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CIRCLE ACTIONS ON ORIENTED 4-MANIFOLDS

DONGHOON JANG AND OLEG R. MUSIN

Abstract. In the present paper, we consider an action of the circle group on a compact oriented 4-manifold. We derive the Atiyah-Hirzebruch formula for the manifold, and associate a graph in terms of data on the fixed point set. We show in the case of isolated fixed points that if an abstract graph satisfies the Atiyah-Hirzebruch formula, then there exists a corresponding 4-dimensional oriented S^1-manifold.

Contents

Keywords: oriented 4-manifolds, circle actions, Atiyah-Hirzebruch formula, graphs, weights, equivariant connected sum, equivariant splitting.

1. Introduction

The problem of classification of torus and circle actions on 4-manifolds was considered in 1970's by Orlik and Raymond [30, 31], Fintushel [9, 10, 11], Pao [32, 33], and Yoshida [34], and later by Melvin [24], Melvin and Parker [25], Huck [14], Huck and Puppe [15], etc. We refer to [8] for a survey on group actions on 4-manifolds. Circle actions on different types of 4-manifolds have been also studied; Carrell, Howard, and Kosniowski [7] studied complex manifolds, Ahara and Hattori [1], Audin [4], and Karshon [20] studied symplectic manifolds, and the first named author studied almost complex manifolds [17].

Orlik and Raymond proved that a T^2-action on a closed simply connected oriented 4-manifold is an equivariant connected sum of copies of $\pm\mathbb{CP}^2$ and S^2-bundles over S^2 [30]. Fintushel [10] and Yoshida [34] showed that an S^1-action on a closed simply connected 4-manifold is a connected sum of a homotopy S^4 and copies of $\pm\mathbb{CP}^2$ and $S^2 \times S^2$; their connected sums are not equivariant in general. Orlik and Raymond showed that a T^2-action on a closed orientable 4-manifold is determined by its orbit data [31]; Fintushel proved a similar result for S^1-actions [11].

Let M be a 4-dimensional compact oriented S^1-manifold. After reviewing necessary background in Section 2, in Section 3 we derive the Atiyah-Hirzebruch formula for M (Theorem 3.2), that is, the Atiyah-Singer index formula for the equivariant index of the signature operator on M. The formula is expressed in terms of signs and weights at isolated fixed points and the Euler numbers of the normal bundles of fixed surfaces. We discuss several consequences of Theorem 3.2.

In Section 4, using the Atiyah-Hirzebruch formula, in Theorem 4.3 we associate a certain type of graph to M that we call a graph of weights (Definitions 4.1 and 4.2). The graph contains information on the data on its fixed point set, signs and weights at isolated fixed points and the Euler numbers of the normal bundles of fixed surfaces. Moreover, any edge with label w bigger than 1 corresponds to an invariant 2-sphere of weight w containing two fixed points that are the vertices of the edge; see (2) of Theorem 4.3. We note that a graph of weights does not determine an S^1-manifold uniquely; see Remark 4.6.

In Section 5, we discuss equivariant connected sum and splitting. A traditional connected sum of two m-manifolds removes a ball of each manifold and glue the boundary spheres. We will also take a connected sum along neighborhoods of submanifolds, and also call its reverse operation a splitting. Utilizing the graph that we associate to M, in Theorem 5.2 we show that we can equivariantly split M into

M_0 and copies of $\pm\mathbb{CP}^2$, where M_0 is another S^1-manifold which is minimal in the sense that every weight in the normal bundle of any fixed component is 1. Therefore, for a further topological classification, it is natural to ask if there also exist invariant 2-spheres for weight 1, see Question 4.13; one may ask the same question for higher dimensional oriented S^1-manifolds.

In Section 6, we prove a partial converse to a combination of Theorem 3.2 and Theorem 4.3, that if an abstract graph of weights whose vertices all correspond to isolated fixed points (see Definition 4.1) satisfies the Atiyah-Hirzebruch formula (Theorem 3.2), then there exists a corresponding 4-dimensional compact oriented S^1-manifold; see Theorem 6.1.

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2. Background and preliminaries

Let a group G act on a manifold M. Throughout this paper, any group action on a manifold is assumed to be effective. We denote its fixed point set by M^G, that is,

$$M^G = \{m \in M \mid g \cdot m = m \text{ for all } g \in G\}.$$

If H is a subgroup of G, H also acts on M, and its fixed point set M^H is defined analogously.

Let the circle group S^1 act on a $2n$-dimensional oriented manifold M. Let F be a fixed component of dimension $2m$, and let q be a point in F. The normal space N_qF of F to M at q decomposes into the sum of 2-dimensional irreducibles

$$N_qF = L_{F,1} \oplus \cdots \oplus L_{F,n-m},$$

where on each $L_{F,i}$ the circle acts by multiplication by $g^{w_{F,i}}$ for some non-zero integer $w_{F,i}$. These integers $w_{F,i}$ are the same for all $q \in F$ and called the **weights** of F. Throughout this paper, we choose an orientation of $L_{F,i}$ so that $w_{F,i}$ is positive for each i. The choice of the orientation of each $L_{F,i}$ then determines an orientation of F.

Let p be an isolated fixed point. Then the tangent space at p has two orientations; one induced from the orientation of M and the other that we chose on $L_{p,1} \oplus \cdots \oplus L_{p,n}$. We define the **sign** of p, denoted by ε_p, to be $+1$ if the two orientations agree and -1 otherwise.

Let the circle group S^1 act on a manifold M. As a subgroup, the cyclic subgroup $\mathbb{Z}_w$ also acts on M. Its fixed point set $M^{\mathbb{Z}_w}$ is a submanifold of M.

Suppose that M is orientable and $w > 2$. Let F be a component of $M^{\mathbb{Z}_w}$. Let $p \in F$. The normal space N_pF of F decomposes into irreducible $\mathbb{Z}_w$-representations $L_1 \oplus \cdots \oplus L_m$. On each L_i, the group $\mathbb{Z}_w$ acts as multiplication by a root of unity, and thus each L_i has (real) dimension 2 and has an orientation induced and preserved by the group $\mathbb{Z}_w$. Since this holds for all points in F, the normal bundle of F is orientable. Because M is orientable, it follows that F is also orientable. For references, see [21, Lemma 12], [35, Theorem 3.5.2].

Theorem 2.1. *Let the circle group S^1 act effectively on an oriented manifold M. Let $w \geq 3$ be an integer. Then the $\mathbb{Z}_w$-fixed point set $M^{\mathbb{Z}_w}$ is orientable.*

If $\dim M = 4$, then a component of the $\mathbb{Z}_2$-fixed point set containing an S^1-fixed point is also orientable.

Lemma 2.2. *Let M be a 4-dimensional compact oriented S^1-manifold. Suppose that an isolated fixed point p has a weight 2. Then the component of $M^{\mathbb{Z}_2}$ containing p is orientable.*

Proof. Let F denote the component of $M^{\mathbb{Z}_2}$ that contains p. Assume on the contrary that F is not orientable. The component F is a compact submanifold of M, and the S^1-action on M acts on F as a restriction with a fixed point p. This implies that $\dim F = 2$. As shown in [5], by the classification of S^1-actions on 2-dimensional compact manifolds, these imply that F is $\mathbb{RP}^2$; moreover, the action of $S^1/\mathbb{Z}_2$ on F has $Z := \mathbb{RP}^1$ as a component of the $\mathbb{Z}_2$-fixed point set, and the normal bundle N_1 of Z in F is isomorphic to the canonical line bundle over $Z = \mathbb{RP}^1$, that is, a neighborhood of Z in F is diffeomorphic to an open Möbius strip. Therefore, Z is a $\mathbb{Z}_4$-fixed component of M. The normal bundle of Z in M splits into the Whitney sum of the normal bundle N_1 of Z in F and a real 2-dimensional $\mathbb{Z}_4$-vector bundle N_2 over Z, on which $\mathbb{Z}_4$ acts on fibers freely outside the zero section. Then N_1 is non-orientable while N_2 is orientable because the $\mathbb{Z}_4$-action on fibers makes N_2 a complex line bundle. This contradicts the orientability of M. □

A fixed point with weight $w > 1$ lies in an invariant 2-sphere with weight w, and the 2-sphere contains another S^1-fixed point.

Lemma 2.3. *Let M be a 4-dimensional compact oriented S^1-manifold. Let p be an isolated fixed point and let $w > 1$ be a weight at p. Then* *the component of $M^{\mathbb{Z}_w}$ containing p is a 2-sphere, invariant under the S^1-action, and it contains another fixed point q that has a weight w.*

Proof. Let F be the component of $M^{\mathbb{Z}_w}$ that contains p. Because M is compact, the component F is a closed submanifold of M. Since the action is effective, two weights at p are relatively prime and F has dimension 2. By Theorem 2.1 (for $w > 2$) and Lemma 2.2 (for $w = 2$), F is orientable. The S^1-action on M restricts to an action on this 2-dimensional closed orientable manifold F and has a fixed point p. This implies that F is a 2-sphere and has another fixed point q with a weight w. □

3. Atiyah–Hirzebruch formula in dimension 4

3.1. **Rigidity of genera.** By rigidity of a genus of a manifold we mean that if a group (torus) acts on a certain type of (compact) manifold, then its equivariant genus is equal to its ordinary genus; in particular, the equivariant genus is a constant.

