-a}, & \text{if } a^2+b^2 < 1. \end{cases}$$

Symbols for Theorem 7

$E := 1 + (a^2+b^2+c^2)$;

$P_{22} := I - \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix}$;

$x_1 := \frac{-c(ab+bc) \pm \sqrt{(1-a-b^2-c+ac)(1+a-b^2+c+ac)P_{13}}}{1-a^2-b^2}$;

$x_2 := \frac{-bc(a+b) - \sqrt{P_{13} \det P_{22}}}{1-a^2-b^2}$;

$x_3 := \frac{-bc(a+b) + \sqrt{P_{13} \det P_{22}}}{1-a^2-b^2}$;

$y_1 := -\sqrt{\left(1 - \frac{x^2}{a^2}\right)(1-x^2)}$;

$y_2 := \sqrt{\left(1 - \frac{x^2}{a^2}\right)(1-x^2)}$;

$y_3 := \frac{c^2(-a+a^3+2ab^2+b^2c) - (1-a^2-b^2)(1-a^2-b^2-c^2)}{(1-a^2-b^2)^2}$;

$y_4 := \frac{c^2(-a+a^3+2ab^2+b^2c) + (1-a^2-b^2)(1-a^2-b^2-c^2)}{(1-a^2-b^2)^2}$;

$y_5 := -\frac{(1-a^2-b^2-c^2)(bc+cx) - (ac+bx)(ab+bc+cx)}{-abc-bc^2-x+a^2x+b^2x}$;

$y_6 := \frac{-1-a+b^2-c-ac+c^2+c^3-2bcx+x^2+ax^2}{1+a-b^2+c+ac}$;

$y_7 := \frac{1-a-b^2-c+ac-c^2+c^3-2bcx-x^2+ax^2}{1-a-b^2-c+ac}$.

Theorem 7

Let $H_3 := H(a,b,c;x,y) = \begin{pmatrix} a & b & c \\ b & c & x \\ c & x & y \end{pmatrix} \in M_3(\mathbb{R})$.

If H_3 is well-posed, then H_3 has a contractive completion.

Moreover, the exact values x and y given by

$$\begin{cases} x = 0 \text{ and } -1 \leq y \leq 1, & \text{if } E = 0,\, b \neq 0,\, c = 0 \\ |x| \leq |a| \text{ and } y_1 \leq y \leq y_2, & \text{if } E = 0,\, b = c = 0 \\ x = -\frac{b(a+c)}{c} \text{ and } y = \frac{b^2(a+c)-ac^2}{c^2}, & \text{if } E = 0,\, c \neq 0 \\ x = -\frac{bc(a+c)}{1-a^2-b^2} \text{ and } y_3 \leq y \leq y_4, & \text{if } E > 0,\, \det P_{22} = 0 \\ \{x = x_1,\, y = y_5\} \text{ or} & \text{if } E > 0,\, \det P_{22} > 0. \\ \{x_2 \leq x \leq x_3,\, y_6 \leq y \leq y_7\} & \end{cases}$$

Proof of Theorem 7

Observe that H_3 has a contractive completion if and only

if there exist x and y such that

$$P_{33}(x,y) := I - H_3^* H_3 =$$

$$\begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cx & -ac-bx-cy \\ -ab-bc-cx & 1-b^2-c^2-x^2 & -bc-cx-xy \\ -ac-bx-cy & -bc-cx-xy & 1-c^2-x^2-y^2 \end{pmatrix} \geq 0.$$

Case 1: $\{a,b,c\}$ is extremal $c \neq 0$.

In order to $P_{33}(x,y) \geq 0$, we have $-ab-bc-cx = -ac-bx-cy = 0$.

These equations define $x = -\frac{b(a+c)}{c}$, and $y = \frac{b^2(a+c)-ac^2}{c^2}$.

Since H_3 is well-posed, $\det P_{22} = a^2c^2 - b^2(a+c)^2 \geq 0$.

Thus,

$$P_{33}\left(-\frac{b(a+c)}{c}, \frac{b^2(a+c)-ac^2}{c^2}\right)$$

$$= \frac{a^2c^2-b^2(a+c)^2}{c^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{b}{c} \\ 0 & -\frac{b}{c} & \frac{b^2}{c^2} \end{pmatrix} \geq 0.$$

Case 2: $\{a,b,c\}$ is extremal and $c=0$.

