3-29-2023

Lacunary eta quotients with identically vanishing coefficients

Timothy Huber
The University of Texas Rio Grande Valley, timothy.huber@utrgv.edu

James McLaughlin

Dongxi Ye

Recommended Citation

Huber, Timothy, James McLaughlin, and Dongxi Ye. "Lacunary eta quotients with identically vanishing coefficients." International Journal of Number Theory 19, no. 07 (2023): 1639-1670. https://doi.org/10.1142/S179304212350080X

LACUNARY ETA QUOTIENTS WITH IDENTICALLY VANISHING COEFFICIENTS

TIM HUBER, JAMES MCLAUGHLIN AND DONGXI YE

ABSTRACT. For $|q| < 1$, define $f_i = \prod_{n=1}^{\infty}(1 - q^{in})$, and let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}.$$

For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n)q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then for each pair it is shown that $a(n)$ vanishes if and only if $b(n)$ vanishes. In each case, a criterion is given which states precisely when $a(n) = b(n) = 0$. Moreover, for the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1}\right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6}\right) \right\}$$

it is shown that $a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$.

1. INTRODUCTION

The work in the present paper was motivated by a result of Han and Ono [4] about vanishing coefficients in the series expansion of a specific pair of eta quotients. To state their result, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$f_1^8 =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_3^3}{f_1} =: \sum_{n=0}^{\infty} b(n)q^n, \qquad f_i := \prod_{n=1}^{\infty}(1 - q^{in}), \quad i \in \mathbb{Z}^+. \tag{1.1}$$

Han and Ono proved the following result.

Theorem 1.1. (*Han and Ono,* [4, Theorem 1.4, page 307]) *Assuming the notation above, we have that*

$$a(n) = 0 \quad \textit{if and only if} \quad b(n) = 0. \tag{1.2}$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $ord_p(3n + 1)$ is odd for some prime $p \equiv 2 \pmod 3$.

If $A(q)$ and $B(q)$ are two functions for which the coefficients in the series expansions satisfy the condition (1.2) in the theorem, then for ease of discussion, we say that *the coefficients vanish identically*, or that $A(q)$ and $B(q)$ have *identically vanishing coefficients*.

Date: January 7, 2023.

2000 *Mathematics Subject Classification.* 11F11; 11B65 .

Key words and phrases. modular forms, infinite product, eta quotient, vanishing coefficients, lacunary q-series, Dedekind eta function, theta series .

Dongxi Ye is supported by the Natural Science Foundation of China (Grant No. 11901586), the Natural Science Foundation of Guangdong Province (Grant No. 2019A1515011323), the Fundamental Research Project of Guangzhou (Grant No. 202102080195) and the Fundamental Research Funds for the Central Universities, Sun Yat-sen University (Grant No. 22qntd2901).

Theorem 1.1 motivated the second author to investigate experimentally if similar results held for other pairs of eta quotients. What was discovered as a result of these computer algebra experiments is summarized as follows.

Let $(A(q), B(q))$ be any of the pairs

$$\left\{ \left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right) \right\}. \tag{1.3}$$

For any such pair $(A(q), B(q))$, define the sequences $\{a(n)\}$ and $\{b(n)\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a(n) q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b(n) q^n. \tag{1.4}$$

Then, for each pair, experiment suggested $a(n)$ vanishes if and only if $b(n)$ vanishes. For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1}\right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6}\right) \right\}$$

experiment suggested that $a(n) = b(n) = 0$ if $12n + 13$ satisfies a criteria of Serre for $a(n) = 0$ ([8] or see Lemma 2.1). By determining conditions for when the coefficients vanish, Serre proved that all of the first components of each pair in (1.3) and (1.4) are *lacunary*, with series expansions $\sum_{n=0}^{\infty} c(n) q^n$ satisfying

$$\lim_{x \to \infty} \frac{|\{0 \le n \le x \mid c(n) = 0\}|}{x} = 1.$$

In this paper we extend Serre's work [8] to prove all of the statements suggested above by experimental evidence are indeed true and that the vanishing of coefficients in the second component of each pair aligns with Serre's criterion for the vanishing of the coefficients of the first components.

We next describe some of the theory of modular forms needed for the proofs of these conjectures. Let f_i be as defined at (1.1). In [8], Serre proved that for even positive integers s, f_1^s is lacunary if and only if $s \in \{2, 4, 6, 8, 10, 14, 26\}$. In each of these cases, Serre provided a characterization of the vanishing coefficients $c(n)$ in terms of the factorization of coefficients in a dilation of the appropriate power of the Dedekind eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z} \text{ with } \mathrm{Im}(z) > 0.$$

Serre's criteria and more general characterizations for lacunary holomorphic modular forms my be used to identify quotients of eta functions that vanish according to the conditions Serre derived for f_1^s. In particular, our work makes use of the result of Ribet [6, c.f. Theorems (4.4) and (4.5)] that a newform has complex multiplication (CM) by an imaginary quadratic field K if and only if it is representable as a theta series corresponding to Hecke characters associated to the field K.

A critical first step in proving the lacunarity and common vanishing of the eta quotients involves demonstrating the second components of each pair are CM newforms by some imaginary quadratic extension of $\mathbb{Q}$. This is done by representing the eta quotients in terms of CM newforms from the L-functions and Modular Forms Database (LMFDB). The LMFDB does not give us explicit expressions for the CM newforms involved, but it does help us make guesses to compute these CM newforms. Based on the speculation that the coefficients of the eta quotients vanish simultaneously with the even eta powers, the LMFDB was used to collect CM newforms of appropriate weight and level. In each case, the Fourier coefficients of the eta quotients are shown to agree up to the Sturm bound with linear combinations of the CM newforms. Ribet's result allows for expansion of the eta quotients as linear combinations of Hecke theta series. The theta representations serve as bridges

between the eta-quotients and CM newforms, so that the multiplicativity properties and recursive formulas satisfied by the coefficients can be applied to prove conditions for vanishing.

For the cases considered here, these theta series have the form $\sum_{m,n}(m+n\sqrt{-D})^k q^{m^2+Dn^2}$, where D is a positive integer and the m and n run over all the integers or certain arithmetic progressions. Since the equation $p=m^2+Dn^2$, where p is a prime, has just finitely many solutions, the coefficient of q^p in each of the theta series can be determined explicitly in terms of the integers m and n in the representation $p=m^2+Dn^2$. Since newforms are Hecke eigenforms, the vanishing of the coefficients follows from the recursion formulas for the coefficients at powers of primes. The multiplicative property of the coefficients of the newform is then used to derive information about the coefficient of q^{jn+t} in the series expansion of these forms from the prime factorization of $jn+t$, and hence about the coefficient of q^{jn+t} in the series expansion of $\eta^u(sjz)/\eta^v(rjz)$. This provides information about when this coefficient vanishes, A proof that the coefficients of the products in each pair vanish simultaneously is realized by showing the conditions for vanishing are precisely the criteria found by Serre for coefficients of the even powers of the eta function.

Acknowledgements. The authors thank the anonymous referee for his/her useful comments, suggestions and corrections.

2. Preliminary results

2.1. **Review of CM newforms.** Let $F(z)=\sum_{n=1}^{\infty}a(n)q^n$ with $q=e^{2\pi iz}$ be a newform of weight $k>1$ for some $\Gamma_0(N)$ with some Nebentypus ε, and let ϕ denote some Dirichlet character modulo d. Then $F(z)$ is said to have CM by ϕ if $\phi(p)a(p)=a(p)$ for all but finitely many primes p. By a result of Deligne and Serre [2, (6.3)], it is known that ϕ must be a quadratic character associated with some quadratic field K. In particular, when K is imaginary, it is unique, in which case we call $F(z)$ a CM newform by K. One of the incredible applications of CM newforms is the characterization of the lacunarity of a cusp form proved by Serre [7, Theorem 17] stating that a cusp form of weight $k>1$ is lacunary if and only if it is a linear combination of CM newforms, and this leads Serre to obtain the following explicit characterization of the vanishing of the Fourier coefficients of the even powers of the Dedekind eta function that are lacunary.

Lemma 2.1. [8, Section 2] *Let $f_i=f_i(q)$ be defined as in Section 1.*

(1) If

$$qf_6^4=\sum_{n=0}^{\infty}a(n)q^{6n+1},$$

then we have that $a(n)=0$ if and only if $6n+1$ has a prime factor $p\equiv -1 \pmod 3$ with odd exponent.

(2) If

$$qf_4^6=\sum_{n=0}^{\infty}a(n)q^{4n+1},$$

then we have that $a(n)=0$ if and only if $4n+1$ has a prime factor $p\equiv -1 \pmod 4$ with odd exponent.

(3) If

$$q^5f_{12}^{10}=\sum_{n=0}^{\infty}a(n)q^{12n+5},$$

then we have that $a(n)=0$ if and only if $12n+5$ has a prime factor $p\equiv -1 \pmod 4$ with odd exponent.

(4) *If*

$$q^7 f_{12}^{14} = \sum_{n=0}^{\infty} a(n) q^{12n+7},$$

then we have that $a(n) = 0$ *if and only if* $12n+7$ *has a prime factor* $p \equiv -1 \pmod 3$ *with odd exponent.*

(5) *If*

$$q^{13} f_{12}^{26} = \sum_{n=0}^{\infty} a(n) q^{12n+13},$$

then we have that $a(n) = 0$ *if either of the following holds:*

(a) $12n+13$ *has a prime factor* $p_1 \equiv -1 \pmod 3$ *with odd exponent* *and* *a prime* $p_2 \equiv -1 \pmod 4$ *with odd exponent (it may be that* $p_1 = p_2$*),*

(b) $12n+13$ *is a square and all prime factors* p *satisfy* $p \equiv -1 \pmod{12}$.

Moreover, in [6], Ribet uses the theory of Galois representations to show that such a newform arises in the form of a theta function associated with a so-called Hecke character associated with K which is defined as follows. Let $K = \mathbb{Q}[\sqrt{-d}]$ be an imaginary quadratic field of discriminant $-d < 0$. Denote by $\mathcal{O}_K$ the ring of integers of K. Take a nonzero integral ideal $\mathfrak{m}$ of $\mathcal{O}_K$. Let $I_K(\mathfrak{m})$ be the group of fractional ideals generated by all integral ideals coprime to $\mathfrak{m}$. For $k > 1$, a Hecke character $\psi_{\mathfrak{m}}$ modulo $\mathfrak{m}$ of infinite type $\left(\frac{\alpha}{|\alpha|}\right)^{k-1}$ is a character on $I_K(\mathfrak{m})$ such that

$$\psi_{\mathfrak{m}}(\alpha\mathcal{O}_K) = \left(\frac{\alpha}{|\alpha|}\right)^{k-1}$$

for any $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\mathfrak{m}}$, i.e., $v_{\mathfrak{p}}((\alpha-1)\mathcal{O}_K) \geq v_{\mathfrak{p}}(\mathfrak{m})$ whenever $\mathfrak{p}|\mathfrak{m}$. Clearly, if we write $\psi_{f,\mathfrak{m}}(\alpha\mathcal{O}_K) = \psi_{\mathfrak{m}}(\alpha\mathcal{O}_K)|\alpha|^{k-1}/\alpha^{k-1}$, this defines a character on $(\mathcal{O}_K/\mathfrak{m})^\times$, and therefore on $(\mathbb{Z}/N(\mathfrak{m})\mathbb{Z})^\times$ called the finite part of $\psi_{\mathfrak{m}}$. Now extend $\psi_{\mathfrak{m}}$ to I_K by setting $\psi_{\mathfrak{m}}(\mathfrak{a}) = 0$ for $\mathfrak{a}$ with $v_{\mathfrak{p}}(\mathfrak{a}) \neq 0$ for some $\mathfrak{p}|\mathfrak{m}$. Then Ribet [6, Theorems (4.4) and (4.5)] showed that a CM newform $F(z)$ by K satisfies

$$F(z) = \sum_{\substack{\mathfrak{a} \in I_K \\ \textit{integral}}} \psi_{\mathfrak{m}}(\mathfrak{a}) N(\mathfrak{a})^{\frac{k-1}{2}} q^{N(\mathfrak{a})}$$

for some nonzero integral ideal $\mathfrak{m}$, which is of level $\Gamma_0(dN(\mathfrak{m}))$ with character $\chi_d\psi_{f,\mathfrak{m}}$. By such a construction, it is clear that $a(mn) = a(m)a(n)$ for $(m,n) = 1$, $a(p^\ell) = a(p)a(p^{\ell-1}) - (\chi_d\psi_{f,\mathfrak{m}})(p)p^{k-1}a(p^{\ell-2})$, and $\overline{F}(z) := \sum_{n=1}^{\infty} \overline{a(n)} q^n$ is also a CM newform provided that $F(z) = \sum_{n=1}^{\infty} a(n)q^n$ is a CM newform.