Atiyah and Hirzebruch first proved such a rigidity result that if the circle group acts on a compact (oriented) spin manifol, then its equivariant $\hat{A}$-genus vanishes, by using the Atiyah-Singer index theorem [2]. Krichever used the equivariant cobordism theory to prove that for a unitary S^1-manifold, the Hirzebruch $T_{x,y}$-genus is rigid [23]. Rigidity of genera has been studied extensively. We refer to [6] for a summury of the history on the theory of equivariant genera.

3.2. **Oriented manifold, dimension 4.** For a compact oriented manifold, the L-genus is the genus of the power series $\frac{\sqrt{x}}{\tanh \sqrt{x}}$. The Hirzebruch signature theorem states that for a compact oriented manifold M, its L-genus evaluated on the fundamental class of M is equal to the signature of M. The Atiyah-Signer index theorem states that the signature of M is equal to the (analytic) index of the signature operator on M [3].

Let the circle act on a $2n$-dimensional compact oriented manifold M. For each element z of the circle group S^1, its equivariant index of the signature operator on M is defined. Let F be a fixed component. Let $\dim F = 2m$ and $TM|_F = TF \oplus L_1 \oplus \cdots \oplus L_{n-m}$, where the circle acts on each L_j with weight w_j. Denote by $c_1(L_j)$ the first Chern class of L_j, $P(F) = \prod_{i=1}^{m}(1 + x_i^2)$ the total Pontryagin class of F, and $[F]$ the fundamental homology class of F.

The signature (in other words, the L-genus) of a compact oriented S^1-manifold is rigid; moreover, the following formula holds for the equivariant signature.

Theorem 3.1. [12, p. 72] *Let the circle act on a $2n$-dimensional compact oriented manifold M. The equivariant signature sign(z, M) of M is*

$$\operatorname{sign}(M) = \operatorname{sign}(z, M) = \sum_{F \subset M^{S^1}} \left\{ \left(\prod_{i=1}^{m} x_i \frac{1+e^{-x_i}}{1-e^{-x_i}} \right) \left(\prod_{j=1}^{n-m} \frac{1+z^{w_j} e^{-c_1(L_j)}}{1-z^{w_j} e^{-c_1(L_j)}} \right) \right\} [F]$$

for all indeterminates z, and is a constant.

Note that the signature is the only rigid Hirzebruch genera for oriented manifolds [28, Theorem 4.2].

Now let M have dimension 4. We label isolated fixed points by p_1, $\cdots$, p_m and fixed surfaces by F_1, $\cdots$, F_k. Let w_{i1} and w_{i2} denote the weights at p_i, for $1 \leq i \leq m$. Let n_j denote the Euler number of the normal bundle of F_j, for $1 \leq j \leq k$. With these notations, we derive the Atiyah-Hirzebruch formula for a 4-dimensional compact oriented S^1-manifold.

Theorem 3.2. *Let S^1 act on a 4-dimensional compact oriented manifold M. The signature* sign(M) *of M satisfies*

$$\operatorname{sign}(M) = \sum_{i=1}^{m} \varepsilon_i \frac{(1+z^{w_{i1}})(1+z^{w_{i2}})}{(1-z^{w_{i1}})(1-z^{w_{i2}})} - \sum_{j=1}^{k} \frac{4zn_j}{(1-z)^2} = \sum_{i=1}^{m} \varepsilon_i \tag{1}$$

for all indeterminates z, and

$$\operatorname{sign}(M) = \frac{1}{3} p_1(M),$$

where $p_1(M)$ is the first Pontryagin class of M.

Proof. First, we consider an isolated fixed point p_i. Its total Pontryagin class is $P(p_i) = 1$ and $c_1(L_j) = 0$ for $j = 1, 2$. Thus, in the formula of Theorem 3.1 the fixed point p_i contributes the term

$$\varepsilon_i \frac{(1+z^{w_{i1}})(1+z^{w_{i2}})}{(1-z^{w_{i1}})(1-z^{w_{i2}})}.$$

Let F_j be a fixed surface. Let $P(F_j) = 1+x^2$ be the total Pontryagin class of F_j. Then

$$x \frac{1+e^{-x}}{1-e^{-x}} = x \frac{2-x+\cdots}{x-\frac{x^2}{2}+\cdots} = \frac{2-x+\cdots}{1-\frac{x}{2}+\cdots} =$$
$$(2-x+\cdots)\left(1+\frac{x}{2}+\cdots\right) = 2+x-x+\cdots = 2+0\cdot x+O(x^2).$$

Next, let L denote the normal bundle of F_j, $u \in H^2(F_j)$ a generator, and $c_1(L) = n_j u$. The weight w_j of F_j is 1. Then

$$\frac{1+z^{w_j}e^{-c_1(L_j)}}{1-z^{w_j}e^{-c_1(L_j)}} = \frac{1+ze^{-n_j u}}{1-ze^{-n_j u}} = \frac{(1+z)+z(-n_j u)+\cdots}{(1-z)+zn_j u+\cdots}.$$

Using $(1-z)+zn_j u+\cdots = (1-z)(1-(-\frac{zn_j u}{1-z})+\cdots)$, this is equal to

$$\frac{1}{1-z}\cdot\frac{(1+z)-zn_j u+\cdots}{1-(-\frac{zn_j u}{1-z})+\cdots} =$$

$$\frac{1+z}{1-z}+\left(\frac{-zn_j u}{1-z}+\frac{1+z}{1-z}\cdot(-\frac{zn_j u}{1-z})\right)+\cdots = \frac{1+z}{1-z}+\frac{-2n_j z}{(1-z)^2}u+\cdots.$$

Thus,

$$\left\{x\frac{1+e^{-x}}{1-e^{-x}}\cdot\frac{1+ze^{-c_1(L_j)}}{1-ze^{-c_1(L_j)}}\right\}[F_j] = \frac{-4zn_j}{(1-z)^2}.$$

Therefore, the first equation follows. Taking $z=0$, the second equation holds. The last equation $\text{sign}(M) = \frac{1}{3}p_1(M)$ follows by the Hirzebruch signature theorem. □

Note that the Atiyah-Hirzebruch formula above does not see the genus of a fixed surface. Also, note that the Atiyah-Hirzebruch formula is derived when a finite group of odd order acts on a compact oriented 4-manifold [3, Proposition 6.18, p. 585] and [13, p. 176]. Using the above formula, we obtain the following formulas.

Theorem 3.3. *Let S^1 act on a 4-dimensional compact oriented manifold M. The following equations hold.*

$$(1)\ \ L(M) = \sum_{i=1}^m \varepsilon_i \frac{w_{i1}^2+w_{i2}^2+1}{3w_{i1}w_{i2}}.$$

$$(2)\ \ 0 = \sum_{i=1}^m \varepsilon_i \frac{1}{w_{i1}w_{i2}} + \sum_{j=1}^k (-n_j).$$

$$(3)\ \ 3L(M) = \sum_{i=1}^m \varepsilon_i \frac{w_{i1}^2+w_{i2}^2}{w_{i1}w_{i2}} + \sum_{j=1}^k n_j.$$

Proof. We consider Equation (1). Let $t = z-1$. First we show that

$$\frac{(1+z^a)(1+z^b)}{(1-z^a)(1-z^b)} = \frac{4}{ab}t^{-2} + \frac{4}{ab}t^{-1} + \frac{a^2+b^2+1}{3ab} + O(t),$$

where a and b are positive integers. Indeed,

$$\frac{(1+z^a)(1+z^b)}{(1-z^a)(1-z^b)} = \frac{(1+(1+t)^a)(1+(1+t)^b)}{(1-(1+t)^a)(1-(1+t)^b)} =$$
$$\frac{(2+at+\frac{a(a-1)}{2}t^2+O(t^3))(2+bt+\frac{b(b-1)}{2}t^2+O(t^3))}{abt^2(1+\frac{a-1}{2}t+\frac{(a-1)(a-2)}{6}t^2+O(t^3))(1+\frac{b-1}{2}t+\frac{(b-1)(b-2)}{6}t^2+O(t^3))}.$$

Letting $-A = \frac{a-1}{2}t + \frac{(a-1)(a-2)}{6}t^2 + O(t^3)$ and using the geometric series expansion $\frac{1}{1-A} = 1 + A + A^2 + \cdots$,

$$\begin{aligned}
&\frac{2+at+\frac{a(a-1)}{2}t^2+O(t^3)}{1+\frac{a-1}{2}t+\frac{(a-1)(a-2)}{6}t^2+O(t^3)} \\
&= \left(2+at+\frac{a(a-1)}{2}t^2+O(t^3)\right)(1+A+A^2+\cdots) \\
&= 2-(a-1)t+at-\frac{(a-1)(a-2)}{3}t^2+\frac{(a-1)^2}{2}t^2-\frac{a(a-1)}{2}t^2+ \\
&\qquad \frac{a(a-1)}{2}t^2+O(t^3) \\
&= 2+t+\frac{a^2-1}{6}t^2+O(t^3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{(1+z^a)(1+z^b)}{(1-z^a)(1-z^b)} \\
&= \frac{1}{abt^2}\left(2+t+\frac{a^2-1}{6}t^2+O(t^3)\right)\left(2+t+\frac{b^2-1}{6}t^2+O(t^3)\right) \\
&= \frac{1}{abt^2}\left(4+4t+\frac{1}{3}(a^2+b^2+1)t^2+O(t^3)\right).
\end{aligned}$$

Moreover,

$$\frac{z}{(1-z)^2} = \frac{1}{t^2}+\frac{1}{t}.$$

Therefore, comparing the constant terms in Equation (1) proves the first equation. Moreover, comparing the coefficients of the t^{-2}-terms in Equation (1), we get

$$0 = \sum_{i=1}^{m} \varepsilon_i \frac{4}{w_{i1}w_{i2}} + \sum_{j=1}^{k}(-4n_j),$$

which proves the second equation. The first and second equations imply the third equation. □

If there are only isolated fixed points, Theorem 3.3 implies the following formulas.