In order to $P_{33}(x,y) \geq 0$, we have $ab = bx = 0$.

In this case,

$$P_{33}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2-x^2 & -xy \\ 0 & -xy & 1-x^2-y^2 \end{pmatrix}.$$

If $b \neq 0$, then $a = x = 0$ and hence $P_{33}(0,y) \geq 0 \Longleftrightarrow -1 \leq y \leq 1$.

If $b = 0$, then $P_{33}(x,y) \geq 0$ if and only if $|x| \leq |a|$ and $y_1 \leq y \leq y_2$.

Thus H_3 has a contractive completion.

Recall that for a $m \times n$ matrix A, a *Moore-Penrose inverse* of A is defined as a matrix as a $n \times m$ matrix $A^\dagger$ satisfying all of the following four conditions:

(i) $AA^\dagger A = A$; (ii) $A^\dagger AA^\dagger = A^\dagger$; (iii) $\left(AA^\dagger\right)^* = AA^\dagger$;

(iv) $\left(A^\dagger A\right)^* = A^\dagger A$.

Lemma 8: Let $P \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a finite matrix.

Then $P \geq 0$ if and only if the following conditions hold:

(i) $A \geq 0$;

(ii) $\operatorname{ran} B \subseteq \operatorname{ran} A$

(where, for a matrix T, ran T means the range of T as an operator); and

(iii) $C \geq B^* A^\dagger B$, where $A^\dagger$ is the Moore-Penrose inverse of A.

2.2 Moore-Penrose Inverse

An example of the Moore-Penrose inverse:

For $A := \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $A^\dagger = \begin{pmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{pmatrix}$

■ Let $P \equiv \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & 1 \end{pmatrix} \end{pmatrix} =: \begin{pmatrix} D & E \\ E^* & F \end{pmatrix}.$

An example of the Moore-Penrose inverse: continue

■ $P \geq 0 \Longleftrightarrow F \geq E^* D^\dagger E$

$$\Longleftrightarrow \begin{pmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & 1 \end{pmatrix} - \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{21}{25} & \frac{67}{100} \\ \frac{67}{100} & \frac{24}{25} \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \det \begin{pmatrix} \frac{21}{25} & \frac{67}{100} \\ \frac{67}{100} & \frac{24}{25} \end{pmatrix} = \frac{143}{400} \geq 0.$$

Case 3: $\{a, b, c\}$ is not extremal and $\det P_{22} = 0$.

Let $d_{11} := 1 - b^2 - c^2 - x^2 - \frac{(ab+bc+cx)^2}{1-a^2-b^2-c^2}$;

$d_{12} := -(bc + cx + xy) - \frac{(ab+bc+cx)(ac+bx+cy)}{1-a^2-b^2-c^2}$;

$d_{22} := 1 - c^2 - x^2 - y^2 - \frac{(ac+bx+cy)^2}{1-a^2-b^2-c^2}$.

By Lemma 8(iii), H_3 admits a contractive completion

if and only if there exist x and y such that

$$D \equiv \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix} \geq 0.$$

Since $\det P_{22} = 0$, $d_{11} \leq 0$ for every x.

Indeed, for every x,

$$\begin{aligned} &1 - b^2 - c^2 - x^2 - \frac{(ab+bc+cx)^2}{1-a^2-b^2-c^2} \\ &= -\frac{(1-a^2-b^2)x^2 + 2bc(a+c)x + c^2(1-b^2-c^2)}{1-a^2-b^2-c^2} \\ &= -\frac{(1-a^2-b^2)\left(x + \frac{bc(a+c)}{1-a^2-b^2}\right)^2}{1-a^2-b^2-c^2} \leq 0. \end{aligned}$$

Thus, the only possible x is $x = -\frac{bc(a+c)}{1-a^2-b^2}$ and in the case we can choose y in the interval $y_3 \leq y \leq y_4$.

Case 4: $\{a,b,c\}$ is not extremal and $\det P_{22} \neq 0$.

By Lemma 8(ii), we have that H_3 admits a contractive completion if and only if $D \geq 0$.

