11-2023

Geometric aspects of quantization and relationship to integrability

Paul Bracken

Physica Scripta

PAPER • OPEN ACCESS

Geometric aspects of quantization and relationship to integrability

To cite this article: Paul Bracken 2023 *Phys. Scr.* **98** 125202

Physica Scripta

PAPER

RECEIVED
4 June 2023

REVISED
6 October 2023

ACCEPTED FOR PUBLICATION
19 October 2023

PUBLISHED
1 November 2023

Geometric aspects of quantization and relationship to integrability

Paul Bracken
Department of Mathematics, University of Texas, Edinburg, TX 78541-2999, United States of America
E-mail: paul.bracken@utrgv.edu

Keywords: quantum, geometry, integrability, conservation, curvature, evolution, connection[1], integrable,

Abstract
It is the case that quantum mechanics has a deep geometric structure and can be presented accordingly. Quantum mechanics is to a certain degree foreshadowed by the geometry inherent in the geometric structure of classical mechanics. The purpose here is to present some new results and proofs which impact the mathematical structure of quantum mechanics. The relationship between integrability and quantum physics is investigated in terms of geometric ideas and structures. Two physical examples are drawn from these mathematical ideas which directly relate to physics. An introduction as to how these ideas can be extended to infinite degrees of freedom is also described.

1. Introduction

Quantization is a process that refers to the construction of a quantum mechanical system with respect to a particular classical system. It is this process and its implications for the relationship of quantum mechanics and integrability that is the subject studied here. We attempt to justify the machinery of geometric quantization on something like physical grounds.

Classical systems have a deep geometric structure at their root as they are founded on symplectic manifolds. A symplectic manifold is a pair (M, Ω) such that M is a manifold and Ω is a closed non-degenerate two-form defined on M. In general, Ω is required to be closed which means $d\Omega = 0$ and d is exterior differentiation. This means the induced Poisson bracket satisfies the Jacobi identity and so the flows of Hamiltonian vector fields consist of canonical transformations [1–6]. A vector field X on M is called Hamiltonian if there exists a function $H\colon M \to \mathbb{R}$ such that $i_X \Omega = dH$ for all $w \in T_x M$ then $\Omega_x(X, w) = dH(x)\, w$ and X_H is written for X. Hamilton's equations are the evolution equations

$$\dot{z} = X_H(z). \tag{1.1}$$

In finite dimensions, these are given in canonical coordinates by

$$\frac{dq^i}{\partial t} = \frac{dH}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{1.2}$$

The classical phase space M is oriented and carries a natural volume element

$$dv_\omega = (-1)^{n(n-1)/2}\,\frac{1}{n!}\omega^n = dp_1 \wedge \cdots \wedge dp_n \wedge dq^1 \wedge \cdots \wedge dq^n. \tag{1.3}$$

This is the Liouville form and there is an obvious Hilbert space associated with M. This is the space $L^2(M)$ of square integrable complex valued functions on M with inner product

$$(\varphi, \psi) = \int \varphi\bar{\psi}\, dv_\omega, \qquad \varphi, \psi \in L^2(M). \tag{1.4}$$

Many of the original paths to quantization can be formulated as what is called Dirac's quantization rule. This rule associates operators to the simplest classical observable which appear in dynamical problems such as position and momentum functions. Thus it is a map from classical observables to symmetric operators which preserves the bracket relations and sends the constant function 1 to the identity operator **I**. The basic geometric construction is called pre-quantization and was originally presented by Souriau and Kostant [7–10]. The construction of the physical quantum Hilbert space makes use of an additional structure which is called a polarization. The idea of a Lax pair can also be extended to quantum systems. As an example, if a symplectic formulation of the theory of constraints is given, the geometric significance of the classification of constraints and the Dirac bracket can be given. Two other extended physical examples which are physical in nature are given. Other ways of introducing the quantum integrability exist and overlap with this framework as well [11–14]. A well posed form for quantum integrability is necessary in looking at quantum chaos. A similar but more algebraic approach is to let $S = \{Q_j | [Q_j, Q_{j'}] = 0,\ j,\ j' = 1,\ldots,N\}$ be a complete set of commuting observables of a quantum system. A basis set $\{|\alpha\rangle\}$ of its Hilbert space can be labeled completely by M quantum numbers $\{\alpha_j | j = 1,\ldots,M\}$ which are related to the observables of the non-fully degenerate observables $\{Q_j: j = 1,\ldots,M\}$ with $M \leqslant N$ a subset of S. Then M is defined as the number of quantum dynamical degrees of freedom. Dynamical degrees of freedom means that the quantum degrees of freedom depend on the Hilbert-space structure which is determined by the system's dynamical properties and include the system's internal quantum degrees of freedom and intrinsic properties.