Corollary 3.4. *Let S^1 act on a 4-dimensional compact oriented manifold M. Suppose that there are only isolated fixed points. The following formulas hold.*

$$(1)\ L(M) = \sum_{i=1}^{m} \varepsilon_i \frac{w_{i1}^2 + w_{i2}^2}{3w_{i1}w_{i2}}.$$
$$(2)\ 0 = \sum_{i=1}^{m} \varepsilon_i \frac{1}{w_{i1}w_{i2}}.$$

Proof. Since there are only isolated fixed points, (2) of Theorem 3.3 implies the second formula of this corollary. Together with (1) of Theorem 3.3, this implies the first formula. □

Let M be a $2n$-dimensional compact oriented S^1-manifold. For an isolated fixed point p_i, denote by k_i the number of weights at p_i that are equal to 1. If the fixed point set of M consists of isolated points only, [26, Lemma 1.1] states that

$$\sum_i \varepsilon_i k_i = 0;$$

also see [16]. Now let M have dimension 4, and suppose M has isolated fixed points $p_1, \cdots, p_m$ and fixed surfaces $F_1, \cdots, F_k$, Then k_i can be 0, 1, or 2. The below lemma, which is an extension in dimension 4 of the above fact, can be derived from Equation (1) by the same way as [26, Lemma 1.1], as we prove below.

Lemma 3.5. *Let M be a 4-dimensional compact oriented S^1-manifold with isolated fixed points $p_1, \cdots, p_m$ and fixed surfaces $F_1, \cdots, F_k$. Then*

$$\sum_{i=1}^{m} \varepsilon_i k_i - 2\sum_{j=1}^{k} n_j = 0, \tag{2}$$

where k_i is the multiplicity of weight 1 at p_i.

Proof. By Theorem 3.2, the signature of M satisfies

$$\mathrm{sign}(M) = \sum_{i=1}^{m} \varepsilon_i \frac{(z^{w_{i1}}+1)(z^{w_{i2}}+1)}{(z^{w_{i1}}-1)(z^{w_{i2}}-1)} - \sum_{j=1}^{k} \frac{4zn_j}{(1-z)^2}.$$

This formula holds for all indeterminate z, and is a constant.

For an isolated fixed point p_i,

$$\frac{d}{dz}\left(\frac{(z^{w_{i1}}+1)(z^{w_{i2}}+1)}{(z^{w_{i1}}-1)(z^{w_{i2}}-1)}\right)\Bigg|_{z=0}$$

$$= \begin{cases} 0 & \text{if none of } w_{i1} \text{ and } w_{i2} \text{ are equal to 1} \\ 2 & \text{if exactly one of } w_{i1} \text{ and } w_{i2} \text{ is equal to 1} \\ 4 & \text{if both } w_{i1} \text{ and } w_{i2} \text{ are equal to 1} \end{cases}$$

For a fixed surface F_j,

$$\frac{d}{dz}\left(\frac{4zn_j}{(1-z)^2}\right)\Big|_{z=0} = 4n_j.$$

Therefore, taking the derivative of the above formula and evaluating at $z=0$, the lemma follows. □

Lemma 3.5 enables us to associate a graph to a 4-dimensional compact oriented S^1-manifold, as we do so in the next section.

4. Graph

4.1. **Virtual graph of weights.** To a 4-dimensional compact oriented S^1-manifold, we shall associate a graph that contains information on data on the fixed point set. For this, we introduce terminologies.

Definition 4.1. *A (4-dimensional virtual)* ***graph of weights*** *is a graph defined as follows.*

1. *Its vertex set consists of two types of vertices, called* ***points*** $p_1, \cdots, p_m$ *and* ***surfaces*** $F_1, \cdots, F_k$.
2. *To each point p_i is associated sign $+1$ or -1, denoted ε_i.*
3. *To each surface F_j is associated an integer n_j.*
4. *Each point p has two edges, and the edges are labeled by positive integers that are relatively prime, called* ***weights*** *of p.*
5. *Each surface has $2|n_j|$ edges, each of which has label 1.*

Definition 4.2. *Let M be a 4-dimensional compact oriented S^1-manifold. We say that G_W is a* ***graph of weights of*** *M if the following hold.*

1. *The vertices of G_W are the fixed components of M.*
2. *The sign of a point in G_W is the sign of the corresponding isolated fixed point.*
3. *The labels of edges of a point of G_W are the weights of the corresponding isolated fixed point.*
4. *For a surface F_j in G_W, the number n_j is the Euler number of the normal bundle of the corresponding fixed surface.*
5. *For a fixed surface F_j in M, the corresponding surface of G_W has $2|n_j|$ edges, all of label 1.*
6. *If two vertices are connected by an edge whose label w is bigger than 1, then the corresponding fixed points are in the same*

component of $M^{\mathbb{Z}_w}$, which is an invariant 2-sphere with weight w.

With the terminologies, we show that for a 4-dimensional compact oriented S^1-manifold, we can encode its fixed point data in a graph.

Theorem 4.3. *Let M be a 4-dimensional compact oriented S^1-manifold. Then a graph G_W of weights for M exists. Moreover, there is one that has no self-loops with the following property: if there is an edge of label 1 between two vertices, then the corresponding fixed components contribute the terms with different signs in Equation (2) of Lemma 3.5.*

Proof. Let $p_1, \cdots, p_m$ denote isolated fixed points and let $F_1, \cdots, F_k$ be fixed surfaces. To each p_i we assign a vertex, also denoted by p_i, which is a point; the sign of the vertex p_i is the sign ε_i of the corresponding fixed point p_i. To each F_j we assign a vertex, also denoted F_j, which is a surface.

Suppose that a fixed point p_i has weight w that is bigger than 1. By Lemma 2.3, p_i lies in a 2-sphere, which is a component of $M^{\mathbb{Z}_w}$ and contains another fixed point p_l that has a weight w. We draw an edge between p_i and p_l and give the edge a label w.

Next, we draw edges for weight 1. We rewrite Equation (2) as

$$\sum_{\varepsilon_i>0} k_i - 2\sum_{n_j<0} n_j = \sum_{\varepsilon_i<0} k_i + 2\sum_{n_j>0} n_j, \tag{3}$$

so that each fixed component contributes Equation (3) in one side of the equation with a positive coefficient. If $\varepsilon_i > 0$ ($\varepsilon_i < 0$), then p_i contributes to Equation (3) in the left side (right side) by k_i, which is positive. Similarly, if $n_j < 0$ ($n_j > 0$) then F_j contributes to Equation (3) in the left side (right side) by $-2n_j$ ($2n_j$), which is positive. Now, to each isolated fixed point p_i we assign k_i edges of label 1 and to each fixed surface F_j, we assign $2n_j$ edges of label 1, so that an edge of label 1 has vertices (fixed components) v and v' whose contributions as fixed components in Equation (3) are in the opposite sides. □

We illustrate a graph of weights with an example.

Example 4.4. *Let S^1 act on the complex projective space $\mathbb{CP}^2$ by*

$$g \cdot [z_0 : z_1 : z_2] = [gz_0 : z_1 : z_2]$$

for all $g \in S^1 \subset \mathbb{C}$ and $[z_0 : z_1 : z_2] \in \mathbb{CP}^2$. There are two fixed components, $p = [1 : 0 : 0]$ and $F = [0 : z_1 : z_2]$. The complex weights at p are $-1, -1$ and thus $\varepsilon_p = +1$ and $w_{p1} = w_{p2} = 1$.

The Euler number of the normal bundle of F is 1. Then Figure 1 is a graph of weights for $\mathbb{CP}^2$ with the action. The fixed point p has weights

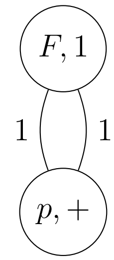


FIGURE 1. Graph of weights for Example 4.4

$\{1,1\}$ and sign $+1$ and so a corresponding vertex has two edges of label 1 and has sign $+$. The fixed surface F has Euler number 1 for its normal bundle, and so its corresponding vertex is assigned a number 1 and has 2 edges of label 1. In this case, the Atiyah-Hirzebruch formula (Theorem 3.2) is

$$\operatorname{sign}(\mathbb{CP}^2) = \frac{(z+1)(z+1)}{(z-1)(z-1)} - \frac{4z}{(z-1)^2} = \frac{z^2-2z+1}{(z-1)^2} = 1.$$

There are two 2-spheres $F_1 = [z_0 : z_1 : 0]$ and $F_2 = [z_0 : 0 : z_2]$ connecting the fixed components p and F, which are both invariant 2-spheres with weight 1.