Since $\det P_{22} \cdot P_{13} \geq 0$,

if $d_{11} = 0$, that is $x = x_1 := \frac{-c(ab+bc) \pm \sqrt{(1-a-b^2-c+ac)(1+a-b^2+c+ac)P_{13}}}{1-a^2-b^2}$,

then we have

$$D \geq 0 \Longrightarrow d_{12} = 0 \text{ and } d_{22} \geq 0.$$

Note that

$$d_{12} = 0 \Longleftrightarrow y = y_5.$$

Thus, when $x = x_1$, we have

$$D \geq 0 \Longleftrightarrow y = y_5 \text{ and } d_{22} \geq 0.$$

Because $x = x_1$ and $y = y_5$ imply $d_{22} \geq 0$,

H_3 admits a contractive completion if and only if $x = x_1$ and $y = y_5$.

Let $d_{11} > 0$.

Then, the Moore-Penrose inverse of

$$\begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cx \\ -ab-bc-cx & 1-b^2-c^2-x^2 \end{pmatrix} \text{ is}$$

$$\begin{pmatrix} \frac{1-b^2-c^2-x^2}{(1-a^2-b^2-c^2)d_{11}} & \frac{ab+bc+cx}{(1-a^2-b^2-c^2)d_{11}} \\ \frac{ab+bc+cx}{(1-a^2-b^2-c^2)d_{11}} & \frac{1-a^2-b^2-c^2}{(1-a^2-b^2-c^2)d_{11}} \end{pmatrix}.$$

Thus, by Lemma 8(ii) again, we have that H_3 admits a contractive completion if and only if $d_{11}d_{22} \geq d_{12}^2$.

Since $y_7 - y_6 = \frac{2\det P_{23}}{\det P_{22}} \geq 0$, we get that

$$D \quad \geq \quad 0 \Longleftrightarrow d_{11} \geq 0 \text{ and } d_{11}d_{22} \geq d_{12}^2$$

$$\Longleftrightarrow \quad x_2 \leq x \leq x_3 \text{ and } y_6 \leq y \leq y_7.$$

Thus, if $a^2 + b^2 + c^2 < 1$ and $\det P_{22} \neq 0$, then we have

$$\{x = x_1, y = y_5\} \text{ or } \{x_2 \leq x \leq x_3, y_6 \leq y \leq y_7\}.$$

Therefore, by the argument above, we have that H_3 has a contractive completion.

Lemma 9

If $\begin{pmatrix} A \\ C \end{pmatrix}$ and $\begin{pmatrix} A & B \end{pmatrix}$ are rectangular contractions,

then there exists a matrix D such that the matrix

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a contraction as well.

Theorem 10 Assume $d = 0$.

Then Problem 1 is soluble for H_4 if and only if $b(a+c) = 0$.

Proof of Theorem 10

Lemma 9 implies that Problem 1 is soluble for H_4

if and only if there exist x such that we simultaneously have

$$\left\| \begin{pmatrix} a & b & c & 0 \\ b & c & 0 & x \end{pmatrix} \right\| \leq 1 \text{ and } \left\| \begin{pmatrix} a & b & c \\ b & c & 0 \\ c & 0 & x \end{pmatrix} \right\| \leq 1.$$

By a direct calculation, we have

$$\left\| \begin{pmatrix} a & b & c & 0 \\ b & c & 0 & x \end{pmatrix} \right\| \leq 1$$

$$\iff \begin{pmatrix} 0 & -b(a+c) \\ -b(a+c) & a^2 - x^2 \end{pmatrix} \geq 0 \iff \begin{cases} x^2 \leq a^2 \\ b(a+c) = 0. \end{cases}$$

We see at once that $b(a+c) = 0$ is a necessary condition for solubility.

On the other hand, if $b(a+c)=0$, then it suffices to choose $|x| \leq |a|$ to ensure that $\left\| \begin{pmatrix} a & b & c & 0 \\ b & c & 0 & x \end{pmatrix} \right\| \leq$ 1.