2. The problem of quantization

The kinematic relationship between the classical and the quantum scales brings one to the first major issue. Given manifold M and ω, is it possible to reconstruct Hilbert space H with observables by a set of symmetric operators **O** which act on H. The quantum condition enunciated by Dirac provides a link between the classical and quantum domains. It states first that operators in set **O** form a Hilbert space representation of a subalgebra $\mathcal{C}$ of the classical observables. The selection of a subalgebra $\mathcal{C}$ such that the following postulates hold involves picking out some additional structure on M which could be thought of as a precursor of quantum mechanics. To each $f \in \mathcal{C}$ (*i*) the map $f \to \hat{f}$ is linear over $\mathbb{R}$ (*ii*) for such $f,\ g \in \mathcal{C}$ it holds that the commutator is $[\hat{f}, \hat{g}] = -i\hbar\ \hat{k}$ where $k = \{f, g\}$ and $[\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}$. In Dirac's view, the Poisson bracket is the classical analogue of the quantum commutator. This condition is not in itself going to determine uniquely the underlying quantum system. As well, not every Hilbert space and set of operators that satisfy the algebra represent a physically reasonable quantization. The constant functions in $C^\infty(M)$ must be present in $\mathcal{C}$ and must be represented by corresponding multiples of the identity operator which has to be in **O**.

If M is an elementary symplectic manifold for some Lie algebra $\mathbf{g}$ and ϑ: $\mathbf{g} \to C^\infty(M)$, then if H is the Hilbert space of an elementary quantum system, $\mathcal{C}$ must contain $\vartheta(\mathbf{g})$ and the corresponding operators in **O** must form an irreducible representation of $\mathbf{g}$ [15, 16]. The central idea of quantum integrability from this point of view is dynamical symmetry. Integrability is a fundamental concept in the study of dynamical systems. Symmetry restricts the possible forms of Lagrangian, but not the dynamics. A quantum system with a dynamical group **G** has a dynamical symmetry if and only if the Hamiltonian of the system can be written in terms of the Casimir operators C_{ki}^{α} of any particular subgroup chain $\mathbf{G}^a$ of **G**: $\hat{H} = \vartheta(C_{ki}^{\alpha})$ where $k = s^\alpha,\ldots,1;\ i = 1,\ldots,l_k^\alpha\ l_k^\alpha$ the rank of subgroup G_k^α and α is fixed and labels the particular subgroup chain [17].

It does not seem possible to have a one-to-one correspondence between **O** and all of $C^\infty(M)$ without making Hilbert space H too large. The choice of a subalgebra $\mathcal{C}$ for which both (*i*) and (*ii*) hold selects additional structure on M. A dynamical relationship exists between the classical and quantum pictures. First of all, the canonical flows generated by the elements of $\mathcal{C}$ by one-parameter unitary groups acting on M have to be represented. There should be a link between the evolution generated by the classical Hamiltonian and the corresponding time development in H. Formally the canonical flow generated by any $f \in \mathcal{C}$ can be replaced by the unitary group generated by $\hat{f}$. However to construct a group of unitary transformations from an observable $\hat{f} \in \mathbf{O}$, it must be that $\hat{f}$ is actually self-adjoint. Both a formal expression for $\hat{f}$ and a specific domain for it have to be specified. A construction for Hilbert space H and set of operators **O** could be sought where only the symplectic structure of M is used and an operator in **O** can be associated to every classical observable in $C^\infty(M)$. A subalgebra is usually taken such that (*i*) and (*ii*) both hold and involves developing more structure on M. A vestige of quantum mechanics already originates in the classical system possibly involving an element of choice along the way [14, 18, 19].