Example 4.5. *Let $S^4 = \{(z_1, z_2, x) \in \mathbb{C}\times\mathbb{C}\times\mathbb{R} : |z_1|^2+|z_2|^2+x^2 = 1\}$ be the 4-sphere. Let S^1 act on the 4-sphere S^4 by*

$$g \cdot (z_1, z_2, x) = (gz_1, z_2, x)$$

for all $g \in S^1 \subset \mathbb{C}$ and for all $(z_1, z_2, x) \in S^4$. The action has one fixed component $F := \{(0, z_2, x) \in S^4\}$, which is a 2-sphere. The Euler number of the normal bundle of F is 0. Thus, a graph of weights for this action on S^4 consists of one vertex that is a surface, with 0 assigned to the vertex.

Remark 4.6. *A graph of weights does not determine an S^1-manifold uniquely. Here is an example. Consider the S^1-action on $\mathbb{CP}^2$ in Example 4.4 and its graph of weights Figure 1. Let N be any 4-dimensional compact connected oriented S^1-manifold that has no fixed points. Take an equivariant connected sum along free orbits of Example 4.4 and N; let M denote the connected sum. Then Figure 1 is also a graph of weights for M, but Example 4.4 and M are in general not (equivariantly) diffeomorphic.*

4.2. **Euler number of an edge.** Given a graph of weights, for an edge we shall define a number and call it the Euler number of the edge. If the graph is of a 4-dimensional compact oriented S^1-manifold, for an edge of label bigger than 1, we shall see that the Euler number of an edge is exactly the Euler number of the normal bundle of the invariant 2-sphere corresponding to the edge.

Definition 4.7. *Let G_W be a graph of weights. Let e be an edge with label w_e, and let v be a vertex of e. We define a number, called the* ***other weight*** *of w_e at v, as follows.*

(1) *If v is a point, then the other weight of w_e at v is the label of the other edge of v.*
(2) *If v is a surface, then the other weight of v is 0.*

Definition 4.8. *Let G_W be a graph of weights. Let e be an edge with label w_e. Let v and v' be the vertices of e, and let w and w' be the other weight of w_e at v and v', respectively. We define a rational number, denoted n_e and called the* ***Euler number of*** *e, by*

$$n_e := \frac{\varepsilon_v w + \varepsilon_{v'} w'}{w_e}.$$

Here, if v is a surface, we define ε_v to be 1 and similarly for v'.

Definition 4.9. *We say that a graph of weights is* ***admissible*** *if the Euler number of each edge is an integer.*

For an integer n, we consider a complex line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$ with Euler number n, the quotient of $(\mathbb{C}^2 - \{0\}) \times \mathbb{C}$ by a C^*-action

$$z \cdot (z_1, z_2, w) = (zz_1, zz_2, z^n w).$$

Let S^1 act on $\mathcal{O}(n)$ by

$$g \cdot [z_1, z_2, w] = [z_1, g^{u_1} z_2, g^{u_2} w]$$

for some integers u_1 and u_2.

Suppose that $u_1 \neq 0$. Then this action has two fixed points $q_1 = [1, 0, 0]$ and $q_2 = [0, 1, 0]$ that have (complex) weights $\{u_1, u_2\}$ and $\{-u_1, -nu_1 + u_2\}$. If we let $u_1 = w_e$, $u_2 = \epsilon_i w_{i2}$, $-nu_1 + u_2 = -\epsilon_k w_{k2}$, then we have

$$n = \frac{\epsilon_i w_{i2} + \epsilon_k w_{k2}}{w_e}.$$

Theorem 4.10. *Let M be a 4-dimensional compact oriented S^1-manifold. Let G_W be its graph of weights. Let e be an edge whose weight is bigger than 1. Then the Euler number n_e of the edge is equal to the Euler number of the normal bundle of the invariant 2-sphere of weight w_e, containing two fixed points that correspond to the vertices of e.*

Proof. Let p_i and p_k be the vertices (fixed points) of e and let $w = w_{i1} = w_{k1}$ be the weight of the edge e. By (1) of Theorem 4.3, p_i and p_k are in an invariant 2-sphere S_e of weight w. Choose an orientation of S_e, so that S^1 acts on $T_{p_i}S_e$ with weight w_{i1} and on $T_{p_k}S_e$ with weight $-w_{k1}$. Then S^1 acts on $N_{p_i}S_e$ with weight $\epsilon_i w_{i2}$ and on $N_{p_k}S_e$ with weight $-\epsilon_k w_{k2}$.

The normal bundle of S_e is orientation preserving equivariantly diffeomorphic to $\mathcal{O}(n)$ for some integer n with $u_1 = w_{i1}$ and $u_2 = \epsilon_i w_{i2}$. Thus, $-\epsilon_k w_{k2}$ is equal to $-nu_1 + u_2 = -nw_{i1} + \epsilon_i w_{i2}$, that is, $-\epsilon_k w_{k2} = -nw_{i1} + \epsilon_i w_{i2}$. Therefore, the Euler number n of the normal bundle of S_e is $n = \frac{\varepsilon_i w_{i2} + \varepsilon_k w_{k2}}{w_{i1}}$, which is the Euler number n_e of the edge e. □

Any graph of weights of a 4-dimensional compact oriented S^1-manifold is admissible.

Proposition 4.11. *Let M be a 4-dimensional compact oriented S^1-manifold and let G_W be its graph of weights. Then G_W is admissible.*

Proof. Let e be an edge of G_W. Let v and v' be the vertices of e, and let w (w') be the other weight of v (v').

If its label w_e is 1, then its Euler number $n_e = \frac{\varepsilon_v w + \varepsilon_{v'} w'}{w_e} = \varepsilon_v w + \varepsilon_{v'} w'$ is an integer.

If its label w_e is bigger than 1, then Theorem 4.10 implies that its Euler number n_e is an integer. □

With the above theorem, we give another description of the L-genus of a 4-dimensional oriented S^1-manifold in terms of n_e and n_j.

Theorem 4.12. *Let M be a 4-dimensional compact oriented S^1-manifold and G_W its graph of weights. Then*

$$3\,L(M) = \sum_{e \in E(G_W)} n_e + \sum_{j=1}^{k} n_j,$$

where $E(G_W)$ denotes the set of edges of G_W.

Proof. By the last equation of Theorem 3.3,

$$3L(M) = \sum_{i=1}^{m} \varepsilon_i \left(\frac{w_{i1}}{w_{i2}} + \frac{w_{i2}}{w_{i1}} \right) + \sum_{j=1}^{k} n_j.$$

Let e be an edge of G_W. Let w_e denote its label.

First, suppose that the edge is between two isolated fixed points p_i and p_k. Let w and w' be the other weight at p_i and p_k, respectively. Then $\varepsilon_i \frac{w}{w_e}$ is one of the two terms $\varepsilon_i(\frac{w_{i1}}{w_{i2}} + \frac{w_{i2}}{w_{i1}})$ and similarly $\varepsilon_k \frac{w'}{w_e}$ is one of the two terms $\varepsilon_k(\frac{w_{k1}}{w_{k2}}+\frac{w_{k2}}{w_{k1}})$ in (3) of Theorem 3.3. By Definition 4.8, their sum $\frac{\varepsilon_v w+\varepsilon_{v'} w'}{w_e}$ is the Euler number n_e of the edge e.

Second, suppose that the edge is between an isolated fixed point p_i and a fixed surface F_k. Let w be the other weight at p_i. The term $\varepsilon_i \frac{w}{w_e}$ is one of the two terms $\varepsilon_i(\frac{w_{i1}}{w_{i2}}+\frac{w_{i2}}{w_{i1}})$ in (3) of Theorem 3.3. The weight of the tangent bundle of F_k is zero, and this agrees with the other weight w' of F_k as a vertex of G_W. By Definition 4.8, the sum $\frac{\varepsilon_v w+\varepsilon_{v'} w'}{w_e}=\frac{\varepsilon_v w}{w_e}$ is the Euler number n_e of the edge e.

Third, suppose that the edge is between two fixed surfaces F_i and F_k. The weight of the tangent bundle of F_i (F_k) is zero, which is the other weight w (w') of the vertex F_i (F_k, respectively). Thus $0=\frac{\varepsilon_v w+\varepsilon_{v'} w'}{w_e}$ is the Euler number n_e of the edge e.

To sum up, by rearranging the terms $\varepsilon_i(\frac{w_{i1}}{w_{i2}}+\frac{w_{i2}}{w_{i1}})$ for the edges, we get

$$\sum_{i=1}^{m}\varepsilon_i\left(\frac{w_{i1}}{w_{i2}}+\frac{w_{i2}}{w_{i1}}\right)=\sum_{e\in E(G)}\frac{\varepsilon_i w_{i2}+\varepsilon_k w_{k2}}{w_e}=\sum_{e\in E(G)} n_e.$$

This proves the theorem. □

Let M be a 4-dimensional compact oriented manifold and let G_W be its graph of weights. By Theorem 4.10, every edge with label bigger than 1 corresponds to an invariant 2-sphere. It is a natural question to ask, if there exists a graph of weights, in which an edge with label 1 also corresponds to an invariant 2-sphere.