Looking now at the 3×3 matrix, we have

$$\begin{aligned} &\left\| \begin{pmatrix} a & b & c \\ b & c & 0 \\ c & 0 & x \end{pmatrix} \right\| \leq 1 \\ \iff &\begin{pmatrix} 0 & -b(a+c) & -c(a+x) \\ -b(a+c) & a^2 & -bc \\ -c(a+x) & -bc & a^2+b^2-x^2 \end{pmatrix} \geq 0 \end{aligned} \tag{2.1}$$

which is equivalent to the conditions

$$\begin{cases} b(a+c)=0 \\ c(a+x)=0 \\ \begin{pmatrix} a^2 & -bc \\ -bc & a^2+b^2-x^2 \end{pmatrix} \geq 0. \end{cases} \tag{2.2}$$

Thus, if $c=0$, then these conditions reduce to $ab=0$ and $x^2 \leq 1$,

so choosing $x^2 \leq a^2$ fulfills condition 1.

On the other hand, if $c \neq 0$, then we must choose $x=-a$ to meet and with this choice we also have condition 2.

From the above analysis, it follows that Problem 1 is soluble for H_4 if and only if $b(a+c)=0$.

2.3 Problem

Recall:

■ Conjecture (Curto, Lee, JYoon (JMP 12): The solution set

$$\left\{(x,y,z) \in \mathbb{R}^3 : \text{the Problem 1 is soluble for } H_4\right\}$$

is a prism in $\mathbb{R}^3$.

■ Result (Kim, Yoo, JYoon, (JKMS 15)): The solution set

$$\left\{(x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for } H_4\right\}$$

needs not a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2 \neq 1$

Based on Theorem 10, we try to find examples which show that the set

$$\left\{(x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for } H_4\right\}$$

needs NOT a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2 = 1$.

Given real numbers $a_1, a_2, \cdots, a_n$, find real numbers $x_1, \cdots, x_{n-1}$ such that

$$H(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$$

is contractive.

2.4 Counter Example

A counter example to show that the solution set

$$\left\{(x,y,z)\in\mathbb{R}^3 : \text{Problem 1 is soluble for} H_4\right\}$$

is a prism in $\mathbb{R}^3$, the extremal case.

Proof:

Let H be a 4×4 Hankel matrix where only the upper left triangular values are known.

$$H=\begin{bmatrix} a & \frac{1}{2}\sqrt{1-4a^2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2}\sqrt{1-4a^2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & x \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & x & y \\ \frac{1}{\sqrt{2}} & x & y & z \end{bmatrix}$$

Lets represent H as four 2×2, now we have

$$H=\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

where

$$A=\begin{bmatrix} a & \frac{1}{2}\sqrt{1-4a^2} \\ \frac{1}{2}\sqrt{1-4a^2} & \frac{1}{2} \end{bmatrix} B=\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & x \end{bmatrix} C=\begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

Lets calculate A^{-1} since we will need it next.

$$
\begin{aligned}
A^{-1} &= \frac{1}{det(A)} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{1-4a^2} \\ -\frac{1}{2}\sqrt{1-4a^2} & a \end{bmatrix} \\
&= \frac{1}{a^2+\frac{a}{2}-\frac{1}{4}} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{1-4a^2} \\ -\frac{1}{2}\sqrt{1-4a^2} & a \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{4a^2+2a-1} & -\frac{2\sqrt{1-4a^2}}{4a^2+2a-1} \\ -\frac{2\sqrt{1-4a^2}}{4a^2+2a-1} & \frac{4a}{4a^2+2a-1} \end{bmatrix}
\end{aligned}
$$

Now, let $D = C - BA^{-1}B \geq 0$. which gives us

$$
\begin{aligned}
&D = C - BA^{-1}B \\
&= \begin{bmatrix} x & y \\ y & z \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & x \end{bmatrix} \begin{bmatrix} \frac{2}{4a^2+2a-1} & -\frac{2\sqrt{1-4a^2}}{4a^2+2a-1} \\ -\frac{2\sqrt{1-4a^2}}{4a^2+2a-1} & \frac{4a}{4a^2+2a-1} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & x \end{bmatrix} \\
&= \begin{bmatrix} \frac{8a^2x+4ax-2x-4a-1+2\sqrt{2}\sqrt{-4a^2+1}}{8a^2+4a-2} & \frac{4\sqrt{2}a^2y+2\sqrt{2}ay-\sqrt{2}y-4ax+x\sqrt{2}\sqrt{-4a^2+1}-1+\sqrt{2}\sqrt{-4a^2+1}}{\sqrt{2}(4a^2+2a-1)} \\ \frac{8a^2y+4ay-2y-4\sqrt{2}ax+2\sqrt{-4a^2+1}-\sqrt{2}+2x\sqrt{-4a^2+1}}{8a^2+4a-2} & \frac{4a^2z+2az-z-4ax^2+2x\sqrt{2}\sqrt{-4a^2+1}-1}{4a^2+2a-1} \end{bmatrix} \\
&\geq 0
\end{aligned}
$$