3. Construction of hilbert space *H* and operators O

It is natural to begin with a construction for H and **O** in which only the symplectic structure of M is used and there is an operator in **O** for every classical observable in $C^\infty(M)$. This is often called pre-quantization. A classical observable f is thought to act on $L^2(M)$ in the form of a symmetric operator $\varphi \to -i\hbar X_f \varphi$ where $\varphi \in L^2(M)$ or in a dense subset. The vector field X_f is discussed shortly. This correspondence does satisfy the quantum condition. Since X_f vanishes when $df = 0$, it is however physically insufficient. It can be modified to the form $\varphi \to -i\hbar X_f(\varphi) + f\varphi$, but the constants do not act on $L^2(M)$ by multiplication. However, property (*ii*) no longer holds, since for $f, g \in C^\infty(M)$

$$[-i\hbar\ X_f + f,\, -i\hbar\ X_g + g]$$
$$=-\hbar^2(X_fX_g - X_gX_f)\,\varphi - i\hbar\ fX_g\varphi - i\hbar\,(X_fg)\,\varphi - i\hbar g\ X_f\ \varphi + i\hbar\,(X_g f)\,\varphi + i\hbar\ f\ X_g\ \varphi \tag{3.1}$$
$$=-\hbar^2(X_fX_g - X_gX_f)\,\varphi - i\hbar\,(X_fg)\varphi + i\hbar\,(X_g f)\varphi - \hbar^2\ X_{\{f,g\}}\varphi - i\hbar\,(X_fg - X_g f)\,\varphi.$$

In terms of coordinates $X_f$acts in the following way

$$X_f g = \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} - \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} = \{f, g\}. \tag{3.2}$$

As well (3.2) lets us simplify (3.1) to

$$[-i\hbar\ X_f + f,\, -i\hbar\ X_g + g] = -i\hbar\,(i\hbar\ X_{\{f,g\}} + 2\,\{f, g\}). \tag{3.3}$$

If this is modified slightly, the following operator results

$$\hat{f}(\varphi) = \frac{\hbar}{i}\,[X_f(\varphi) - \frac{i}{\hbar}(X_f \rfloor \theta)\,\varphi] + f\ \varphi. \tag{3.4}$$

The constants still act by multiplication and the quantum condition holds, as we show, noting that $X_f\rfloor\theta$ is a scalar quantity and vector fields are derivations. Starting with

$$\hat{f}\hat{g}\ \psi = -i\hbar\,[X_f(\hat{g}\psi) - \frac{i}{\hbar}(X_f \rfloor \theta)\hat{g}\psi] + f\hat{g}\psi, \qquad \hat{g}\hat{f}\,\psi = -i\hbar\,[X_g(\hat{f}\,\psi) - \frac{i}{\hbar}(X_g \rfloor \theta)\hat{f}\,\psi] + g\hat{f}\ \psi. \tag{3.5}$$

and observing that many terms cancel, what remains is

$$[\hat{f}, \hat{g}]\psi = -\hbar^2\,(X_fX_g - X_gX_f)\psi - i\hbar X_f(X_g \rfloor \theta)\psi + i\hbar X_g(X_f \rfloor \theta)\psi - i\hbar\ \psi X_f g\ +\ i\hbar\ \psi X_g f$$
$$=-\hbar^2[X_f, X_g]\ \psi - i\hbar\psi(X_fg - X_f(X_g \rfloor \theta) - X_g f + X_g(X_f \rfloor \theta))$$
$$=-\hbar^2[X_f, X_g]\psi - i\hbar\ (X_fg - X_f(X_g \rfloor \theta) - X_g f + X_g(X_f \rfloor \theta))$$
$$=-\hbar^2[X_f, X_g]\ \psi - i\hbar\psi\ (X_f(g - X_g \rfloor \theta) - X_g(f - X_f \rfloor \theta)) = i\hbar\ \hat{\eta}\ \psi,$$

where $\eta = \{f, g\}$.

The construction of $\hat{f}$ depends on the choice of a symplectic potential, so $\hat{f}$ is defined in a global way when θ is exact giving the quantization an unnatural aspect. This can be avoided by admitting gauge transformations. If θ is replaced by $\theta' = \theta + du$, then φ is replaced by

$$\varphi' = e^{i\ u/\hbar}\ \varphi \tag{3.6}$$

in which case the transition amplitudes remain invariant. Thus $\hat{f}$ becomes a well-defined operator independent of the θ chosen, and admits the following relation

$$e^{iu/\hbar}\{-i\hbar\ [X_f\varphi - \frac{i}{\hbar}(X_f \rfloor \theta)\varphi] + f\varphi\} = -i\hbar\ [X_f\varphi' - \frac{i}{\hbar}(X_f \rfloor \theta')\varphi'] + f\varphi'. \tag{3.7}$$