In fact, one may rewrite this series as a theta type series over elements of $\mathcal{O}_K$ as follows. First suppose the ideal class group $\mathrm{Cl}_K = I_K/P_K = \{[\mathfrak{b}]\}$ with $\mathfrak{b}$'s all integral. Then the set of integral ideals $\mathfrak{a} \in [\mathfrak{b}]$ with $N(\mathfrak{a}) = n$ is in bijection with $\alpha \in \mathfrak{b}/\mathcal{O}_K^\times$ with norm $N(\alpha) = nN(\mathfrak{b})$, since if $\mathfrak{a} = x\mathfrak{b}$ then $\overline{\mathfrak{a}}\mathfrak{b} = \overline{x}N(\mathfrak{b}) \subset \mathfrak{b}$, and one can take $\alpha = \overline{x}N(\mathfrak{b})$, and on the other hand, for an $\alpha \in \mathfrak{b}$ with $\frac{N(\alpha)}{N(\mathfrak{b})} = n$, take $\mathfrak{a} = \frac{\overline{\alpha}}{N(\mathfrak{b})}\mathfrak{b}$ which is integral since $\mathfrak{b}|\alpha\mathcal{O}_K$ by assumption. Therefore, the series above can be rewritten as

$$\begin{aligned} &\sum_{\substack{\mathfrak{a} \in I_K \\ \textit{integral}}} \psi_{\mathfrak{m}}(\mathfrak{a}) N(\mathfrak{a})^{\frac{k-1}{2}} q^{N(\mathfrak{a})} \\ &= \sum_{[\mathfrak{b}] \in \mathrm{Cl}_K} \sum_{n=1}^{\infty} \sum_{\substack{\mathfrak{a} \in [\mathfrak{b}] \\ N(\mathfrak{a}) = n}} \psi_{\mathfrak{m}}(\mathfrak{a}) N(\mathfrak{a})^{\frac{k-1}{2}} q^{N(\mathfrak{a})} \end{aligned}$$

$$= \sum_{[\mathfrak{b}]\in \mathrm{Cl}_K} \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathfrak{b}/O_K^\times \\ N(\alpha)=nN(\mathfrak{b})}} \psi_{\mathfrak{m}}(\overline{\alpha}N(\mathfrak{b})^{-1}\mathfrak{b}) \frac{N(\alpha)^{\frac{k-1}{2}}}{N(\mathfrak{b})^{\frac{k-1}{2}}} q^{\frac{N(\alpha)}{N(\mathfrak{b})}}$$

$$= \sum_{[\mathfrak{b}]\in \mathrm{Cl}_K} \frac{\psi_{\mathfrak{m}}(N(\mathfrak{b})^{-1}\mathfrak{b})}{N(\mathfrak{b})^{\frac{k-1}{2}}} \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathfrak{b}/O_K^\times \\ N(\alpha)=nN(\mathfrak{b})}} \psi_{\mathfrak{m}}(\overline{\alpha}\mathcal{O}_K) N(\alpha)^{\frac{k-1}{2}} q^{\frac{N(\alpha)}{N(\mathfrak{b})}}$$

$$= \frac{1}{|\mathcal{O}_K^\times|} \sum_{[\mathfrak{b}]\in \mathrm{Cl}_K} \frac{\psi_{\mathfrak{m}}(N(\mathfrak{b})^{-1}\mathfrak{b})}{N(\mathfrak{b})^{\frac{k-1}{2}}} \sum_{n=1}^{\infty} \sum_{\substack{\alpha\in\mathfrak{b} \\ N(\alpha)=nN(\mathfrak{b})}} \psi_{f,\mathfrak{m}}(\overline{\alpha}) \alpha^{k-1} q^{\frac{N(\alpha)}{N(\mathfrak{b})}}.$$

In particular, when K is of class number 1, this can be reduced to the following formulation.

Corollary 2.2. *Let $K = \mathbb{Q}[\sqrt{-d}]$ be an imaginary quadratic field of discriminant $-d < 0$ and class number 1. Then for any nonzero integral ideal $\mathfrak{m}$ and a Hecke character $\psi_{\mathfrak{m}}$ modulo $\mathfrak{m}$,*

$$\sum_{\substack{\mathfrak{a}\in I_K \\ integral}} \psi_{\mathfrak{m}}(\mathfrak{a}) N(\mathfrak{a})^{\frac{k-1}{2}} q^{N(\mathfrak{a})} = \frac{1}{|\mathcal{O}_K^\times|} \sum_{[\beta]\in(\mathcal{O}_K/\mathfrak{m})^\times} \psi_{f,\mathfrak{m}}(\overline{\beta}) \sum_{\alpha\in\beta+\mathfrak{m}} \alpha^{k-1} q^{N(\alpha)}.$$

Therefore, when $F(z)$ is a CM newform by an imaginary quadratic field K of class number 1, it is a linear combination of the spherical theta series $\sum_{\alpha\in\beta+\mathfrak{m}} \alpha^{k-1}q^{N(\alpha)}$ as β runs over $(\mathcal{O}_K/\mathfrak{m})^\times$ for some nonzero integral ideal $\mathfrak{m}$.

The spherical theta series $\sum_{\alpha\in\beta+\mathfrak{m}} \alpha^{k-1}q^{N(\alpha)}$ are known to be holomorphic modular forms. The reader is referred to [1, Corollary 14.3.16] for more details.

In the following lemma, theta representations for certain CM newforms given in the LMFDB are specified. The newforms in the lemma are those that linearly interpolate the eta quotients. The newforms were identified from the LMFDB through a comprehensive search of CM newforms of appropriate weight and level that interpolate the eta-quotients. The theta function expansions for the newforms will later help with analyzing the vanishingness of the coefficients of the eta-quotients.

Lemma 2.3. *Let $q = e^{2\pi i z}$ with $\mathrm{Im}(z) > 0$. Then as functions in z,*

(1) the functions

$$S(q) = H_1 - H_2 - H_3 + H_4$$

and

$$\overline{S}(q) = H_1 - H_2 + H_3 - H_4$$

are CM newforms of weight 3 for $\Gamma_0(144)$ with Nebentypus $\left(\frac{-4}{\cdot}\right)$ by $K = \mathbb{Q}[\sqrt{-3}]$ labelled 144.3.g.c in the LMFDB, where

$$H_1 = \sum_{m,n} (-6n+1+(4m-2n)\sqrt{-3})^2 q^{(-6n+1)^2+3(4m-2n)^2},$$

$$H_2 = \sum_{m,n} (-6n+5+(4m-2n)\sqrt{-3})^2 q^{(-6n+5)^2+3(4m-2n)^2},$$

$$H_3 = \sum_{m,n} (-6n-2+(4m-2n+3)\sqrt{-3})^2 q^{(-6n-2)^2+3(4m-2n+3)^2},$$

$$H_4 = \sum_{m,n} (-6n+2+(4m-2n+3)\sqrt{-3})^2 q^{(-6n+2)^2+3(4m-2n+3)^2},$$

(2) *the functions*

$$S_1(q)=\frac{1}{2}\left(1+\sqrt{-3}\right)H_1+\frac{1}{2}\left(-1+\sqrt{-3}\right)H_2-H_3$$

and

$$S_2(q)=\frac{1}{2}\left(1-\sqrt{-3}\right)H_1+\frac{1}{2}\left(-1-i\sqrt{3}\right)H_2-H_3$$

are CM newforms of weight 4 *for* $\Gamma_0(108)$ *with trivial Nebentypus by* $K=\mathbb{Q}[\sqrt{-3}]$ *labelled 108.4.a.b and 108.4.a.c, respectively, in LMFDB, where*

$$H_1=\sum_{m,n}\left((6m-3n-1)+\sqrt{-3}(3n-2)\right)^3 q^{(6m-3n-1)^2+3(3n-2)^2},$$
$$H_2=\sum_{m,n}\left((6m-3n+1)+\sqrt{-3}(3n-2)\right)^3 q^{(6m-3n+1)^2+3(3n-2)^2},$$
$$H_3=\sum_{m,n}\left((6m-3n-1)+\sqrt{-3}(3n)\right)^3 q^{(6m-3n-1)^2+3(3n)^2},$$

(3) *the functions*

$$S_1(q)=iH_1-iH_2-iH_3+iH_4+H_5+H_6-H_7-H_8$$

and

$$S_2(q)=-iH_1+iH_2+iH_3-iH_4+H_5+H_6-H_7-H_8,$$

are CM newforms of weight 2 *for* $\Gamma_0(288)$ *with trivial Nebentypus by* $K=\mathbb{Q}[i]$ *labelled 288.2.a.a and 288.2.a.e, respectively, in LMFDB, where*

$$H_1=\sum_{m,n}(6m-6n+1+i(6m+6n+4))q^{(6m-6n+1)^2+(6m+6n+4)^2},$$
$$H_2=\sum_{m,n}(6m-6n+1+i(6m+6n-4))q^{(6m-6n+1)^2+(6m+6n-4)^2},$$
$$H_3=\sum_{m,n}(6m-6n+5+i(6m+6n+4))q^{(6m-6n+5)^2+(6m+6n+4)^2},$$
$$H_4=\sum_{m,n}(6m-6n+5+i(6m+6n-4))q^{(6m-6n+5)^2+(6m+6n-4)^2},$$
$$H_5=\sum_{m,n}(6m-6n+1+i(6m+6n))q^{(6m-6n+1)^2+(6m+6n)^2},$$
$$H_6=\sum_{m,n}(6m-6n+5+i(6m+6n))q^{(6m-6n+5)^2+(6m+6n)^2},$$
$$H_7=\sum_{m,n}(6m-6n-3+i(6m+6n+4))q^{(6m-6n-3)^2+(6m+6n+4)^2},$$
$$H_8=\sum_{m,n}(6m-6n-3+i(6m+6n-4))q^{(6m-6n-3)^2+(6m+6n-4)^2},$$