Lets also check that this matrix is extremal by verifying that $a^2+b^2+c^2+d^2 = 1$ For $a = a, b = \frac{1}{2}\sqrt{1-4a^2}, c = \frac{1}{2}, d = \frac{1}{\sqrt{2}}$, we have

$$
\begin{aligned}
a^2+b^2+c^2+d^2 &= 1 \\
a^2+(\frac{1}{2}\sqrt{1-4a^2})^2+(\frac{1}{2})^2+(\frac{1}{\sqrt{2}})^2 &= 1 \\
a^2+\frac{1}{4}(1-4a^2)+\frac{1}{4}+\frac{1}{2} &= 1 \\
a^2+\frac{1}{4}-a^2+\frac{1}{4}+\frac{1}{2} &= 1 \\
1 &= 1
\end{aligned}
$$

Thus, we see that H is extremal. Now lets check to see if H is a contraction by verifying that $I - H^*H \geq 0$ since $||H|| \leq 1$ iff $I - H^*H \geq 0$. Now we have

$$
\begin{aligned}
H_2 &= I - H^*H \\
&= \begin{bmatrix} 1-a^2 & \frac{1}{4}(-1+4a^2) & -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4}(-1+4a^2) & \frac{3}{4} & -\frac{1}{2} & -x^2 \\ -\frac{1}{4} & -\frac{1}{2} & 1-x^2 & -y^2 \\ -\frac{1}{2} & -x^2 & -y^2 & 1-z^2 \end{bmatrix}
\end{aligned}
$$

where a is $0 < a < \frac{1}{2}$. Since $0 < a < \frac{1}{2}$, lets set $a = \frac{2}{5}$ and $x = 0$ where $0 \leq y \leq \frac{1}{2}$ and take the determinate of $H_2 > 0$, we have

$$
\begin{aligned}
2 - 3y^4 - 2z^2 &> 0 \\
2 - 3y^4 &> 2z^2 \\
z^2 &< 1 - \frac{3}{2} \\
z^2 &\leq \frac{341 - 340y^2 - 956y^4}{541} \\
-\sqrt{\frac{341 - 340y^2 - 956y^4}{541}} \leq z &\leq \sqrt{\frac{341 - 340y^2 - 956y^4}{541}}
\end{aligned}
$$

Since z is positive, we will only consider

$$z \leq \sqrt{\frac{341 - 340y^2 - 956y^4}{541}}$$

Since z is bounded by $\sqrt{\frac{341-340y^2-956y^4}{541}}$, we see that it is a curve, not a prism. Thus, we have shown that

$$\left\{(x,y,z) \in \mathbb{R}^3 : \text{Problem 1 is soluble for } H_4\right\}$$

needs NOT a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2=1$.

■

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BIOGRAPHICAL SKETCH

Manuel A. Villarreal Jr. was born in McAllen Texas 1987 to parents Manuel and Elizabeth Villarreal. Manuel's father was in the ARMY so he was raised in many places such as South Carolina, North Carolina, Germany, Colorado and Texas. At an early age, Manuel was interested in outer space and our place in it. Manuel would later get accepted to The Science Academy of South Texas High School where he realized that he had a love for mathematics. After high school, Manuel started his associates degree in math at South Texas College. While at South Texas College, he worked for the tutoring center and tutored math. After South Texas College, he transferred to The University of Texas Pan American where he received a Bachelors in Science with a concentration in pure mathematics. During his time at The University of Texas Pan American, he joined The Society of Industrial and Applied Mathematics where he served as president for 2 years and was involved with The Experimental Algebra and Geometry Lab doing research and community outreach. Manuel then went on to study for a Masters in Science with a concentration in Pure Math. During his masters, he did research with Dr. Yoon in functional analysis and held volunteer review sessions for undergraduate students while working as a graduate teaching assistant.

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