There is a problem in that the Hilbert space it generates is too large to represent the phase space of a physically acceptable, plausible quantum system. This can be approached by working with a line bundle B and to admit sections of B that are parallel along a polarization of the classical phase space. A real polarization then of a symplectic manifold (P, ω) is a foliation of P by Lagrangian submanifolds called leaves. An uncomplicated case to start with is a symplectic manifold (M, ω) with a Kähler polarization P. Suppose (M, ω) is quantizeable. A line bundle B is kept fixed over M with connection $\nabla$and Hermitian structure $(\cdot, \cdot)$. Let $\mathcal{H}$ denote the square integrable sections of B.

Let $C_B^\infty(M)$ be the smooth sections with subspace $C_B^\infty(M,\ P)$ which satisfy for all $X \in V(M, P)$

$$\nabla_X\ s = 0, \tag{3.8}$$

the polarized sections, and define $\mathcal{H}_D = \mathcal{H} \cap C_B^\infty(M,\ P)$, the set of square integrable polarized sections of B. The space $\mathcal{H}_P$ is a Hilbert space and can act as a model for the quantum mechanical phase space.

A circle bundle can also be used over M. This is a principle bundle over M or fiber bundle π: $Q \to M$ with structure group $S^1 = \{e^{i\tau} | \tau \in \mathbb{R}\}$, which means that each fiber $\pi^{-1}(p)$ is a circle and there is a consistent action on $S^1 \times Q \to Q$ which is a multiplication on each fiber. This gives a quotient structure $M = Q/S^1$ such that S^1 acts freely on Q.

This allows us to define what we mean by the process of quantization. Let (P, ω) be a symplectic manifold. It is quantizeable if and only if there is a principle circle bundle π: $Q \to P$ over P and a one-form α on Q such that two conditions hold (a) α is invariant under the action of S^1 (b) $\pi^*\omega = d\,\alpha$. Then α is called a connection and ω is the corresponding curvature with Q the quantizing manifold. On fibers, write $\alpha|\pi^{-1}(m) = \hbar\, ds$ such that $\hbar$ is constant and can be identified with Planck's constant. The following theorem allows us to state when a system arising as a result of fiber bundle theory is quantizeable.

Theorem 3.1. *(P, ω) is quantizeable if and only if $\omega/h \in H^2(P, Z)$, so ω/h is an integral cohomology class. The inequivalent quantizing manifolds are classified by $H^1(P, \mathbb{Z})$.* □

The importance of the theorem is that $\omega/h \in H^2(P, \mathbb{Z})$ implies that when ω/h is integrated over a compact four-manifold $K \subset P$ without boundary an integer results. As an example, suppose ω is exact, then $P = T^*M$ is quantizeable. If P is simply connected, then Q is unique so $\omega = d\vartheta$. Let $Q = P \times S^1$ and let $\alpha = \vartheta + \hbar\, ds$. Since a principle bundle is involved and S^1 acts on $\mathbb{C}$, based on Q a complex line bundle L over P can be constructed out of general methods that come from fiber bundle theory. Each fiber S is replaced by the space $\mathbb{C}$ on which it acts. The connection α on Q gives a connection ∇on the line. The condition $\pi^* \omega = d\alpha$ implies that ω is the curvature form $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = i\omega(X, Y)$. In terms of coordinates

$$(\nabla_X s)^a = \frac{\partial s^a}{\partial x^i} X^i + \alpha^a_{ib}\, s^b\, X^i. \tag{3.9}$$

Theorem 3.2. *Let (P, ω) be a quantizeable symplectic manifold. Then there is a quantizeable map $f \to \hat{f}$, where $\hat{f}$ is an operator on the space of sections of B satisfying (i) $(f+g)\hat{} = \hat{f} + \hat{g}$ (ii) $(\lambda f)\hat{} = \lambda\hat{f}$ with $\lambda \in \mathbb{R}$ (iii) $\{f, g\}\hat{} = -i[\hat{f}, \hat{g}]$ (iv) $\hat{1} = I$ where 1 is the constant function and I is the identity operator. Moreover, the action of $\hat{f}$ is given by*