(4) *the functions*

$$S(q)=-H_1-H_2+\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_3+\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_4$$

$$
\begin{aligned}
&+\left(\frac{1}{2}-\frac{\sqrt{-3}}{2}\right)H_5+\left(-\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_6\\
&+H_7+H_8+\left(-\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_9\\
&-H_{10}-H_{11}+\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_{12}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{S}(q)=H_1+H_2&-\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_3-\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_4\\
&-\left(\frac{1}{2}-\frac{\sqrt{-3}}{2}\right)H_5-\left(-\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_6\\
&+H_7+H_8+\left(-\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_9\\
&-H_{10}-H_{11}+\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)H_{12}
\end{aligned}
$$

are CM newforms of weight 2 for $\Gamma_0(432)$ with $\left(\frac{3}{\cdot}\right)$ by $K=\mathbb{Q}[\sqrt{-3}]$ labelled 432.2.c.a in LMFDB, where

$$
\begin{aligned}
H_1&=\sum_{m,n}\left(12m-6n-\frac{1}{2}+\left(6n+\frac{3}{2}\right)\sqrt{-3}\right)q^{(12m-6n-1/2)^2+3(6n+3/2)^2},\\
H_2&=\sum_{m,n}\left(12m-6n+\frac{7}{2}+\left(6n+\frac{3}{2}\right)\sqrt{-3}\right)q^{(12m-6n+7/2)^2+3(6n+3/2)^2},\\
H_3&=\sum_{m,n}\left(12m-6n-\frac{1}{2}+\left(6n-\frac{5}{2}\right)\sqrt{-3}\right)q^{(12m-6n-1/2)^2+3(6n-5/2)^2},\\
H_4&=\sum_{m,n}\left(12m-6n-\frac{5}{2}+\left(6n-\frac{1}{2}\right)\sqrt{-3}\right)q^{(12m-6n-5/2)^2+3(6n-1/2)^2},\\
H_5&=\sum_{m,n}\left(12m-6n+\frac{7}{2}+\left(6n-\frac{5}{2}\right)\sqrt{-3}\right)q^{(12m-6n+7/2)^2+3(6n-5/2)^2},\\
H_6&=\sum_{m,n}\left(12m-6n-\frac{11}{2}+\left(6n+\frac{1}{2}\right)\sqrt{-3}\right)q^{(12m-6n-11/2)^2+3(6n+1/2)^2},\\
H_7&=\sum_{m,n}\left(12m-6n+1+(6n)\sqrt{-3}\right)q^{(12m-6n+1)^2+3(6n)^2},\\
H_8&=\sum_{m,n}\left(12m-6n+5+(6n)\sqrt{-3}\right)q^{(12m-6n+5)^2+3(6n)^2},\\
H_9&=\sum_{m,n}\left(12m-6n+\frac{7}{2}+\left(6n+\frac{1}{2}\right)\sqrt{-3}\right)q^{(12m-6n+7/2)^2+3(6n+1/2)^2},\\
H_{10}&=\sum_{m,n}\left(12m-6n-\frac{11}{2}+\left(6n+\frac{3}{2}\right)\sqrt{-3}\right)q^{(12m-6n-11/2)^2+3(6n+3/2)^2},
\end{aligned}
$$

$$H_{11} = \sum_{m,n} \left(12m - 6n + \frac{13}{2} + \left(6n - \frac{3}{2}\right)\sqrt{-3}\right) q^{(12m-6n+13/2)^2+3(6n-3/2)^2},$$
$$H_{12} = \sum_{m,n} \left(12m - 6n - \frac{7}{2} + \left(6n + \frac{1}{2}\right)\sqrt{-3}\right) q^{(12m-6n-7/2)^2+3(6n+1/2)^2},$$

(5) the functions

$$S_1(q) = \sum_{m,n} (-1)^n \left(3m + 1 + (2n - 3m)\sqrt{-3}\right)^3 q^{(3m+1)^2+3(2n-3m)^2},$$
$$\overline{S}_1(q) = \sum_{m,n} (-1)^{3m-n} \left(3m + 1 + (2n - 3m)\sqrt{-3}\right)^3 q^{(3m+1)^2+3(2n-3m)^2}$$

are CM newforms of weight 4 for $\Gamma_0(144)$ *with Nebentypus* $\left(\frac{3}{\cdot}\right)$ *by* $K = \mathbb{Q}[\sqrt{-3}]$*, which are respectively twists of 48.4.c.a and its conjugate by* $\left(\frac{-3}{\cdot}\right)$*, and*

$$S_2(q) = H_3 - iH_4 + \frac{(1+i)H_7}{\sqrt{2}} + \frac{(1-i)H_8}{\sqrt{2}} - \frac{(1+i)H_9}{\sqrt{2}} - \frac{(1-i)H_{10}}{\sqrt{2}}$$

and

$$\overline{S}_2(q) = H_3 - iH_4 - \frac{(1+i)H_7}{\sqrt{2}} - \frac{(1-i)H_8}{\sqrt{2}} + \frac{(1+i)H_9}{\sqrt{2}} + \frac{(1-i)H_{10}}{\sqrt{2}}$$

are CM newforms of weight 4 for $\Gamma_0(144)$ *with Nebentypus* $\left(\frac{3}{\cdot}\right)$ *by* $K = \mathbb{Q}[i]$ *labelled 144.4.c.a in LMFDB, where*

$$H_3 = \sum_{m,n} (6m + 1 + 6ni)^3 q^{(6m+1)^2+(6n)^2},$$
$$H_4 = \sum_{m,n} (6m + 3 + (6n - 2)i)^3 q^{(6m+3)^2+(6n-2)^2},$$
$$H_7 = \sum_{m,n} ((6m + 1)i + (6(m + 2n) - 2))^3 q^{(6m+1)^2+(6(m+2n)-2)^2},$$
$$H_8 = \sum_{m,n} ((6m + 1)i + (6(m + 2n) + 2))^3 q^{(6m+1)^2+(6(m+2n)+2)^2},$$
$$H_9 = \sum_{m,n} ((6m + 5)i + (6(m + 2n) + 2))^3 q^{(6m+5)^2+(6(m+2n)+2)^2},$$
$$H_{10} = \sum_{m,n} ((6m + 5)i + (6(m + 2n) - 2))^3 q^{(6m+5)^2+(6(m+2n)-2)^2},$$

(6) the functions

$$S_1(q) = H_3 - H_4 + iH_7 - iH_8$$

and

$$S_2(q) = H_3 - H_4 - iH_7 + iH_8$$

are CM newforms of weight 5 for $\Gamma_0(144)$ *with Nebentypus* $\left(\frac{-4}{\cdot}\right)$ *by* $K = \mathbb{Q}[i]$ *labelled 144.5.g.a and 144.5.g.b, respectively, in LMFDB, and*

$$S_3(q) = H_1 - H_2 - H_5 + H_6$$

and

$$\overline{S}_3(q) = H_1 - H_2 + H_5 - H_6$$

are CM newforms of weight 5 for $\Gamma_0(144)$ with Nebentypus $\left(\frac{-4}{\cdot}\right)$ by $K = \mathbb{Q}[\sqrt{-3}]$ labelled 144.5.g.e in LMFDB, where

$$H_1 = \sum_{m,n}(-6n+1+(4m-2n)\sqrt{-3})^4 q^{((-6n+1)^2+3(4m-2n)^2)},$$

$$H_2 = \sum_{m,n}(-6n+5+(4m-2n)\sqrt{-3})^4 q^{((-6n+5)^2+3(4m-2n)^2)},$$

$$H_3 = \sum_{m,n}(6m+1+6ni)^4 q^{(6m+1)^2+(6n)^2},$$

$$H_4 = \sum_{m,n}(6m+3+(6n-2)i)^4 q^{(6m+3)^2+(6n-2)^2},$$

$$H_5 = \sum_{m,n}(-6n-2+(4m-2n+3)\sqrt{-3})^4 q^{((-6n-2)^2+3(4m-2n+3)^2)},$$

$$H_6 = \sum_{m,n}(-6n+2+(4m-2n+3)\sqrt{-3})^4 q^{((-6n+2)^2+3(4m-2n+3)^2)},$$

$$H_7 = \sum_{m,n}(6m+1+(6n-2)i)^4 q^{(6m+1)^2+(6n-2)^2},$$

$$H_8 = \sum_{m,n}(6m+1+(6n+2)i)^4 q^{(6m+1)^2+(6n+2)^2}.$$

2.2. **Theta representations for eta-quotients.** The series in Lemma 2.1 are called eta quotients and are modular forms with character. In order to use the theory of modular forms to study these and other relevant quotients of f_i, we write them in terms of eta quotients.

Theorem 2.4 ([1, Proposition 5.9.2])**.** *Let $f = \prod_{m|N} \eta^{r_m}(mz)$, $r_m \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, with $k = \sum_{m|N} r_m/2 \in \mathbb{Z}$. Define M as the least common multiple of N and the denominator of $\sum_{m|N} r_m/(24m)$. Then $f \in M_k(\Gamma_0(M), \chi)$, where*

$$\chi(d) = \left(\frac{(-1)^k \prod_{m|N} m^{r_m}}{d}\right)$$

if and only if

$$\sum_{m|N} m r_m \equiv \sum_{m|N} (N/m) r_m \equiv 0 \pmod{24}.$$

In what follows, using Theorem 2.4 and Sturm's bound we find alternative representations in terms of CM newforms or theta functions for the eta-quotients considered in this work. Together with Lemma 2.3 and basic properties of newforms, these yield explicit formulations for the coefficients of those eta-quotients and show that the coefficients enjoy some multiplicativity and recursion relations, which serve as key ingredients in the verification of our observations.

Lemma 2.5. *Let $f_i = f_i(q)$ be defined as in Section 1. Then one has that*

(1)

$$q\frac{f_1^8}{f_2^2}(q^6) = q\frac{f_6^8}{f_{12}^2} = \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{-3}}\right)S(q) + \left(1-\frac{1}{\sqrt{-3}}\right)\overline{S}(q)\right]$$

and

$$q^7\frac{f_2^8}{f_1^2}(q^{12}) = q^7\frac{f_{24}^8}{f_{12}^2} = -\frac{1}{16\sqrt{-3}}S(q) + \frac{1}{16\sqrt{-3}}\overline{S}(q),$$

where $S(q)$ and $\overline{S}(q)$ are given as in Lemma 2.3 (1),

(2)

$$q\frac{f_1^{10}}{f_3^2}(q^6) = q\frac{f_6^{10}}{f_{18}^2} = \frac{1}{2}S_1(q) + \frac{1}{2}S_2(q),$$

where $S_1(q)$ and $S_2(q)$ are given as in Lemma 2.3 (2),

(3)

$$\frac{f_2^4}{f_1^2} = \sum_{m,n=0}^{\infty} q^{m(m+1)/2+n(n+1)/2},$$

(4)

$$q\frac{f_1^{14}}{f_2^4}(q^4) = q\frac{f_4^{14}}{f_8^4} = \sum_{\substack{m\equiv 1 \pmod 4\\ n\equiv 0 \pmod 2}} (m+in)^4 q^{m^2+n^2}$$

is the twist of the CM newform labelled 64.5.c.a in LMFDB by the quadratic character $\left(\frac{8}{\cdot}\right)$,

(5)

$$q^5\frac{f_2^6}{f_1^2}(q^{12}) = q^5\frac{f_{24}^6}{f_{12}^2} = -\frac{1}{8}S_1(q) + \frac{1}{8}S_2(q),$$

where $S_1(q)$ and $S_2(q)$ are given as in Lemma 2.3 (3),

(6)

$$q^7\frac{f_3^5}{f_1}(q^{12}) = q^7\frac{f_{36}^5}{f_{12}} = -\frac{1}{6\sqrt{-3}}S(q) + \frac{1}{6\sqrt{-3}}\overline{S}(q),$$

where $S(q)$ and $\overline{S}(q)$ are given as in Lemma 2.3 (4),

(7)

$$q^7\frac{f_2^8}{f_1^2}(q^{12}) = q^7\frac{f_{24}^5}{f_{12}^2} = \frac{i}{8\sqrt{3}}\left(\frac{S(q)-\overline{S}(q)}{2}\right),$$

where $S(q)$ and $\overline{S}(q)$ are given as in Lemma 2.3 (1),

(8)

$$q^{13}\frac{f_3^9}{f_1}(q^{12}) = q^{13}\frac{f_{36}^9}{f_{12}} = \frac{S_1(q)+\overline{S}_1(q)}{324} - \frac{S_2(q)+\overline{S}_2(q)}{324},$$

where $S_1(q)$, $\overline{S}_1(q)$, $S_2(q)$ and $\overline{S}_2(q)$ are given as in Lemma 2.3 (5),

(9)

$$q^{13}\frac{f_2^{16}}{f_1^6}(q^{12}) = q^{13}\frac{f_{24}^{16}}{f_{12}^6} = \frac{1}{384}\left(\frac{S_1(q)+S_2(q)}{2} - \frac{S_3(q)+\overline{S}_3(q)}{2}\right),$$

where $S_1(q)$, $S_2(q)$, $S_3(q)$ and $\overline{S}_3(q)$ are given as in Lemma 2.3 (6).