Question 4.13. *Let M be a 4-dimensional compact oriented S^1-manifold. Does there exists a graph G_W of weights, such that any edge e of label 1 between vertices p and q corresponds to an invariant 2-sphere with weight 1 between fixed components p and q?*

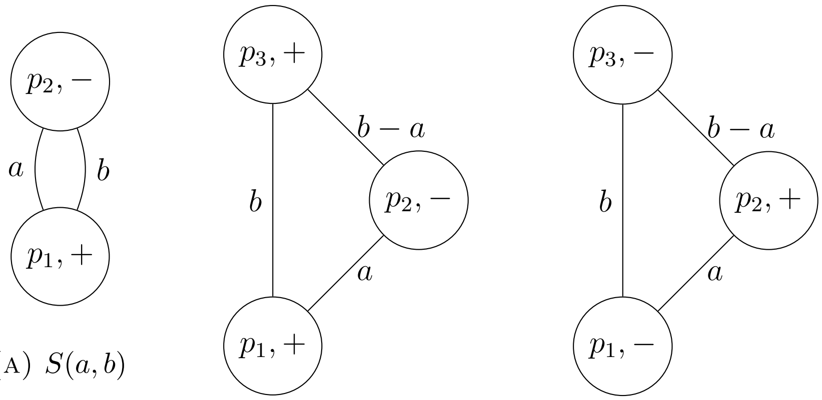
For instance, the answer to Question 4.13 is affirmative in Example 4.4.

4.3. **Graphs for basic manifolds.** We illustrate graphs of weights for basic 4-dimensional oriented S^1-manifolds.

Example 4.14. *Let a and b be positive integers. Let $S(a,b)$ denote an S^1-action on S^4 with weights a and b. That is, let S^1 act on the 4-sphere $S^4=\{(z_1,z_2,x)\in\mathbb{C}^2\times\mathbb{R}:|z_1|^2+|z_2|^2+x^2=1\}$ by*

$$g\cdot(z_1,z_2,x)=(g^a z_1,g^b z_2,x).$$

The action has two fixed points $p_1=(0,0,1)$ and $p_2=(0,0,-1)$. Both fixed points have weights $\{a,b\}$, and the sign of p_1 is $+1$ and the sign of p_2 is -1. Figure 2a is the graph of weights for S^4 with this action.



(A) $S(a, b)$ (B) $P(a, b)$ (C) $Q(a, b)$

FIGURE 2. Graphs for $S(a, b)$, $P(a, b)$, $Q(a, b)$

The above example and its graph can be extended to any even sphere; see [16, 19]. Any S^1-action on a compact oriented manifold with two fixed points have the same signs and weights at the fixed points as a linear S^1-action on S^{2n} [16, 18, 22]. In addition, see [29] for S^1-actions with two fixed points on unitary manifolds.

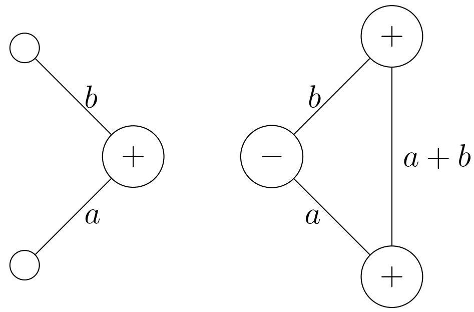
Example 4.15. *Let $0 < a < b$ be integers. Consider an action of S^1 on the complex projective space $\mathbb{CP}^2$ by*

$$g \cdot [z_0 : z_1 : z_2] = [z_0 : g^a z_1 : g^b z_2]$$

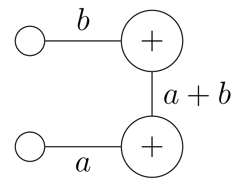
for all $g \in S^1 \subset \mathbb{C}$. The action has 3 fixed points, $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, and $p_3 = [0 : 0 : 1]$, whose signs are $+1$, -1, and $+1$, and weights are $\{a, b\}$, $\{a, b - a\}$, and $\{b - a, b\}$, as their complex weights are $\{a, b\}$, $\{-a, b - a\}$, and $\{-b, a - b\}$, respectively. Denote $\mathbb{CP}^2$ with this action by $P(a, b)$ and $-\mathbb{CP}^2$ by $Q(a, b)$, where $-\mathbb{CP}^2$ denotes $\mathbb{CP}^2$ with the opposite orientation. Figure 2b is a graph of weights for $P(a, b)$, and Figure 2c is a graph of weights for $Q(a, b)$.

5. Equivariant connected sum and splitting

Connected sum and splitting are classical operations in topology. If two manifolds of same dimension admit torus actions, then we may take a connected sum equivariantly, so that the connected sum is equipped with a torus action and similarly for splitting. Orlik and Raymond considered equivariant splitting of a T^2-action on a compact, simply connected oriented 4-manifold into a rather minimal manifold [30] and Fintushel considered that of an S^1-action [10]. They showed that such a manifold is an equivariant connected sum of a homology S^4 and copies of $\pm\mathbb{CP}^2$ and $S^2 \times S^2$. Via equivariant splitting, the second author



(A) Graphs of weights for M and $P(a, a+b)$



(B) Connected sum

FIGURE 3. Equivariant connected sum of M and $P(a, a+b)$ at p and $[0:1:0]$; its converse is equivariant splitting

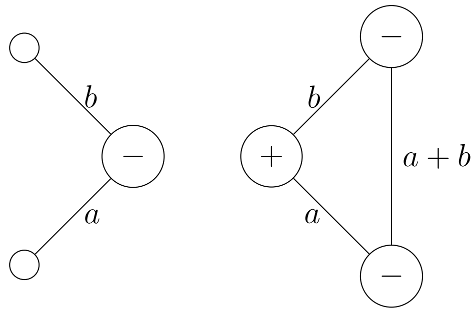
studied generators of unitary S^1-bordism ring [27]. In this section, we discuss equivariant connected sum and splitting, of a circle action on an oriented 4-manifold.

5.1. **Connected sum and splitting.** Let M be a 4-dimensional compact oriented S^1-manifold. Let p be an isolated fixed point. Let a and b be the weights at p.

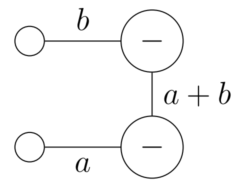
Suppose that the sign of p is $+1$. Consider the manifold $P(a, a+b)$; its fixed point $[0:1:0]$ has weights $\{a, b\}$ and sign -1. Thus, there is an orientation reversing S^1-equivariant diffeomorphism between a neighborhood of p in M and a neighborhood of $[0:1:0]$ in $P(a, a+b)$. By using the equivariant diffeomorphism, we can take an equivariant connected sum of M and $P(a, a+b)$ at p and $[0:1:0]$, to get another 4-dimensional compact oriented S^1-manifold N. Figure 3a is a graph of weights for $M \sqcup P(a, a+b)$, and Figure 3b is one for N.

The converse of the above equivariant connected sum is equivariant splitting. Let N be a 4-dimensional compact oriented S^1-manifold and suppose that there is an invariant 2-sphere with weight $a+b$ between two fixed points q and q', which have weights $\{a, a+b\}$ and $\{b, a+b\}$ and both have sign $+1$. Then by the reverse operation of the above equivariant connected sum, we can equivariantly split N into another 4-dimensional compact oriented S^1-manifold M and $P(a, a+b)$, whose graphs of weights are the left and the right of Figure 3a, respectively.