$$\hat{f}: s \to -i\,(\nabla_{X_f})\, s + f\, s. \tag{3.10}$$

To prove (i) write $-i(\nabla_{X_{f}+X_{g}})s + (f+g)\,s = -i\nabla_{X_f}\, s + fs - i\nabla_{X_g}\, s + g\, s = \hat{f} + \hat{g}$ (ii) proceeds in a similar way as (iv). To prove (iii) we know X_f preserves the phase space volume so $\hat{f}$ is a symmetric operator such that

$$[\hat{f}, \hat{g}]\, s = \hat{f}\hat{g}\, s - \hat{g}\hat{f}\, s = \hat{f}(-i\nabla_{X_g}s + g\, s) - \hat{g}(-i\nabla_{X_f}\, s + f\, s)$$

$$= \frac{1}{i}\{X_f\left(\frac{1}{i}\nabla_{X_g}s + g\, s\right) - \nabla_{X_g}\left(\frac{1}{i}\nabla_{X_f}s + fs\right) + \frac{1}{i}f\nabla_{X_g}\, s + fs - \frac{1}{i}g\nabla_{X_g}s - gf\, s\}$$

$$= -(\nabla_{X_f}\nabla_{X_g}\, s - \nabla_{X_g}\nabla_{X_f}\, s) + \frac{1}{i}\nabla_{X_f}\, gs - \frac{1}{i}\nabla_{X_g}\, fs + \frac{1}{i}f\, \nabla_{X_g}\, s - \frac{1}{i}g\nabla_{X_f}\, s$$

$$= -(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})\, s + \frac{1}{i}(g\, \nabla_{X_f}s + X_f(g)\, s) - \frac{1}{i}(f\nabla_{X_g}s + X_g(f)\, s) + \frac{1}{i}f\nabla_{X_g}s - \frac{1}{i}g\nabla_{X_f}\, s$$

$$= -(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})\, s + \frac{1}{i}(X_f(g)\, s - X_g(f)\, s).$$

This follows from the Leibnitz property $\nabla_X(gs) = (Xg)\, s + g\nabla_X s$ and a similar formula with f and g interchanged. On account of the fact that $\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f} - \nabla_{[X_g,X_f]} = -i\,\omega(X_f, X_g)$, it follows that

$$[\hat{f}, \hat{g}]\, s = -\nabla_{[X_f,X_g]}s + \frac{1}{i}\omega(X_f, X_g)\, s + \frac{1}{i}(X_f(g) - X_g(f))\, s$$

$$= -\nabla_{[X_f,X_g]}\, s + \frac{1}{i}\{f, g\}\, s + \frac{1}{i}(\{g, f\} - \{f, g\})\, s.$$

Since $[X_f, X_g] = -X_{\{f,g\}}$ the required result follows

$$[\hat{f}, \hat{g}]\, s = \nabla_{X_{\{f,g\}}}\, s - \frac{1}{i}\{f, g\}s = -\frac{1}{i}\widehat{\{f, g\}}\, s. \tag{3.11}$$

□

If $\omega = -d\theta$ is exact then locally it is the case that $\vartheta = \sum_i p_i\, dq^i$ and $Q = P \times S^1$ so $\hat{f}\, s$ takes the form

$$\hat{f}\, s = \frac{\hbar}{i} X_f\, s + \left(f - \sum_i p_i \frac{\partial f}{\partial p_i}\right) s,$$

where s: $P \to \mathbb{C}$. For all f, then $\hat{f}$ is a first-order operator and that the functions are on P with $H = L^2(P, \omega)$.

The expression (3.3) is certainly not arbitrary and has rigorous origin in a Hermitian line bundle with connection over M with potentials of the form $\beta = (1/\hbar)\vartheta$ or with curvature $\Omega = \omega/\hbar$. This follows from the fact that if (U, ψ) is a local trivialization of B and $s = \psi(\cdot\,, 1)$, the unit section over U and β the complex 1-form defined on U by

$$(X \rfloor \beta)\, s = -i\nabla_X\, s. \tag{3.12}$$

Then β is called a potential 1-form for ∇. If $s' = f\ s$ is any other smooth section, applying ∇_X to it gives

$$\nabla_X\, s' = (X(f) - i(X\rfloor\beta) f)\, s. \tag{3.13}$$

Substituting this into the equation

$$\Omega(X,\ Y) = \frac{i}{2}([\nabla_X,\ \nabla_Y] - \nabla_{[X,Y]}) \tag{3.14}$$

produces the result $\Omega = d\beta$. The following theorem gives a nice picture of what has been put forward.