3. Proof of the conjecture for the pair $\left(f_1^4, \frac{f_1^8}{f_2^2}\right)$

The proof of this case of the conjecture provides a good illustration of the method of proof outlined in the introduction.

Theorem 3.1. *Define the sequences* $\{a(n)\}$ *and* $\{b(n)\}$ *by*

$$f_1^4 =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_1^8}{f_2^2} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then

$$a(n) = 0 \quad \text{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ *precisely for those non-negative* n *for which* $ord_p(6n+1)$ *is odd for some prime* $p \equiv -1 \pmod 3$.

Proof. From Serre's criterion at Lemma 2.1, item (1), it can be seen that all that is required to prove the theorem is to show that $b(n) = 0$ only under the conditions stated in the lemma. Moreover, by Lemma 2.5, item (1),

$$\sum_{n=0}^{\infty} b(n)q^{6n+1} = \frac{1}{2}\left[\left(1 + \frac{1}{\sqrt{-3}}\right)\sum_{m=0}^{\infty} s(m)q^m + \left(1 - \frac{1}{\sqrt{-3}}\right)\sum_{m=0}^{\infty} \overline{s(m)}q^m\right],$$

where $\sum_{m=0}^{\infty} s(m)q^m = S(q)$ and $\sum_{m=0}^{\infty} \overline{s(m)}q^m = \overline{S}(q)$ are the CM newforms defined as in Lemma 2.3, item (1). Observe by Lemma 2.3, item (1), the theta series representations for $S(q)$ and $\overline{S}(q)$, that

$$b(2n) = s(12n+1) = \overline{s(12n+1)}, \qquad b(2n+1) = \frac{s(12n+7)}{\sqrt{-3}} = -\frac{\overline{s(12n+7)}}{\sqrt{-3}}. \tag{3.1}$$

Note that $s(2) = s(3) = 0$, and if p is a prime, $p \equiv -1 \pmod 3$, then $s(p) = 0$. Define the sequences $\{h_i(n)\}$, $i = 1, \ldots, 4$ by

$$H_i = \sum_{n=0}^{\infty} h_i(n)q^n, \qquad i = 1, \ldots, 4,$$

where H_i are defined as in Lemma 2.3, item (1).

If $p \equiv 1 \pmod{12}$, then $p = x^2 + 3y^2$, for unique positive integers x and y with x odd and y even. Thus $h_3(p) = h_4(p) = 0$. It will be shown that only one of H_1 and H_2 contributes to $s(p)q^p$, and whichever contributes, it contributes exactly two terms.

If $4|y$, then it can be seen from the exponent of q in the formulae for both H_1 and H_2, that n must be even, since $4m - 2n = y$ or $4m - 2n = -y$. If H_1 contributes to $s(p)q^p$, then $-6n+1 = \pm x$ for some even n so $x \equiv \pm 1 \pmod{12}$. If H_2 contributes to $s(p)q^p$, then $-6n+5 = \pm x$ for some even n so $x \equiv \pm 5 \pmod{12}$. Since these are incompatible, only one of H_1 or H_2 contributes to $s(p)q^p$.

If H_2 contributes, then there are exactly two pairs of integers (m_1, n), (m_2, n) that contribute to $s(p)q^p$, where n is even and either $-6n+5 = x$ or $-6n+5 = -x$ (only one of the two equations is solvable for n even) and $4m_1 - 2n = y$ and $4m_2 - 2n = -y$ (so $m_2 = n - m_1$).

Thus, after simplifying,

$$\begin{aligned} h_2(p) &= \left(-6n+5+(4m_1-2n)\sqrt{-3}\right)^2 + \left(-6n+5+(4(n-m_1)-2n)\sqrt{-3}\right)^2 \\ &= 2\left((-6n+5)^2 - 3(4m_1-2n)^2\right) = 2(x^2-3y^2). \end{aligned}$$

Again, by Lemma 2.3, item (1), one finds that

$$s(p) = 2(x^2 - 3y^2).$$

A similar analysis of the case where H_1 contributes to $s(p)q^p$ when $4|y$, and also of the situation where $4 \not| y$ (whichever of H_1 or H_2 contribute), gives that if $p \equiv 1 \pmod{12}$ is prime, then

$$s(p) = 2(x^2 - 3y^2) \qquad \text{or} \qquad s(p) = -2(x^2 - 3y^2).$$

For our calculations, the key implication in this case ($p \equiv 1 \pmod{12}$) is that,

$$s(p) = \pm 2(x^2 - 3y^2) = \pm 2(x^2 - (p - x^2)) \equiv \pm 4x^2 \pmod{p}$$
$$\implies s(p) \not\equiv 0 \pmod{p}.$$

Similarly, if $p \equiv 7 \pmod{12}$, then $p = x^2 + 3y^2$, for unique positive integers x and y with x even and y odd. This time H_1 and H_2 contribute nothing to $s(p)q^p$, but H_3 and H_4 contribute exactly one term each to $s(p)x^p$. An analysis similar to that carried out in the case $p \equiv 1 \pmod{12}$ gives in this case, $p \equiv 7 \pmod{12}$, that

$$s(p) = \pm 4xy\sqrt{-3} \implies s(p)^k \not\equiv 0 \pmod{p}, \forall k \in \mathbb{N}.$$

The recurrence formula for $s(n)$ at prime powers is

$$s(p^k) = s(p)s(p^{k-1}) - \chi(p)p^2 s(p^{k-2}), \tag{3.2}$$

where $\chi(p) = (-1)^{(p-1)/2}$. This gives that if $p \equiv -1 \pmod{3}$ is prime, then $|s(p^{2k})| = p^{2k} \neq 0$ and $s(p^{2k+1}) = 0$ for all integers $k \geq 0$.

If $p \equiv 1 \pmod{3}$ (equivalently, $p \equiv 1 \pmod{12}$ or $p \equiv 7 \pmod{12}$), then from (3.2),

$$s(p^k) \equiv s(p)^k \pmod{p} \not\equiv 0 \pmod{p},$$

by the remarks above. Thus if $p \equiv 1 \pmod{3}$, $s(p^k) \neq 0$ for any non-negative integer k.

The multiplicative property thus gives that $s(6n+1) = 0$ if and only if there exists a prime number $p \equiv -1 \pmod{3}$ whose exponent in $1 + 6n$ is odd. The relation at (3.1) between $b(n)$ and $s(6n+1)$ then give that $b(n) = 0$ if and only if there exists a prime number $p \equiv -1 \pmod{3}$ whose exponent in $1 + 6n$ is odd, giving the result.

□

4. Proof of the conjecture for the pair $\left(f_1^4, \frac{f_1^{10}}{f_3^2}\right)$

Theorem 4.1. *Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by*

$$f_1^4 =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_1^{10}}{f_3^2} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then

$$a(n) = 0 \quad \textit{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $ord_p(6n+1)$ is odd for some prime $p \equiv -1 \pmod{3}$.

Proof. Similar to the proof of Theorem 3.1, by Lemma 2.1, item (1), it suffices to prove that $b(n) = 0$ if and only if there is a prime $p \equiv -1 \pmod{3}$ such that $\mathrm{ord}_p(6n+1)$ is odd.

Recall by Lemma 2.3, item (2) that

$$\sum_{n=0}^{\infty} b(n)q^{6n+1} = \frac{1}{2}\left(\sum_{m=0}^{\infty} s_1(m)q^m + \sum_{m=0}^{\infty} s_2(m)q^m\right),$$

where $\sum_{m=0}^{\infty} s_i(m)q^m = S_i(q)$ for $i = 1, 2$ are the CM newforms given as in Lemma 2.5, item (2). Then note by Lemma 2.5, item (2) that $s_1(2) = s_1(3) = s_2(2) = s_2(3) = 0$, and for p prime, $p \equiv -1 \pmod{3}$, then $s_1(p) = s_2(p) = 0$, since primes $p > 3$, $p \equiv -1 \pmod{3}$ are not representable by the quadratic form $x^2 + 3y^2$. For the case $p \equiv 1 \pmod{3}$, it is convenient to consider the case $p \equiv 1 \pmod{6}$.

If $p \equiv 1 \pmod 6$, with $p = x^2 + 3y^2$, and $3 \mid y$, then H_1 and H_2 contribute 0 to the coefficients of q^p in S_1 and S_2, and H_3 contributes

$$\left(x+\sqrt{-3}y\right)^3 + \left(x-\sqrt{-3}y\right)^3 = 2\left(x^3 - 9xy^2\right),$$

where the rightmost expression in the equation above has the same absolute value independent of the signs of x and y. Therefore, in this case

$$s_1(p) = s_2(p) = -2\left(x^3 - 9xy^2\right) = b((p-1)/6) \equiv -8x^3 \not\equiv 0 \pmod p. \tag{4.1}$$

If $p \equiv 1 \pmod 6$, with $p = x^2 + 3y^2$, and $3 \nmid y$, then the coefficient of q^p in H_3 is 0, and the coefficients of q^p in H_1 and H_2 are respectively,

$$\left(x+\sqrt{-3}y\right)^3, \left(-x+\sqrt{-3}y\right)^3.$$

Thus, in this case,

$$s_1(p) = -x^3 - 9x^2y + 9xy^2 + 9y^3, \quad s_2(p) = -x^3 + 9x^2y + 9xy^2 - 9y^3,$$

and changing signs of x and y maps the expressions above to $\pm\{s_1(p), s_2(p)\}$. Hence,

$$s_1(p) \equiv 12y^2(x+3y) \not\equiv 0 \pmod p, \tag{4.2}$$

$$s_2(p) \equiv 12y^2(x-3y) \not\equiv 0 \pmod p, \tag{4.3}$$

and

$$b((p-1)/6) = \frac{s_1(p)+s_2(p)}{2} = -x^3 + 9xy^2.$$

The recurrence formula for $s_1(n)$ and $s_2(n)$ at prime powers takes the form

$$s_1(p^k) = s_1(p)s_1(p^{k-1}) - p^3 s_1(p^{k-2}), \tag{4.4}$$

Thus, for each integer $k \geq 0$, if $p \equiv -1 \pmod 3$, $s_1(p^{2k+1}) = s_2(p^{2k+1}) = 0$, and $s_1(p^{2k}) \neq 0$, $s_2(p^{2k}) \neq 0$. Moreover, from (4.1) and (4.2)-(4.3), we conclude $s_1(p^{2k}), s_2(p^{2k}) \neq 0$ for all nonnegative integers k. By reasoning as in the last section, we obtain a proof of the conjecture for the pair under consideration.