Suppose that the sign of p is -1. In this case, we can take an equivariant connected sum N of M and $Q(a, a+b)$ at p and $[0:1:0]$; now $[0:1:0]$ has the same weights $\{a, b\}$ but has sign $+1$. Figure 4a is a graph of weights for $M \sqcup Q(a, a+b)$, and Figure 4b is one for N. Conversely, if in a 4-dimensional compact oriented S^1-manifold N

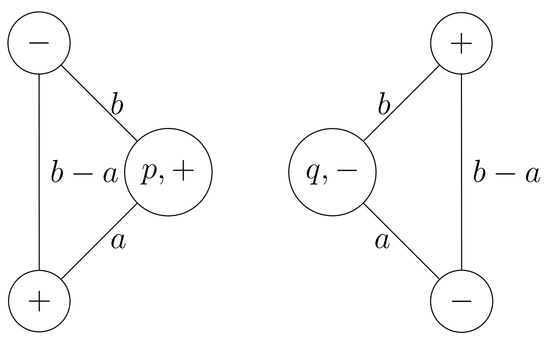


(A) Graphs of weights for M and $Q(a, a+b)$

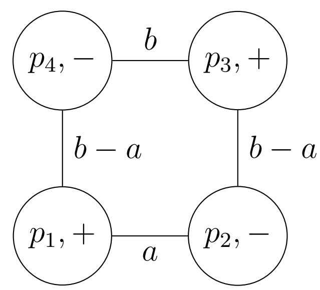


(B) Connected sum

FIGURE 4. Equivariant connected sum of M and $Q(a, a+b)$ at p and $[0:1:0]$; its converse is equivariant splitting



(A) Graphs of weights for $P(a, b)$ and $Q(a, b)$



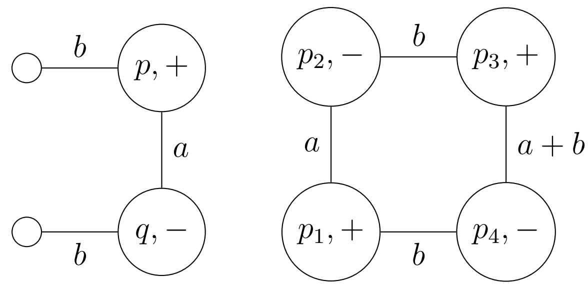
(B) Connected sum of $P(a, b)$ and $Q(a, b)$ at $[1:0:0]$

FIGURE 5. Equivariant connected sum of $P(a, b)$ and $Q(a, b)$ at $[1:0:0]$

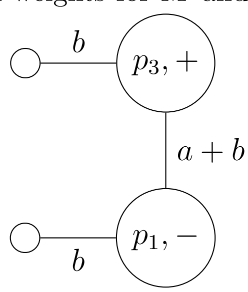
there is an invariant 2-sphere with weight $a+b$ between two fixed points that have weights $\{a, a+b\}$ and $\{b, a+b\}$ and both fixed points have sign -1, we can equivariant split N into M and $Q(a, a+b)$; Figure 4b is a graph of weights for N and Figure 4a is one for $M \sqcup Q(a, a+b)$.

Example 5.1. *Consider the two S^1-manifolds $P(a, b)$ and $Q(a, b)$. Their graphs of weights are the left and right figures of Figure 5; note that the positions of vertices are different from Figures 2b and 2c. The fixed point $p = [1:0:0]$ in $P(a, b)$ has sign $+1$ and weights $\{a, b\}$ and the fixed point $q = [1:0:0]$ in $Q(a, b)$ has sign -1 and weights $\{a, b\}$. Therefore, Let $P\#Q(a, b)$ denote a connected sum of $P(a, b)$ and $Q(a, b)$ at p and q. Then Figure 5b is a graph of weights of $P\#Q(a, b)$.*

Let M be a 4-dimensional compact oriented S^1-manifold. Suppose that two fixed points p and q both have weights $\{a, b\}$, $\varepsilon_p = +1$,



(A) Graphs of weights for M and $P\#Q(a, a+b)$



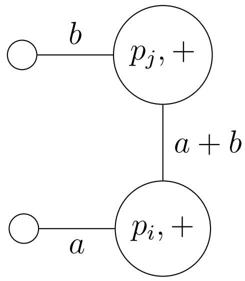
(B) Connected sum

FIGURE 6. Connected sum at the invariant spheres of M and $P\#Q(a, a+b)$

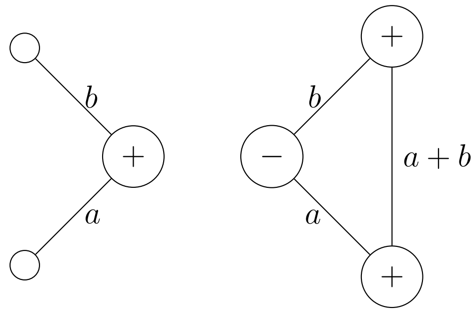
$\varepsilon_q = -1$, and there is an invariant 2-sphere S with weight a between p and q. Now, there is an orientation reversing equivariant diffeomorphism between an equivariant tubular neighborhod of the 2-sphere S and an equivariant tubular neighborhood of the 2-sphere S' of $P\#Q(a, a+b)$ containing p_1 and p_2 of Example 5.1. Thus, we can equivariantly glue M and $P\#Q(a, a+b)$ along neighborhoods of S and S', to construct another S^1-manifold N. Figure 6a is a graph of weights for $M \sqcup P\#Q(a, a+b)$, and Figure 6b is a graph of weights for N.

The converse of the above equivariant connected sum is an equivariant splitting of N into M and $P\#Q(a, a+b)$; Let N be a 4-dimensional compact oriented S^1-manifold and let G_N be its graph of weights. Suppose that there is an invariant sphere of weight $a+b$ between two fixed points p' and q' such that $\varepsilon_{p'} = +1$, $\varepsilon_{q'} = -1$, and both p' and q' have weights $\{a+b, b\}$ for some positive integers a and b; Figure 6b is a graph of weights for N with $p_3 = p'$ and $p_1 = q'$. By the reverse of the above equivariant connected sum, we can equivariantly split N into another 4-dimensional compact oriented S^1-manifold M and $P\#Q(a, a+b)$, where Figure 6a is a graph of weights for $M \sqcup P\#Q(a, a+b)$.

5.2. **Reducing every weight to 1 via splitting.** Given a 4-dimensional compact oriented S^1-manifold, via equivariant splitting we can convert



(A) Graph for M



(B) Graphs for M' and $P(a, a+b)$

FIGURE 7. Equivariant splitting of M into M' and $P(a, a+b)$

it to another S^1-manifold, which is minimal in the sense that weights are small.

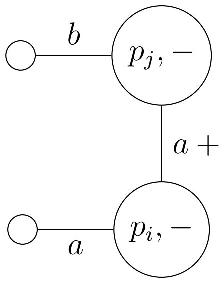
Theorem 5.2. *Let M be a 4-dimensional compact oriented S^1-manifold. Then we can decompose M into an equivariant connected sum of M_0 and copies of $\pm\mathbb{CP}^2$ and Σ_1, where every weight in the normal bundle of any fixed component of M_0 is 1.*

Proof. This theorem holds if the biggest weight over all fixed components of M is 1. Thus, suppose that the biggest weight ℓ is bigger than 1. Every weight in the normal bundle of a fixed surface is 1, and thus any weight larger than 1 occurs at isolated fixed points. Let $G_W(M)$ be a graph of weights that satisfies the properties of Theorem 4.3.

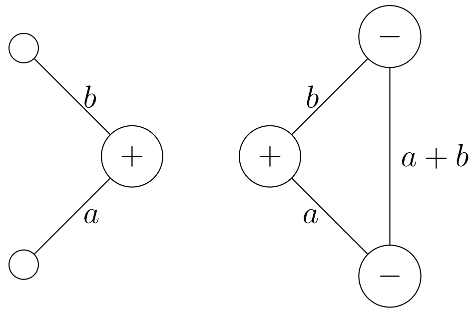
Pick an edge e whose weight is ℓ, and let p_i and p_j be the vertices of e. The fixed points p_i and p_j have weights $\{\ell, a\}$ and $\{\ell, b\}$, respectively, for some positive integers $a, b < \ell$. Because $\ell > 1$, by Theorem 4.10, there is an invariant 2-sphere S_e containing p_i and p_j, and the Euler number of the normal bundle of S_e is the Euler number n_e of the edge, which is $n_e = \frac{\varepsilon_i a + \varepsilon_j b}{\ell}$; in particular, n_e is an integer. Thus, one of the following two cases occurs.

(1) $\varepsilon_i = \varepsilon_j$ and $a + b = \ell$.
(2) $\varepsilon_i = -\varepsilon_j$ and $a = b$.

Assume that Case (1) holds. First, suppose that $\varepsilon_i = \varepsilon_j = +1$. Figure 7a is a graph of weights for M near p_i and p_j. As described in the third paragraph of Section 5.1, we can split M equivariantly into another 4-dimensional compact oriented S^1-manifold M' and $P(a, a+b)$, whose graphs of weights are the left (denoted by $G_W(M')$) and right figures of Figure 7b, respectively.

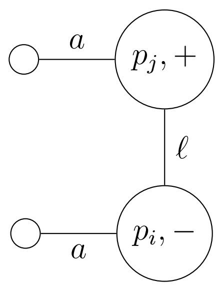


(A) Graph for M

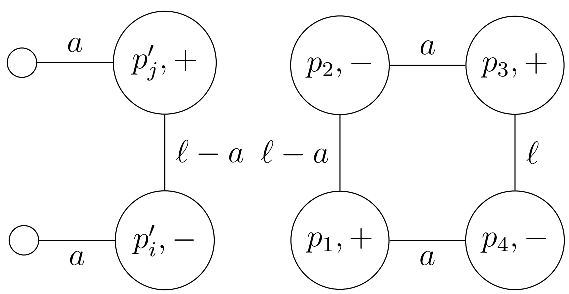


(B) Graphs for M' and $Q(a, a+b)$

FIGURE 8. Equivariant splitting of M into M' and $Q(a, a+b)$



(A) Graph for M



(B) Graphs for M' and $P\#Q(\ell - a, \ell)$

FIGURE 9. Equivariant splitting of M into M' and $P\#Q(\ell - a, \ell)$

The case that $\varepsilon_i = \varepsilon_j = -1$ is analogous. Figure 8a is a graph of weights for M near p_i and p_j. As described in the fourth paragraph of Section 5.1, we can split M equivariantly into another 4-dimensional compact oriented S^1-manifold M' and $Q(a, a+b)$, whose graphs of weights are the left (denoted by $G_W(M')$) and right figures of Figure 8b, respectively.