Theorem 3.3. *The Schrödinger vector field X_H is hamiltonian and $2\hbar\ X_{\hat{H}} = X_H$, so the Schrödinger equation defines a classical hamiltonian system on H.*

Proof. Fix a $\psi \in H$ and let $Y \in TH$, then one finds

$$dH(\psi)(Z) = \frac{d}{dt} H(\psi + t\ Z)|_{t=0} = \frac{d}{dt}\langle\psi + tZ|\hat{H}\ (\psi + tZ)\rangle|_{t=0} = \langle Z|\hat{H}\psi\rangle + \langle\psi|\hat{H}\psi\rangle$$

$$= 2\ G(Z,\ \hat{H}\psi) = 2\hbar\Omega(Z,\ \frac{i}{\hbar}\ \hat{H}\psi) = 2\hbar\Omega(X_{\hat{H}},\ Z)\ \psi. \tag{3.15}$$

This implies that $2\hbar\ X_H$ is Hamiltonian with respect to Ω. □

4. Integrability conditions

There is a necessary and sufficient condition for such a line bundle to exist. It is that Ω should satisfy Weyl's integrability condition. It has the interesting feature that it can be closely related to the quantization rule that appears in earlier quantum theories. Suppose it is possible to find a Hermitian line bundle with connection over M with curvature Ω. Let ∇ be the connection, $(\cdot, \cdot)$ the compatible Hermitian structure on M and γ: $[0, 1] \to M$ a smooth curve with tangent τ. A section s over γ is said to be parallel if $\nabla_\tau s = 0$. In a local trivialization, (U, ψ) with unit section s and connection potential β, this generates a differential equation. With $s' = \varphi\ s$ we see,

$$\tau(\varphi) = i\ (\tau\rfloor\beta)\ \varphi. \tag{4.1}$$

Take curve γ so that it forms the boundary of a 2-sphere surface $\Sigma \subset U$ and $\gamma(0) = \gamma(1)$ and if s' satisfies (4.1), then by Stokes' theorem

$$s'(\gamma(0)) = \exp(i \oint_\gamma \beta)\ s'(\gamma(0)) = \exp(i \int_\Sigma \Omega)\ s'(\gamma(0)). \tag{4.2}$$

If Σ is divided into small pieces, this is holds even if Σ is not contained in U. So if Σ is a closed, oriented 2-surface in M and γ divides Σ into pieces Σ_1 and Σ_2, then on account of the orientations of γ, Σ_1 and Σ_2

$$\exp(i \int_\Sigma \Omega) = \exp(i \int_{\Sigma_1} \Omega) \cdot \exp(-i \int_{\Sigma_2} \Omega) = 1.$$

It is worth stating this condition explicitly. There exists an open cover $U = \{U_j\}$ of M such that the class defined by $\Omega/2\pi$ in $H^2(U, R)$ contains a cocycle z in which z_{ijk} are all integers. Perhaps more simply stated, the integral of Ω over any closed, oriented 2-surface, or over any integer 2-cycle in M, is an integral multiple of 2π.

Suppose θ is a local symplectic potential on a simply connected neighborhood $U \subset M$. A local trivialization (U, ψ) of B can be constructed where $\beta = (1/\hbar)\ \theta$ as the connection potential. To see this, pick a base point $m_0 \in U$ and $v_0 \in B_{m_0}$. A nonvanishing section $s \in C_B^\infty(0)$, $m \in U$ can be defined by

$$s(m) = \exp\left(-\frac{i}{\hbar}\int_\gamma \theta\right) \cdot v,$$

where γ is any path from m_0 to m, and v is obtained from v_0 by parallel translation along γ. Since U is simply connected, $s(m)$ is independent of Γ and ψ: $U \times C \to B$: $(m, z) \to z\, s(m)$ is a local trivialization with s the unit section. For a vector field X,

$$\nabla_X\ s = -\frac{i}{\hbar}\ (X\rfloor\theta)\ s = -i\ (X\rfloor\beta)\ s. \tag{4.3}$$

This means β can be regarded as the connection potential. The case of a symplectic manifold admitting a global symplectic potential θ is quantizeable. However, when it is not simply connected the quantization is not unique.