$\square$

5. Proof of the conjecture for the pair $\left(f_1^6, \frac{f_2^4}{f_1^2}\right)$

Before getting to the proof, recall the following criterion of Ewell ([3, Corollary 8, page 755]).

Proposition 5.1. *A positive integer n can be written as a sum of two triangular numbers if and only if when $4n+1$ is expressed as a product of prime-powers, every prime factor $p \equiv 3 \pmod 4$ occurs with even exponent.*

Here we use a different method of proof from that used to prove other cases, in that we do not proceed by attempting to write $\eta^4(8z)/\eta^2(4z)$ in terms of CM forms and theta series.

Theorem 5.2. *Define the sequences $\{a(n)\}$ and $\{b(n)\}$ as follows:*

$$f_1^6 =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_2^4}{f_1^2} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then

$$a(n) = 0 \quad \text{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $ord_p(4n+1)$ is odd for some prime $p \equiv 3 \pmod 4$.

Proof. As usual, let

$$t(n) = \frac{n(n+1)}{2}, \quad n = 0, 1, 2, 3, \ldots,$$

denote the n-th triangular number. Let

$$T_2 = \{t(m) + t(n) | m, n \geq 0\},$$

the set of non-negative integers representable as a sum of two triangular numbers.

Recall (see, for example, [5, Corollary 6.1, Corollary 6.3]) that

$$(q; q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2},$$

$$\psi(q) = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Then writing $(q; q)_\infty = (q, q^2; q^2)_\infty$, one gets that

$$\frac{f_2^4}{f_1^2} = \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} = \psi(q)^2 = \sum_{m,n=0}^{\infty} q^{m(m+1)/2 + n(n+1)/2}.$$

Thus $b(n) = 0$ if and only if $n \notin T_2$. Likewise

$$f_1^6 = [(q; q)_\infty^3]^2 = \sum_{m,n=0}^{\infty} (-1)^{m+n} (2m+1)(2n+1) q^{m(m+1)/2 + n(n+1)/2}$$

Apply the dilation $q \to q^4$ to f_1^6 and multiply by q to get

$$q f_4^6 = \sum_{n=0}^{\infty} a(n) q^{4n+1}.$$

Next, we recall the criterion of Serre from Lemma 2.1, item (2):

> if $q f_4^6 = \sum_{n=0}^{\infty} a(n) q^{4n+1}$, one has that $a(n) = 0$ if and only if $4n+1$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent.

The statement in the theorem now follows upon combining Ewell's criterion in Proposition 5.1 with Serre's statement above. □

Remark: Note that $a(n) = 0$ if and only if $b(n) = 0$ would also follow if it could be shown directly that for any $N \in T_2$,

$$\sum_{\substack{m,n \geq 0 \\ m(m+1)/2 + n(n+1)/2 = N}} (-1)^{m+n} (2m+1)(2n+1) \neq 0.$$

6. Proof of the conjecture for the pair $\left(f_1^6, \frac{f_1^{14}}{f_2^4}\right)$

In this section, we deal with the case of $\left(f_1^6, \frac{f_1^{14}}{f_2^4}\right)$ and prove the following theorem.

Theorem 6.1. *Define the sequences* $\{a(n)\}$ *and* $\{b(n)\}$ *by*

$$f_1^6 =: \sum_{n=0}^{\infty} a(n) q^n, \qquad \frac{f_1^{14}}{f_2^4} =: \sum_{n=0}^{\infty} b(n) q^n.$$

Then

$$a(n) = 0 \quad \textit{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $ord_p(4n+1)$ is odd for some prime $p \equiv -1 \pmod 4$.

Proof. By Lemma 2.1, item (2), it suffices to prove that $b(n) = 0$ if and only if $4n+1$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent. Now by Lemma 2.5, item (4), one first has that

$$b(n) = \sum_{\substack{x\equiv 1 \pmod 4 \\ y \equiv 0 \pmod 2 \\ x^2+y^2=4n+1}} (x+iy)^4.$$

Then it is clear that

$$b(n) = \prod_{p|(4n+1)} \tilde{b}(p^{e_p})$$

given that $4n+1 = \prod_{p|(4n+1)} p^{e_p}$ is the prime factorization of $4n+1$, where

$$\tilde{b}(p^{e_p}) = \sum_{\substack{x\equiv 1 \pmod 4 \\ y \equiv 0 \pmod 2 \\ x^2+y^2=p^{e_p}}} (x+iy)^4.$$

Suppose that $p|(4n+1)$ with $p \equiv -1 \pmod 4$ and e_p odd. Then $x^2+y^2 = p^{e_p}$ insolvable over $\mathbb{Z}$, since $p^{e_p} \equiv 3 \pmod 4$. Therefore, under the assumption, one must have $\tilde{b}(p^{e_p}) = 0$, which implies that $b(n) = 0$.

On the other hand, suppose that $b(n) = \prod_{p|(4n+1)} \tilde{b}(p^{e_p}) = 0$. We claim that $\tilde{b}(p^{e_p}) \neq 0$ for $p \equiv 1 \pmod 4$, or $p \equiv -1 \pmod 4$ with e_p even. For the former case, one first notes that there is a unique $\alpha = a+ib$ with $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 2$ up to complex conjugate such that the norm $N(\alpha) = p$. Then the solutions to $N(x+iy) = p^{e_p}$ are exactly

$$x + iy = \underbrace{\alpha \times \cdots \times \alpha}_{k\ times} \times \underbrace{\overline{\alpha} \times \cdots \times \overline{\alpha}}_{e_p - k\ times}$$

as k runs from 0 to e_p. Clearly, for $1 \leq k \leq e_p - 1$, $x+iy \in p\mathbb{Z}[i]$ since $\alpha\overline{\alpha} = p$ by assumption. Then

$$\tilde{b}(p^{e_p}) - \left((\alpha^{e_p})^4 + (\overline{\alpha}^{e_p})^4\right) \in p^4\mathbb{Z}[i].$$

If $\tilde{b}(p^{e_p}) = 0$, writing $\alpha^{e_p} = X + iY$ with $X \equiv 1 \pmod 4$ and $Y \equiv 0 \pmod 2$, one can tell by the above that

$$p|(2X^4 + 2Y^4 - 12X^2Y^2) \quad \text{and} \quad p|(X^2+Y^2).$$

These imply that $p|X^2Y^2$, that is to say, $p|X$ or $p|Y$. Either one together with $p|(X^2+Y^2)$ implies that $p|X$ and $p|Y$, and consequently, $p|\alpha^{e_p}$, which is impossible since $p = \alpha\overline{\alpha}$ and the ideals (α) and $(\overline{\alpha})$ are coprime prime ideals. Therefore, $\tilde{b}(p^{e_p}) \neq 0$ for $p \equiv 1 \pmod 4$.

For the latter case that $p \equiv -1 \pmod 4$ with e_p even, since p is inert in $\mathbb{Z}[i]$, it is clear that $\tilde{b}(p^{e_p}) \neq 0$ since the solutions to $N(x+iy) = p^{e_p}$ are exactly

$$x + iy = (-p)^{e_p/2},$$

and thus, $\tilde{b}(p^{e_p}) = p^{2e_p} \neq 0$.

These altogether indicate that $b(n) = 0$ if and only if $4n+1$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent.

$\square$

7. Proof of the conjecture for the pair $\left(f_1^{10}, \frac{f_2^6}{f_1^2}\right)$

In this section, we deal with the case of $\left(f_1^{10}, \frac{f_2^6}{f_1^2}\right)$ and prove the following theorem.

Theorem 7.1. *Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by*

$$f_1^{10} =: \sum_{n=0}^{\infty} a(n)q^n, \qquad\qquad \frac{f_2^6}{f_1^2} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then

$$a(n) = 0 \quad \textit{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $ord_p(12n+5)$ is odd for some prime $p \equiv -1 \pmod 4$.

Proof. By Lemma 2.1, item (3), it suffices to prove that $b(n) = 0$ if and only if $12n+5$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent. Recall by Lemma 2.5, item (5) that

$$q^5 \sum_{n=0}^{\infty} b(n)q^{12n+5} = \frac{1}{8} \sum_{m=1}^{\infty} (s_2(m) - s_1(m))q^m,$$

where $\sum_{m=1}^{\infty} s_i(m)q^m = S_i(q)$ given as in Lemma 2.3, item (3) for $i = 1, 2$, and thus, $b(n) = \frac{1}{8}(s_2(12n+5) - s_1(12n+5))$. We aim to prove that $s_2(12n+5) - s_1(12n+5) = 0$ if and only if $12n+5$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent.

By Lemma 2.3, item (3), we note that H_i for $i = 1, \ldots, 4$ only makes contribution to the coefficient of the term q^m with $m \equiv 5 \pmod{12}$, and H_j for $j = 5, \ldots, 8$ only makes contribution to the coefficient of the term q^m with $m \equiv 1 \pmod{12}$. Now since both $S_1(q)$ and $S_2(q)$ with $q = e^{2\pi i z}$ are CM newforms, for

$$m = 12n+5 = \prod_{\ell \equiv 1 \pmod{12}} \ell^{e_\ell} \prod_{q \equiv 5 \pmod{12}} q^{2e_q} \prod_{q' \equiv 5 \pmod{12}} q'^{2e_{q'}+1} \prod_{p \equiv 3 \pmod 4} p^{e_p},$$

where the number of prime factors $q' \equiv 5 \pmod{12}$ with odd exponent is odd, one has that

$$s_1(12n+5) = \prod_{\ell \equiv 1 \pmod{12}} s_1(\ell^{e_\ell}) \prod_{q \equiv 5 \pmod{12}} s_1(q^{2e_q}) \prod_{q' \equiv 5 \pmod{12}} s_1(q'^{2e_{q'}+1}) \prod_{p \equiv 3 \pmod 4} s_1(p^{e_p})$$

and

$$s_2(12n+5) = \prod_{\ell \equiv 1 \pmod{12}} s_2(\ell^{e_\ell}) \prod_{q \equiv 5 \pmod{12}} s_2(q^{2e_q}) \prod_{q' \equiv 5 \pmod{12}} s_2(q'^{2e_{q'}+1}) \prod_{p \equiv 3 \pmod 4} s_2(p^{e_p}).$$

It is clear by Lemma 2.3, item (3) and the observation above that $s_1(\ell^{e_{p\ell}}) = s_2(\ell^{e_\ell})$, $s_1(q^{2e_q}) = s_2(q^{2e_q})$, $s_1(q'^{2e_{q'}+1}) = -s_2(q'^{2e_{q'}+1})$, and $s_1(p^{e_p}) = s_2(p^{e_p})$, and thus,

$$\begin{aligned} &s_2(12n+5) - s_1(12n+5) \\ &= 2 \prod_{\ell \equiv 1 \pmod{12}} s_2(\ell^{e_\ell}) \prod_{q \equiv 5 \pmod{12}} s_2(q^{2e_q}) \prod_{q' \equiv 5 \pmod{12}} s_2(q'^{2e_{q'}+1}) \prod_{p \equiv 3 \pmod 4} s_2(p^{e_p}). \end{aligned}$$

By some elementary analysis, one can tell that for a prime congruent to $1, 5$ modulo 12, up to complex conjugate there is a unique α of the form $6X \pm 1 + i(6Y \pm 1)$ with $X \equiv Y \pmod 2$ such that $N(\alpha) = p$. Using the same reasoning as that used in the proof of Theorem 6.1, one can show that $s_2(\ell^{e_\ell})$, $s_2(q^{2e_q})$, $s_2(q'^{2e_{q'}+1})$ are all nonzero, and for $p \equiv 3 \pmod 4$, $s_2(p^{e_p}) \neq 0$ if and only if e_p is even. Therefore, $b(n) = 0$ if and only if $12n+5$ is divisible by some odd power of a prime $p \equiv 3 \pmod 4$. $\square$

8. Proof of the conjecture for the pair $\left(f_1^{14}, \frac{f_3^5}{f_1}\right)$

Theorem 8.1. *Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by*

$$f_1^{14} =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_3^5}{f_1} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then

$$a(n) = 0 \quad \text{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $\operatorname{ord}_p(12n+7)$ is odd for some prime $p_1 \equiv -1 \pmod 3$.