(A) $P(1,0)$

(B) $Q(1,0)$

FIGURE 10. Graph for $P(1,0)$ and $Q(1,0)$

Assume that Case (2) holds. Figure 9a is a graph of weights for M near p_i and p_j. Without loss of generality, assume that $\varepsilon_i = -1 = -\varepsilon_j$. As described in the last paragraph of Section 5.1, we can split M equivariantly into another 4-dimensional compact oriented S^1-manifold M' and $P\#Q(\ell - a, \ell)$, whose graphs of weights are the left (denoted by $G_W(M')$) and right figures of Figure 9b, respectively.

We have that $G_W(M')$ is a graph of weights for M'. On M' we consider the biggest weight, and repeat the above prodecure. Continuing the above argument, this theorem follows. □

5.3. **Further: convert to one that has fixed surfaces only.** We can further convert any 4-dimensional compact oriented S^1-manifold into another S^1-manifold, which has only fixed surfaces.

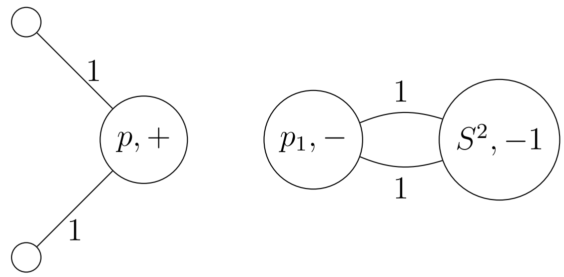
Theorem 5.3. *Let the circle act on a 4-dimensional compact oriented manifold M. Then by taking equivariant splittings with $\pm\mathbb{CP}^2$, Σ_1 and by taking connected sums with $\pm\mathbb{CP}^2$, we can convert M into another 4-dimensional compact oriented S^1-manifold N, which only has fixed surfaces.*

Proof. By Theorem 5.2, we can decompose M into an equivariant connected sum of M_0 and copies of $\pm\mathbb{CP}^2$, where every weight in the normal bundle of any fixed component of M_0 is 1.

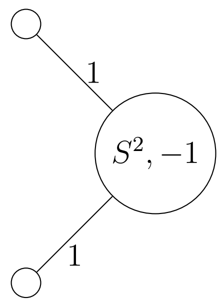
Let p be an isolated fixed point in M_0. It has weights $\{1,1\}$. Suppose that $\varepsilon_p = +1$. In this case, we can take an equivariant connected sum of M_0 and $Q(1,0)$ at p and $[1:0:0]$ to construct another S^1-manifold M', in which p is replaced with a fixed 2-sphere. Figure 11a is a graph of weights for $M \sqcup Q(1,0)$ and Figure 11b is one for M'.

If $\varepsilon_p = -1$, then we take an equivariant connected sum of M_0 and $P(1,0)$ at p and $[1:0:0]$; Figure 12a is a graph of weights for $M \sqcup P(1,0)$ and Figure 12b is one for M'.

Continuing this to every isolated fixed point, the theorem follows. □

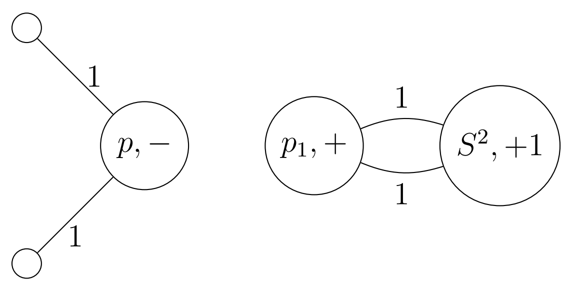


(A) Graphs for M_0 and $Q(1, 0)$

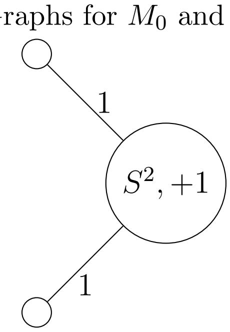


(B) Connected sum

FIGURE 11. Connected sum of M_0 and $Q(1, 0)$ at p and p_1



(A) Graphs for M_0 and $P(1, 0)$



(B) Connected sum

FIGURE 12. Connected sum of M_0 and $P(1, 0)$ at p and p_1

Note that by the second and third equations of Theorem 3.3, the manifold N satisfies

$$\sum_{i=1}^{k} n_j = 0$$

and

$$L(N') = 0.$$

Theorem 5.3 raises the following natural questions.

Question 5.4. *Let N be a 4-dimensional compact oriented S^1-manifold with fixed surfaces only.*

1. *Is the normal bundle of any fixed surface of N trivial?*
2. *If not, can we construct an example of such N that has only fixed surfaces but some normal bundle of a fixed surface is not trivial?*
3. *Or, can we further convert N to another S^1-manifold, in which the normal bundle of any fixed surface has Euler number zero?*

6. Isolated fixed points

6.1. **Converse to Theorems 3.2 and 4.3.** In the case of isolated fixed points, we prove the converse of a combination of Theorem 3.2 and Theorem 4.3 that if an abstract graph of weights satisfies the Atiyah-Hirzebruch formula, then the graph is realized as one for a 4-dimensional oriented S^1-manifold.

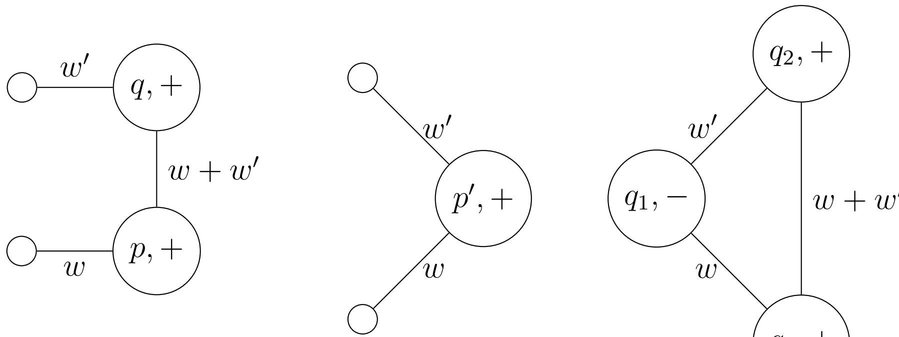
Theorem 6.1. *Let G_W be an admissible graph of weights that has points only. Suppose that G_W satisfies the Atiyah-Hirzebruch formula, Equation (1). Then there exists a 4-dimensional compact connected oriented S^1-manifold, whose graph of weights is G_W.*

Proof. We shall mean that the graph G_W satisfies the Atiyah-Hirzebruch formula (Equation (1)) in the following sense: if G_W has vertices $v_1, \cdots, v_m$ and a vertex v_i has sign ε_i and edges e_{i1} and e_{i2} with labels w_{i1} and w_{i2}, respectively, then

$$\sum_{i=1}^{m} \varepsilon_i \frac{(1+z^{w_{i1}})(1+z^{w_{i2}})}{(1-z^{w_{i1}})(1-z^{w_{i2}})} \tag{4}$$

is a constant for all indeterminates z.

The idea of proof is analogous to that of Theorem 5.2. Let e be an edge whose label w_e is the biggest among the labels of the edges of G_W; let p and q be the vertices of e, and let w (w') be the label of the other edge of p (q, respectively). If $w_e = 1$ or $w_e = 2$, then the theorem follows from Lemma 6.2. Thus, from now on, we assume that $w_e > 2$.



(A) G_W

(B) G'_W and graph of $P(w, w+w')$

FIGURE 13. Splitting of G_W into G'_W and graph of $P(w, w+w')$

By the definition of a graph of weights, w_e and w are relatively prime and similarly for w_e and w'. In particular, $w, w' < w_e$. Because G_W is admissible, the Euler number $n_e = \frac{\varepsilon_p w + \varepsilon_q w'}{w_e}$ of e is an integer. Thus, one of the following cases holds.

(1) $\varepsilon_p = \varepsilon_q = +1$ and $w + w' = w_e$.
(2) $\varepsilon_p = \varepsilon_q = -1$ and $w + w' = w_e$.
(3) $\varepsilon_p = -\varepsilon_q$ and $w = w'$.

Suppose that Case (1) holds. Assume that there exists a 4-dimensional compact connected oriented S^1-manifold M whose graph of weights is G_W; see Figure 13a. As described in the third paragraph of Section 5.1, by equivariant splitting, we can decompose M into another 4-dimensional compact connected oriented S^1-manifold M' and $P(w, w+w')$; their graphs are the left (denoted G'_W) and right figures of Figure 13b.

The right figure of Figure 13b, a graph of weights for $P(w, w+w')$, satisfies Equation (1); it is

(5)
$$\frac{(1+z^w)(1+z^{w+w'})}{(1-z^w)(1-z^{w+w'})} - \frac{(1+z^w)(1+z^{w'})}{(1-z^w)(1-z^{w'})} + \frac{(1+z^{w'})(1+z^{w+w'})}{(1-z^{w'})(1-z^{w+w'})} = 1,$$

which is a constant for all indeterminates z.