Proof. Similarly, by Lemma 2.1, item (4), it is sufficient to prove that $b(n) = 0$ if and only if there is a prime $p \equiv -1 \pmod 3$ such that $\operatorname{ord}_p(12n+7)$ is odd. Recall by Lemma 2.3, item (6) that

$$\sum_{n=0}^{\infty} b(n)q^{12n+7} = \frac{1}{6\sqrt{-3}}\left(-\sum_{m=0}^{\infty} s(m)q^m + \sum_{m=0}^{\infty} \overline{s(m)}q^m\right),$$

where $\sum_{m=0}^{\infty} s(m)q^m = S(q)$ and $\sum_{m=0}^{\infty} \overline{s(m)}q^m = \overline{S}(q)$ are the CM newforms defined as in Lemma 2.5, item (4). Write

$$\begin{aligned} &-H_1 - H_2 + \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)H_3 + \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)H_4 \\ &\quad + \left(\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)H_5 + \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)H_6 = \sum_{n=0}^{\infty} k(n)q^n \end{aligned}$$

and

$$H_7 + H_8 + \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)H_9 - H_{10} - H_{11} + \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)H_{12} = \sum_{n=0}^{\infty} t(n)q^n,$$

where H_i are defined as in Lemma 2.5, item (4), and recall that

$$S(q) = \sum_{n=0}^{\infty} k(n)q^n + \sum_{n=0}^{\infty} t(n)q^n.$$

Then by the definitions of H_i, when $p = x^2 + 3y^2 \equiv 7 \pmod{12}$, the total contribution from the the $t(p)$ is 0, while the total contribution to $s(p)$ from $k(p)$ up to sign is

$$\begin{cases} 2y\sqrt{-3}, & \text{if } 3|y, \\ (y+x)\sqrt{-3}, & \text{if } 3|y+x, \\ (y-x)\sqrt{-3}, & \text{if } 3|y-x. \end{cases}$$

Each claim above follows by rewriting the H_i so the exponents are of the form $x^2 + 3y^2$, where x, y are $\mathbb{Z}$-linear combinations of m, n. For instance, exponents of the theta series H_1 and H_4 may be written, respectively, as

$$\begin{aligned} \left(12m - 6n - \frac{1}{2}\right)^2 + 3\left(6n + \frac{3}{2}\right)^2 &= (6m + 6n + 2)^2 + 3(-6m + 6n + 1)^2, \\ \left(12m - 6n - \frac{5}{2}\right)^2 + 3\left(6n - \frac{1}{2}\right)^2 &= (-6m - 6n + 2)^2 + 3(-6m + 6n + 1)^2. \end{aligned}$$

Corresponding expansions may be obtained for the exponents in the other theta series H_i. From these expansions, we may deduce that, in the case $3 \mid x+y$, the only nonzero contributions to $s(p)$ come from the pair H_1, H_4 and that, up to sign,

$$-H_1 + \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) H_4 = (x+y)\sqrt{-3}.$$

Likewise, in the case $3 \mid x-y$, the only nonzero contributions to $s(p)$ come from the pair H_2, H_3, and

$$-H_2 + \left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) H_3 = (x+y)\sqrt{-3}.$$

When $3 \mid y$, the only contributions to $s(p)$ come from H_5, H_6, and

$$\left(\frac{1}{2} - \frac{\sqrt{-3}}{2}\right) H_5 + \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) H_6 = 2y\sqrt{-3}.$$

A similar argument based on the theta expansions shows that for the case if $p = x^2 + 3y^2 \equiv 1 \pmod{12}$, the total contribution to $s(p)$ from $t(p)$ is, up to sign, $\pm x + 3y$, while the contribution from $k(p)$ is trivial.

By reasoning as in the prior cases, we conclude that the coefficients $s(12n+7)$ vanish if and only if there is a prime $p \equiv -1 \pmod{3}$ such that $\operatorname{ord}_p(12n+7)$ is odd. The claimed vanishing of the coefficients $b(n)$ follows.

□

9. Proof of the conjecture for the pair $\left(f_1^{14}, \frac{f_2^8}{f_1^2}\right)$

Somewhat curiously, the modular form arising from this case of the conjecture is a linear combination of the same two CM forms encountered in the proof of Theorem 3.1 (but of course a different linear combination).

Theorem 9.1. *Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by*

$$f_1^{14} =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_2^8}{f_1^2} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then

$$a(n) = 0 \quad \textit{if and only if} \quad b(n) = 0.$$

Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $\operatorname{ord}_p(12n+7)$ *is odd for some prime $p_1 \equiv -1 \pmod{3}$.*

Proof. Similar to the proofs of the previous cases, this time from Serre's criterion in Lemma 2.1, item (4), we see that it is sufficient to prove that $b(n) = 0$ under the conditions stated in the theorem. By Lemma 2.5, item (7),

$$\sum_{n=0}^{\infty} b(n)q^{12n+7} = -\frac{1}{8\sqrt{-3}} \left[\frac{1}{2}\left(\sum_{m=0}^{\infty} s(m)q^m - \sum_{m=0}^{\infty} \overline{s(m)}q^m\right)\right],$$

where $\sum_{m=0}^{\infty} s(m)q^m = S(q)$ and $\sum_{m=0}^{\infty} \overline{s(m)}q^m = \overline{S}(q)$ are the CM newforms defined as in Lemma 2.3, item (1). Then it is clear that $b(n) = 0$ if and only if $s(12n+7) = 0$.

We summarize the properties of the coefficients $s(p)$ derived in Section 3, where p is a prime. It is clear that $s(2) = s(3) = 0$, and if p is a prime, $p \equiv -1 \pmod{3}$, then $s(p) = 0$.

Using the recursive formula and explicit values computed for the $s(p)$ it was further shown in Section 3 that if $p \equiv -1 \pmod{3}$ is prime, then $|s(p^{2k})| = p^{2k} \neq 0$ and $s(p^{2k+1}) = 0$ for all integers $k \geq 0$.

Likewise, it was shown that if $p \equiv 1 \pmod 3$, $s(p^k) \neq 0$ for any non-negative integer k.

Thus it follows that $s(12n+7) = 0$ if and only if $\mathrm{ord}_p(12n+7)$ is odd for some prime $p \equiv -1 \pmod 3$. This fact together with the remark that $b(n) = 0$ if and only if $s(12n+7) = 0$ gives the desired result. $\square$

10. Proof of the conjecture for the pair $\left(f_1^{26}, \frac{f_3^9}{f_1}\right)$

In [8, page 213], Serre gave just a sufficient condition for $a(n) = 0$, where the sequence $\{a(n)\}$ is as defined in Theorem 10.1. Thus his statement for vanishing coefficients in the expansion of $\eta^{26}(z)$ is in contrast to what he proved for $\eta^r(z)$, when $r = 2, 4, 6, 8, 10, 14$, where he gave necessary and sufficient conditions for the coefficients to vanish (see Lemma 2.1 for details). This difference is reflected in the following theorem, and the theorem in the next section.

Theorem 10.1. *Define the sequences $\{a(n)\}$ and $\{b(n)\}$ by*

$$f_1^{26} =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_3^9}{f_1} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then $a(n) = b(n) = 0$ for those non-negative n for which either
(a) $\mathrm{ord}_{p_1}(12n+13)$ is odd for some prime $p_1 \equiv -1 \pmod 3$ and $\mathrm{ord}_{p_2}(12n+13)$ is odd for some prime $p_2 \equiv -1 \pmod 4$ (it may be that $p_1 = p_2$)

or

(b) $12n+13$ is a square and all prime factors p satisfy $p \equiv -1 \pmod{12}$.

Proof. From Lemma 2.1, item (5), it may be seen it is sufficient to show that $b(n) = 0$ under the conditions stated in the theorem. Recall by Lemma 2.5, item (8) that

$$\sum_{n=0}^{\infty} b(n)q^{12n+13} = \frac{1}{324}\left(\sum_{m=0}^{\infty} s_1(m)q^m + \sum_{m=0}^{\infty} \overline{s_1(m)}q^m\right) - \frac{1}{324}\left(\sum_{m=0}^{\infty} s_2(m)q^m + \sum_{m=0}^{\infty} \overline{s_2(m)}q^m\right),$$

where $\sum_{m=0}^{\infty} s_i(m)q^m = S_i(q)$ and $\sum_{m=0}^{\infty} \overline{s_i(m)}q^m = \overline{S}_i(q)$ are the CM newforms defined as in Lemma 2.3, item (5). Observe that

$$\frac{S_1 + \overline{S}_1}{2} = \sum_{n=0}^{\infty} s_1(12n+1)q^{12n+1}, \tag{10.1}$$

and

$$\frac{S_2 + \overline{S}_2}{2} = \sum_{n=0}^{\infty} s_2(12n+1)q^{12n+1}. \tag{10.2}$$

Thus, from above,

$$b(n) = 0 \quad \text{if and only if} \quad s_1(12n+13) - s_2(12n+13) = 0.$$

We next use the theta series decompositions for S_1 and S_2 from Lemma 2.3, item (5), to determine information about $s_1(n)$ and $s_2(n)$ when n is a prime.

***The coefficients of** S_1.* Recall that the sequence $\{s(n)\}$ is defined by (10.1). If $p \equiv 1 \pmod{12}$ is a prime, then the equation

$$p = (3m+1)^2 + 3(2n-3m)^2 \tag{10.3}$$

is solvable with m even. If (m,n) is one solution, then a second solution is $(m, -n+3m)$, and since m is even, $(-1)^{-n+3m} = (-1)^n$. Hence

$$(10.4)\quad s_1(p) = (-1)^n\left(3m+1+(2n-3m)i\sqrt{3}\right)^3 + (-1)^n\left(3m+1-(2n-3m)\sqrt{-3}\right)^3$$
$$= 2(-1)^n(1+3m)\left(4(3m+1)^2-3p\right) =: s_{p\equiv 1 \pmod{12}}.$$

If $p \equiv 7 \pmod{12}$ is a prime, then (10.3) is solvable with m odd. Once again, if (m,n) is one solution, then $(m,-n+3m)$ gives a second solution, but this time, since m is odd, $(-1)^{-n+3m} = -(-1)^n$, and

$$(10.5)\quad s_1(p) = (-1)^n\left(3m+1+(2n-3m)i\sqrt{3}\right)^3 - (-1)^n\left(3m+1-(2n-3m)\sqrt{-3}\right)^3$$
$$= 2i(-1)^n\left(4(3m+1)^2-p\right)\sqrt{p-(3m+1)^2} =: s_{p\equiv 7 \pmod{12}}.$$

If $p \equiv 5$ or $11 \pmod{12}$ is a prime, then $s_1(p) = 0$.

The coefficients of S_2. Recall that the sequence $\{s_2(n)\}$ is defined by (10.2) and that

$$S_2 = \sum_{n=0}^{\infty} s_2(n)q^n = H_1 - iH_2 + \frac{(1+i)H_3}{\sqrt{2}} + \frac{(1-i)H_4}{\sqrt{2}} - \frac{(1+i)H_5}{\sqrt{2}} - \frac{(1-i)H_6}{\sqrt{2}}.$$

From the fact that that each prime $p \equiv 1 \pmod 4$ has a unique representation of the form $p = x^2+y^2$ (x and y positive integers), it can be seen that each prime $p \equiv 1 \pmod{12}$ has a representation by exactly one of the forms $(6m+1)^2+(6n)^2$ (by H_1) or $(6m+3)^2+(6n-2)^2$ (by H_2).

If p is represented by $H_1 =: \sum_{n=0}^{\infty} h_1(n)q^n$, so that $p = (6m+1)^2+(6n)^2$ for integers m and n, then there are exactly two representations (replace n with $-n$) and then

$$s_2(p) = h_1(p) = (6m+1+6ni)^3 + (6m+1-6ni)^3 = 2(6m+1)(4(6m+1)^2-3p).$$

On the other hand, if p is represented by $H_2 =: \sum_{n=0}^{\infty} h_2(n)q^n$, so that $p = (6m+3)^2+(6n-2)^2$ for integers m and n, then here also there are exactly two representations (replace m with $-m-1$) and then

$$s_2(p) = -ih_2(p) = -i\left[(6m+3+(6n-2)i)^3 + (-6m-3+(6n-2)i)^3\right]$$
$$= 2(6n-2)\left(3p-4(6n-2)^2\right).$$

Similarly, it can be seen that the quadratic forms in the exponent of q in $H_3,\dots,H_6$ all represent primes $p \equiv 5 \pmod{12}$. Further, each such prime is either a) represented exactly once by each of H_3 and H_4 or b) exactly once by each of H_5 and H_6, with no overlap between cases a) and b).

For ease of notation in what follows, we define the sequences $\{h_i(n)\}_{n=0}^{\infty}$, $i = 3,\dots,6$ by

$$H_i =: \sum_{n=0}^{\infty} h_i(n)q^n, \qquad i = 3,\dots,6.$$

If p is represented by H_3 and H_4, so that $p = (6x+1)^2+(6(x+2y)-2)^2 = (6x+1)^2+(6(x+2(-x-y))+2)^2$ for some integers x and y, and then

$$(10.6)\quad s_2(p) = h_{3,4}(p) := \frac{(1+i)h_3(p)}{\sqrt{2}} + \frac{(1-i)h_4(p)}{\sqrt{2}}$$
$$= \frac{(1+i)}{\sqrt{2}}[(6x+1)i+(6(x+2y)-2)]^3 + \frac{(1-i)}{\sqrt{2}}[(6x+1)i+(-6(x+2y)+2)]^3$$
$$= i\sqrt{2}\left[\left(3p-4(6x+1)^2\right)(6x+1) \pm \left(p-4(6x+1)^2\right)\sqrt{p-(6x+1)^2}\right],$$

where the $\pm$ sign depends on whether $6(x+2y)-2$ is positive or negative.

Alternatively, if p is represented by H_5 and H_6, so that $p = (6x+5)^2 + (6(x+2y)+2)^2 = (6x+5)^2 + (6(x+2(-x-y))-2)^2$ for some integers x and y, and then

$$\begin{aligned}
(10.7) \quad s_2(p) = h_{5,6}(p) &:= -\frac{(1+i)h_5(p)}{\sqrt{2}} - \frac{(1-i)h_6(p)}{\sqrt{2}} \\
&= -\frac{(1+i)}{\sqrt{2}}[(6x+5)i + (6(x+2y)+2)]^3 - \frac{(1-i)}{\sqrt{2}}[(6x+5)i + (-6(x+2y)-2)]^3 \\
&= -i\sqrt{2}\left[\left(3p - 4(6x+5)^2\right)(6x+5) \pm \left(p - 4(6x+5)^2\right)\sqrt{p-(6x+5)^2}\right],
\end{aligned}$$

where the $\pm$ sign depends on whether $6(x+2y)+2$ is positive or negative.

If p is prime, $p \equiv 7$ or $11 \pmod{12}$, then $s_2(p) = 0$.

Before using the recurrence formulae to examine $s_1(p^k)$ and $s_2(p^k)$, where k is a positive integer, we summarize the information derived so far.

	$s_1(p)$	$s_2(p)$
$p \equiv 1 \pmod{12}$	$s_{p\equiv 1 \pmod{12}}$ (10.4)	$h_1(p)$ or $-ih_2(p)$
$p \equiv 5 \pmod{12}$	0	$h_{3,4}(p)$ (10.6) or $h_{5,6}(p)$ (10.7)
$p \equiv 7 \pmod{12}$	$s_{p\equiv 7 \pmod{12}}$ (10.5)	0
$p \equiv 11 \pmod{12}$	0	0

TABLE 1. The values of $s_1(p)$ and $s_2(p)$ for p a prime

Next, recall that the sequences $\{s_1(n)\}$ and $\{s_2(n)\}$ satisfy the recurrence relations

$$\begin{aligned}
(10.8) \qquad s_1(p^k) &= s_1(p^{k-1})s_1(p) - \chi(p)p^3 s_1(p^{k-2}), \\
s_2(p^k) &= s_2(p^{k-1})s_2(p) - \chi(p)p^3 s_2(p^{k-2}),
\end{aligned}$$

where

$$\chi(p) = \begin{cases} 1, & p \equiv 1, 11 \pmod{12}, \\ -1, & p \equiv 5, 7 \pmod{12}. \end{cases}$$

We are now ready to prove the claims in the theorem. As remarked previously, $b(n) = 0$ if and only if $s_1(12n+13) - s_2(12n+13) = 0$.

(a) Observe that if $p \equiv 11 \pmod{12}$, then the recurrence formulae (10.8) together with the values $s_1(p) = s_2(p) = 0$ from Table 1 give that, for all integers $k \geq 0$,

$$s_1(p^{2k}) = s_2(p^{2k}) = (-1)^k p^{3k},$$

so that, by the multiplicative property, if $12n+13$ is a square with all prime factors $\equiv -1 \equiv 11 \pmod{12}$, then

$$s_1(12n+13) = s_2(12n+13) \Longrightarrow s_1(12n+13) - s_2(12n+13) = 0 \Longrightarrow b(n) = 0.$$

(b) By similar reasoning, if $p \equiv -1 \pmod 4$ (so $p \equiv 7$ or $11 \pmod{12}$), then the values $s_2(p) = 0$ from Table 1 and the second recurrence formula at (10.8) gives that $s_2(p^{2k+1}) = 0$, for all integers $k \geq 0$. Likewise, if $p \equiv -1 \pmod 3$ (so $p \equiv 5$ or $11 \pmod{12}$), then the values $s_1(p) = 0$ from Table 1 and the first recurrence formula at (10.8) gives that $s_1(p^{2k+1}) = 0$, for all integers $k \geq 0$. The multiplicative property now gives that if $12n+13$ has a prime factor $p_1 \equiv -1 \pmod 4$ and a prime factor $p_2 \equiv -1 \pmod 3$ (it could happen the $p_1 = p_2$), both occurring to odd powers, then

$$s_1(12n+13) = s_2(12n+13) = 0 \Longrightarrow s_1(12n+13) - s_2(12n+13) = 0 \Longrightarrow b(n) = 0.$$

□

11. Proof of the conjecture for the pair $\left(f_1^{26}, \frac{f_2^{16}}{f_1^6}\right)$

As in the previous section, the proof in this sections gives just a <u>sufficient</u> condition for the simultaneous vanishing of the coefficients.

Theorem 11.1. *Define the sequences* $\{a(n)\}$ *and* $\{b(n)\}$ *by*

$$f_1^{26} =: \sum_{n=0}^{\infty} a(n)q^n, \qquad \frac{f_2^{16}}{f_1^6} =: \sum_{n=0}^{\infty} b(n)q^n.$$

Then $a(n) = b(n) = 0$ *for those non-negative* n *for which either*
(a) $\mathrm{ord}_{p_1}(12n + 13)$ *is odd for some prime* $p_1 \equiv -1 \pmod 3$ <u>*and*</u> $\mathrm{ord}_{p_2}(12n + 13)$ *is odd for some prime* $p_2 \equiv -1 \pmod 4$ *(it may be that* $p_1 = p_2$*)*

or

(b) $12n + 13$ *is a square and all prime factors* p *satisfy* $p \equiv -1 \pmod{12}$.

Proof. The proof is quite similar to that of Theorem 10.1; by Lemma 2.1, item (5), it is sufficient to show that $b(n) = 0$ under the conditions stated in the theorem. Recall that from Lemma 2.5, item (9), one has that
(11.1)

$$\sum_{n=0}^{\infty} b(n)q^{12n+13} = \frac{1}{384}\left[\frac{1}{2}\left(\sum_{m=0}^{\infty} s_1(m)q^m + \sum_{m=0}^{\infty} s_2(m)q^m\right) - \frac{1}{2}\left(\sum_{m=0}^{\infty} s_3(m)q^m + \sum_{m=0}^{\infty} \overline{s_3(m)}q^m\right)\right],$$

where $\sum_{m=0}^{\infty} s_1(m)q^m = S_1(q)$, $\sum_{m=0}^{\infty} s_2(m)q^m = S_2(q)$, $\sum_{m=0}^{\infty} s_3(m)q^m = S_3(q)$, and $\sum_{m=0}^{\infty} \overline{s_3(m)}q^m = \overline{S}_3(q)$ are the CM newforms given as in Lemma 2.3, item (6). Recall also that from Lemma 2.3, item (6) (see there for the definitions of the theta series $H_1,\dots,H_8$), one has that

$$\begin{aligned} S_1 &= H_3 - H_4 + iH_7 - iH_8, \\ S_2 &= H_3 - H_4 - iH_7 + iH_8, \\ S_3 &= H_1 - H_2 - H_5 + H_6, \\ \overline{S}_3 &= H_1 - H_2 + H_5 - H_6, \end{aligned} \tag{11.2}$$

Define the sequences $\{h_i(n)\}$, $i = 1, \dots, 8$ by

$$H_i =: \sum_{n=0}^{\infty} h_i(n)q^n, \qquad i = 1, \dots, 8 \qquad .$$

Observe by $h_i(n)$ that

$$\frac{S_1(q) + S_2(q)}{2} = \sum_{n=0}^{\infty} s_1(12n+1)q^{12n+1}.$$

Likewise,

$$\frac{S_3(q) + \overline{S}_3(q)}{2} = \sum_{n=0}^{\infty} s_3(12n+1)q^{12n+1}.$$

Thus, from (11.1), and similarly to the situation in the previous theorem,

$$b(n) = 0 \quad \text{if and only if} \quad s_1(12n + 13) - s_3(12n + 13) = 0.$$

We will use (11.2) and the expansions for the theta series to compute $s_1(p)$ and $s_3(p)$ for p a prime in each of the congruence classes modulo 12.