We consider (1) for G'_W;

(6)
$$\sum_{v'_i \in V(G'_W)} \varepsilon'_i \frac{(1+z^{w'_{i1}})(1+z^{w'_{i2}})}{(1-z^{w'_{i1}})(1-z^{w'_{i2}})}.$$

The graph G'_W is obtained from G_W by shrinking the edge e to a vertex p', removing vertices p and q. The vertex set of G'_W minus p' is equal to the vertex set of G_W minus p and q. In (6), the vertex p' of G'_W contributes the term

$$\frac{(1+z^w)(1+z^{w'})}{(1-z^w)(1-z^{w'})}.$$

The sum of this term with (5) is

$$\frac{(1+z^w)(1+z^{w+w'})}{(1-z^w)(1-z^{w+w'})}+\frac{(1+z^{w'})(1+z^{w+w'})}{(1-z^{w'})(1-z^{w+w'})},$$

which are precisely the terms that p and q contribute in (4). In other words, the sum of (5) and (6) is exactly (4). This means that if G_W satisfies Equation (1), then so does G'_W, and vice versa. The manifold M splits equivariantly into M' and $P(w,w+w')$, and M is an equivariant connected sum of M' and $P(w,w+w')$. Therefore, the existence of such a manifold M for G_W is equivalent to the existence of M' for G'_W, which has one less edge with label w_e. Moreover, G_W is admissible if and only if G'_W is admissible.

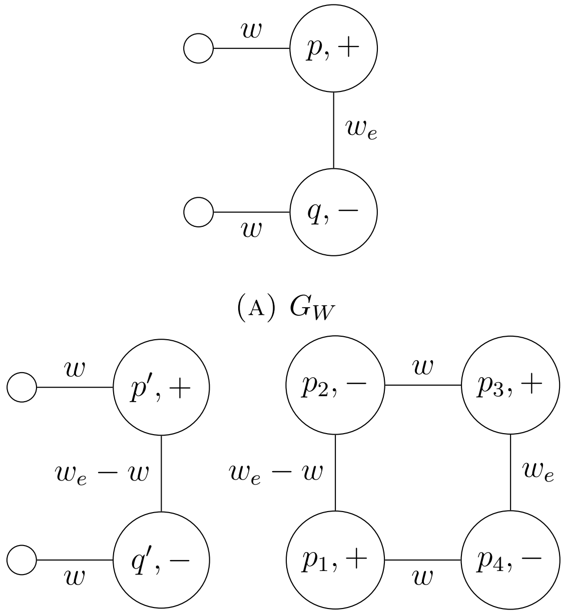
Case (2) is completely analogous. In this case, the labels of edges of p and q are the same as Case (1) but their signs are reversed. Thus instead of $P(w,w+w')$ and its graph (Figure 2b), we use $Q(w,w+w')$ and its graph (Figure 2c).

Suppose that Case (3) holds. Assume that $\varepsilon_p=-\varepsilon_q=+1$; the other case $\varepsilon_p=-\varepsilon_q=-1$ is analogous. Assume that there exists a 4-dimensional compact connected oriented S^1-manifold M whose graph of weights is G_W; see Figure 14a. As described in the last paragraph of Section 5.1, by equivariant splitting, we can decompose M into another 4-dimensional compact connected oriented S^1-manifold M' and $P\#Q(w_e-w,w)$; their graphs are the left (denoted G'_W) and right figures of Figure 14b.

The right figure of Figure 14b, which is a graph of weights for $P\#Q(w_e-w,w)$, satisfies Equation (1); it is

$$-\frac{(1+z^w)(1+z^{w_e})}{(1-z^w)(1-z^{w_e})}+\frac{(1+z^w)(1+z^{w_e-w})}{(1-z^w)(1-z^{w_e-w})}+\frac{(1+z^w)(1+z^{w_e})}{(1-z^w)(1-z^{w_e})}-\frac{(1+z^w)(1+z^{w_e-w})}{(1-z^w)(1-z^{w_e-w})}=0, \tag{7}$$

which is a constant for all indeterminates z.



(A) G_W

(B) G'_W and graph of $P\#Q(w_e - w, w)$

FIGURE 14. Splitting of G_W into G'_W and graph of $P\#Q(w_e - w, w)$

We consider (1) for G'_W;

$$\sum_{v'_i \in V(G'_W)} \varepsilon'_i \frac{(1+z^{w'_{i1}})(1+z^{w'_{i2}})}{(1-z^{w'_{i1}})(1-z^{w'_{i2}})}. \tag{8}$$

The graph G'_W is obtained from G_W by replacing the label w_e of the edge e with $w_e - w$. The vertex set of G'_W is equal to the vertex set of G_W. In (8), the vertices p' and q' (that are p and q in G_W) of G'_W contribute the terms

$$\frac{(1+z^w)(1+z^{w_e-w})}{(1-z^w)(1-z^{w_e-w})} - \frac{(1+z^w)(1+z^{w_e-w})}{(1-z^w)(1-z^{w_e-w})}.$$

The sum of these terms with (7) is

$$\frac{(1+z^w)(1+z^{w_e})}{(1-z^w)(1-z^{w_e})} - \frac{(1+z^w)(1+z^{w_e})}{(1-z^w)(1-z^{w_e})},$$

which are precisely the terms that p and q contribute in (4). In other words, the sum of (7) and (8) is exactly (4). This means that if G_W satisfies Equation (1), then so does G'_W, and vice versa. The manifold M splits equivariantly into M' and $P\#Q(w_e - w, w)$, and M is an equivariant connected sum of M' and $P\#Q(w_e-w,w)$. Therefore, the existence of such a manifold M for G_W is equivalent to the existence of M' for G'_W, which has one less edge with label w_e. Moreover, G_W is admissible if and only if G'_W is admissible.

We repeat the above procedure. Assume that an admissible graph G'_W of weights satisfies (1), consider the edge whose label is the biggest, and so on. Repeating the above, the existence of a 4-dimensional compact connected oriented S^1-manifold whose graph of weights is G_W, reduces to the existence of another 4-dimensional compact connected oriented S^1-manifold M'' whose graph G''_W of weights satisfies Equation (1) and only has edges of label 1. By Lemma 6.2 below, such a manifold M'' exists. □

Lemma 6.2. *Let G_W be a graph of weights that has points only, satisfies (1), and the label of every edge of G_W is 1. Then there exists a 4-dimensional compact connected oriented S^1-manifold M, whose graph of weights is G_W.*

Proof. Let $v_1,\cdots,v_m$ be the vertices of G_W, and let ε_i be the sign of v_i. By the assumption, G_W satisfies (1); because every label is 1, it is

$$\sum_{i=1}^{m}\varepsilon_i\,\frac{(1+z)(1+z)}{(1-z)(1-z)}=\sum_{i=1}^{n}\varepsilon_i\,\{(1+z)(1+z+z^2+\cdots)\}^2=$$
$$\sum_{i=1}^{n}\varepsilon_i\,(1+2\sum_{j=1}^{\infty}z^j)^2,$$

and is a constant for all indeterminates z. For the formula to be a constant for all indeterminates z, we must have that the formula is identically zero and the number k of vertices with sign $+1$ is equal to the number of vertices with sign -1.

Let M be an equivariant connected sum along free orbits of k-copies of $S(1,1)$ of Example 4.14. Each $S(1,1)$ has two fixed points $(0,0,1)$ with sign $+1$ and $(0,0,-1)$ with sign -1 that have weights $\{1,1\}$. Thus, M has k fixed points with sign $+1$ and weights $\{1,1\}$ and k fixed points with sign -1 and weights $\{1,1\}$. By Definition 4.2, G_W is a graph of weights of M. □

6.2. **Minimal manifold's graph has vanishing Euler number.** For a 4-dimensional oriented S^1-manifold has isolated fixed points, its minimal manifold admits a graph of weights whose edges have vanishing Euler numbers.

Corollary 6.3. *Let the circle act on a 4-dimensional compact oriented manifold M with isolated fixed points. Then we can decompose M into* *an equivariant connected sum of M_0 and copies of $\pm\mathbb{CP}^2$, where every weight at a fixed point of M_0 is 1, and M_0 admits a graph G_W of weights for which $n_e = 0$ for all edges e of G_W.*

Proof. Because M has isolated fixed points only, by Theorem 5.2, we can decompose M into an equivariant connected sum of M_0 and copies of $\pm\mathbb{CP}^2$, where every weight at a fixed point of M_0 is 1, and the proof of Theorem 5.2 implies that M_0 also has only isolated fixed points.

Let G_W be a graph of weights of M_0 that satisfies the properties of Theorem 4.3. Let e be any edge of G_W. Let p_i and p_j be the vertices of e. By the property of Theorem 4.3, $\varepsilon_i = -\varepsilon_j$. Since both fixed points p_i and p_j have weights $\{1, 1\}$, by Definition 4.7, the other weight of w_e at p_i (p_j) is 1. By Definition 4.8, the Euler number of e is

$$n_e = \frac{\varepsilon_i \cdot 1 + \varepsilon_j \cdot 1}{1} = 0.$$

□

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