We start with $s_1(p)$. Note that each prime $p \equiv 1 \pmod{12}$ is representable by exactly one of the forms $(6m + 1)^2 + (6n)^2$, $(6m + 3)^2 + (6n - 2)^2$.

The coefficients of H_3. If $p = 12k+1$ is a prime and <u>if</u> there is are solutions $p = (6m+1)^2 + (6n)^2 = (6m+1)^2 + (6(-n))^2$, then these are the only two solutions, and $h_3(p)$ (from this pair of representations) is given by

$$h_3(p) = (6m+1+6ni)^4 + (6m+1-6ni)^4 = 2(8(6n)^4 - 8(6n)^2 p + p^2).$$

If $p \equiv 5, 7,$ or $11 \pmod{12}$ then $p = (6m+1)^2 + (6n)^2$ has no solutions and $h_3(p) = 0$.

The coefficients of H_4. A similar examination of H_4 shows that if $p \equiv 1 \pmod{12}$ and <u>if</u> there is a solution $p = (6m+3)^2 + (6n-2)^2$ then there is exactly one other solution $p = (6(-m-1)+3)^2 + (6n-2)^2$ and $h_4(p)$ is given by

$$h_4(p) = (6m+3+(6n-2)i)^4 + (6(-m-1)+3+(6n-2)i)^4$$
$$= 2(p^2 - 8p(6n-2)^2 + 8(6n-2)^4).$$

Likewise, if $p \equiv 5, 7,$ or $11 \pmod{12}$ then $h_4(p) = 0$.

The coefficients of H_7. If $p \equiv 5 \pmod{12}$, then the equation $p = (6m+1)^2 + (6n-2)^2$ has a unique solution and

$$h_7(p) = (6m+1+(6n-2)i)^4.$$

If $p \equiv 1, 7$ or $11 \pmod{12}$, then $h_7(p) = 0$.

The coefficients of H_8. If $p \equiv 5 \pmod{12}$, then the equation $p = (6m+1)^2 + (6n+2)^2$ has a unique solution and

$$g_4(p) = (6m+1+(6n+2)i)^4.$$

As with H_7, if $p \equiv 1, 7$ or $11 \pmod{12}$, then $g_4(p) = 0$.

If $p \equiv 5 \pmod{12}$, $p = x^2 + y^2$, then we note for later use (in computing $s(p)$) that

$$i(h_7(p) - h_8(p)) = \pm 8xy(x^2 - y^2). \tag{11.3}$$

After using the information above together with formula for S_1 at (11.2), we get the following table of values.

	$s(p)$
$p \equiv 1 \pmod{12}$	$h_3(p)$ or $-h_4(p)$
$p \equiv 5 \pmod{12}$	$\pm 8xy(x^2 - y^2)$ (11.3)
$p \equiv 7 \pmod{12}$	0
$p \equiv 11 \pmod{12}$	0

Table 2. The values of $s(p)$ for p a prime

We next derive formulae for $s_3(p)$, starting with H_1 and H_2. We note that each prime p of the form $p = 12k+1$ is represented by exactly one of the forms $(-6n+1)^2 + 3(4m-2n)^2$ and $(-6n+5)^2 + 3(4m-2n)^2$.

The coefficients of H_1. For $p \equiv 1 \pmod{12}$, <u>if</u> there is a solution $p = (-6n+1)^2 + 3(4m-2n)^2$ then there is exactly one other solution $p = (-6n+1)^2 + 3(4(-m+n)-2n)^2$ and then

$$h_1(p) = (-6n+1+(4m-2n)\sqrt{-3})^4 +$$
$$(-6n+1+(4(-m+n)-2n)\sqrt{-3})^4 = 2(p^2 - 8p(-6n+1)^2 + 8(-6n+1)^4).$$

If $p \equiv 5, 7$ or $11 \pmod{12}$ there is no solution to $p = (-6n+1)^2 + 3(4m-2n)^2$ and $h_1(p) = 0$.

The coefficients of H_2. The situation is similar to that for H_1. For $p \equiv 1 \pmod{12}$, <u>if</u> there is a solution $p = (-6n+5)^2 + 3(4m-2n)^2$ then there is exactly one other solution $p = (-6n+5)^2 + 3(4(-m+n)-2n)^2$ and

$$h_2(p) = (-6n+5+(4m-2n)\sqrt{-3})^4 + (-6n+5+(4(-m+n)-2n)\sqrt{-3})^4$$
$$= 2(p^2 - 8p(-6n+5)^2 + 8(-6n+5)^4).$$

If $p \equiv 5, 7$ or $11 \pmod{12}$ there are no solution to $p = (-6n+5)^2 + 3(4m-2n)^2$ and $h_2(p) = 0$.

The coefficients of H_5. For $p \equiv 7 \pmod{12}$, there is a unique solution to $p = (-6n-2)^2 + 3(4m-2n+3)^2$ and then

$$h_5(p) = (-6n-2+(4m-2n+3)\sqrt{-3})^4.$$

If $p \equiv 1, 5$ or $11 \pmod{12}$ there is no solution to $p = (-6n-2)^2 + 3(4m-2n+3)^2$ and $h_5(p) = 0$.

The coefficients of H_6. The situation is similar to that for H_5. For $p \equiv 7 \pmod{12}$, there is a unique solution to $p = (-6n+2)^2 + 3(4m-2n+3)^2$ and then

$$h_6(p) = (-6n+2+(4m-2n+3)\sqrt{-3})^4.$$

If $p \equiv 1, 5$ or $11 \pmod{12}$ there is no solution to $p = (-6n+2)^2 + 3(4m-2n+3)^2$ and $h_6(p) = 0$.

If $p \equiv 7 \pmod{12}$, $p = x^2 + 3y^2$, then we note for later use (in computing $s_3(p)$) that

$$-h_5(p) + h_6(p) = \pm 8\sqrt{-3}xy(x^2 - 3y^2). \tag{11.4}$$

As with $s_1(p)$ above, the information above together with formula for S_3 at (11.2) is used to compute formulae for $s_3(p)$, for p a prime in the various congruence classes modulo 12. The results are summarized in the following table.

	$s_3(p)$
$p \equiv 1 \pmod{12}$	$h_1(p)$ or $-h_2(p)$
$p \equiv 5 \pmod{12}$	0
$p \equiv 7 \pmod{12}$	$\pm 8\sqrt{-3}xy(x^2-3y^2)$ (11.4)
$p \equiv 11 \pmod{12}$	0

TABLE 3. The values of $s_3(p)$ for p a prime

Next, recall the recurrence relations

$$\begin{aligned} s_1(p^k) &= s_1(p^{k-1})s_1(p) - \chi(p)p^4 s_1(p^{k-2}), \\ s_3(p^k) &= s_3(p^{k-1})s_3(p) - \chi(p)p^4 s_3(p^{k-2}), \end{aligned} \tag{11.5}$$

where

$$\chi(p) = \begin{cases} 1, & p \equiv 1 \pmod{4}, \\ -1, & p \equiv -1 \pmod{4}. \end{cases}$$

We are now ready to prove the claims in the theorem, the remainder of the proof mirroring the last stages in the proof of Theorem 10.1. As remarked previously, $b(n) = 0$ if and only if $s_1(12n+13) - s_3(12n+13) = 0$.

(a) Observe that if $p \equiv -1 \equiv 11 \pmod{12}$, then the recurrence formulae (11.5) together with the values $s_1(p) = s_3(p) = 0$ from the tables give that, for all integers $k \geq 0$,

$$s_1(p^{2k}) = s_3(p^{2k}) = p^{4k},$$

so that, by the multiplicative property, if $12n+13$ is a square with all prime factors $\equiv -1 \pmod{12}$, then

$$s_1(12n+13) = s_3(12n+13) \Longrightarrow s_1(12n+13) - s_3(12n+13) = 0 \Longrightarrow b(n) = 0.$$

(b) By similar reasoning, if $p_1 \equiv -1 \pmod 4$ (so $p_1 \equiv 7$ or $11 \pmod{12}$), then $s_1(p_1^{2k+1}) = 0$, for all integers $k \geq 0$. Likewise, if $p_2 \equiv -1 \pmod 3$ (so $p_2 \equiv 5$ or $11 \pmod{12}$), then $s_3(p_2^{2k+1}) = 0$, for all integers $k \geq 0$. The multiplicative property now gives that if $12n+13$ has prime factor $p_1 \equiv -1 \pmod 4$ and a prime factor $p_2 \equiv -1 \pmod 3$ (it could happen the $p_1 = p_2$), then

$$s_1(12n+13) = s_3(12n+13) = 0 \Longrightarrow s_1(12n+13) - s_3(12n+13) = 0 \Longrightarrow b(n) = 0.$$

□

Remark 11.2. *In both Theorem 10.1 and Theorem 11.1, more details were computed than were strictly necessary for the proofs. In particular explicit values for $s_1(p)$ and $s_3(p)$ were computed when these were non-zero. However, we feel they may be of assistance to some reader in extending the statement of these theorems to the "if and only if" statements, as is the case in the previous theorems.*

12. Concluding Remarks

The replacement $q \to -q$ in an infinite q-series has no effect on the location of the vanishing coefficients. However this replacement can lead to a new infinite product whose coefficients vanish identically with the original, via the identity

$$(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty}.$$

For example, making the replacement $q \to -q$ in the pair $\left(f_1^6, f_1^{14}/f_2^4\right)$ examined in Theorem 6.1 leads to the pair of quotients

$$\left(\frac{f_2^{18}}{f_1^6 f_4^6}, \frac{f_2^{38}}{f_1^{14} f_4^{14}}\right)$$

whose coefficients also vanish identically with those of f_1^6.

This phenomenon of identically vanishing coefficients appears to be even more widespread than we initially expected, even taking into consideration the remarks in the preceding paragraph. We intend to explore this phenomenon further in a subsequent paper.

References

[1] Cohen, H.; Strömberg, F., *Modular forms.* Graduate Studies in Mathematics, Amer. Math. Soc. **179** (2017).

[2] Deligne, P., Serre, J -P., *Formes modulaires de poids 1*, Ann. Scient. Ec. e Norm. Sup., 4 serie, 7 (1974), 507-530.

[3] Ewell, J. A. *On sums of triangular numbers and sums of squares.* Amer. Math. Monthly **99** (1992), no. 8, 752–757.

[4] Han, G.-N.; Ono, K. *Hook lengths and 3-cores.* Ann. Comb. **15** (2011), no. 2, 305–312.

[5] McLaughlin, J. *Topics and methods in q-series.* With a foreword by George E. Andrews. Monographs in Number Theory, **8**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018. x+390 pp.

[6] Ribet, K. A. *Galois representations attached to eigenforms with Nebentypus Modular Functions of One Variable*, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Lecture Notes in Math., vol. 601, Springer, Berlin (1977), 17-51.

[7] Serre, J.-P. *Quelques applications du theoreme de densite de Chebotarev.* Publ. Math. I.H.E.S. **54** (1981), 123–201.

[8] Serre, J.-P. *Sur la lacunarité des puissances de η.* Glasgow Math. J. **27** (1985), 203–221.

School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, Texas 78539, USA

Email address: timothy.huber@utrgv.edu

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383
Email address: jmclaughlin2@wcupa.edu

School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai 519082, Guangdong, People's Republic of China
Email address: yedx3@mail.sysu.edu.cn