Let N be three-dimensional Euclidean space with a closed infinite cylinder parallel to the z-axis removed. If N is thought of as the configuration space of a charge and of the cylinder as the region of Euclidean space in which there is a nonzero magnetic field. Even with the magnetic field vanishing everywhere in N, there is no choice of gauge such that the vector potential A also vanishes in N since

$$\oint A_i \, dq^i$$

around a closed curve in N surrounding the cylinder is nonzero and gauge invariant. Experimentally it is found the total phase change in the wave function of a test charge around a closed loop surrounding such a cylinder is not zero but given by

$$\exp\left(i\frac{e}{\hbar}\int_\gamma A_a \, dq^a\right)$$

even though the field vanishes in N. The prequantum bundle is not $T^*N \times C$ here, but the bundle determined by the phase factor. The connection on the prequantum line bundle is given by potentials of the form $(1/\hbar)(\vartheta + e\pi^*(A))$ rather than $(1/\hbar)\vartheta$.

It is important to note that when the manifold is simply connected, all connections ∇ with curvature $\Omega/\hbar$ are equivalent. This is not necessarily the case when M is not simply connected. There is then an infinite set of inequivalent choices for ∇ and a choice of connection has physical consequences. For example, let $N = \mathbb{R}$ the real line, $M = T^* N = \mathbb{R}^2$ and $B = M \times C$. Let $\{p, q\}$ be the usual Cartesian canonical coordinates on T^*N, the vertical polarization spanned by $\partial/\partial p$. Let s be any nowhere zero smooth global section of B and let ∇, ∇' be connections on B defined by $\nabla_X s = -(i/\hbar)(X \rfloor p\, dq)\, s$ and $\nabla_X{}' s = -(i/\hbar)(X \rfloor \beta)\, s$. Since $M = R^2$ by the Poisson lemma, we can write $\beta' = p\, dq - i\hbar\, d(\log \kappa)$ for some globally smooth function κ. Now s is covariantly constant along $X \in V(M, \gamma)$ with respect to ∇. Hence with ∇ as connection any section s polarized with respect to P in the form $s = \psi(q)\, s$. Equivalence of ∇ and ∇' implies $\nabla_X{}' \kappa\, s = \kappa\, \nabla_X s = 0$ and so $\kappa\, s$ is also covariantly constant along X with respect to ∇'. With ∇' as the connection, any section s polarized with respect to P is of the form $s_1 = \psi(q)\, \kappa\, s = \psi(q)\exp(-i\sigma/\hbar)\, s$.

5. Generalized hilbert space

An arbitrary finite-dimensional manifold does not carry a canonical measure although there is a natural measure on a Riemannian manifold. A natural class of measures are those that are equivalent to Lebesgue measure in every coordinate chart. A measure μ is absolutely continuous with respect to ν if there exists a function ρ such that $\mu(E) = \int_E \rho\, d\nu$ for every Borel set E, and ρ is usually denoted by $d\mu/d\nu$ the Radon-Nikodym derivative of μ with respect to ν. We call μ and ν equivalent if each is absolutely continuous with respect to the other, so $d\nu/d\mu = (d\mu/d\nu)^{-1}$. associated to this choice of measures there is a Hilbert space, called the intrinsic Hilbert space of the manifold M denoted $\mathcal{H}(M)$ and its elements are called *half-forms*.

Consider the set of all pairs (f, μ), where f is a complex measurable function and $\int_M |f|^2\, d\mu < \infty$. Two pairs are equivalent provided that $f\,(\widetilde{d\mu/d\nu}) = g$. An equivalence class can be defined such that the class of (f, μ) is denoted $f\ \widetilde{d\mu}$ and as well $\mathcal{H}(M)$ denotes the set of all such classes. A more useful way of establishing the Hilbert space structure of $\mathcal{H}(M)$ is to choose a natural measure μ. The map U_μ: $f \to f\widetilde{d\mu}$ is a bijection from $L^2(M, \mu)$ onto $\mathcal{H}(M)$. This can be used to transfer the Hilbert space structure from $L^2(M, \mu)$ to $\mathcal{H}(M)$ such that the resulting structure is independent of the choice of μ. If $f\widetilde{d\mu} = g\widetilde{d\nu}$ is a typical vector in $\mathcal{H}(M)$, $||U_\mu^{-1}\xi|| = ||U_\nu^{-1}\xi||^2$, hence $\xi \in \mathcal{H}(M)$ is C^∞ with compact support where $\xi = f\widetilde{d\mu}$. Here f is a $C^\infty(M)$ function with compact support and measure μ is associated with a smooth n-form Ω_μ on M. These C^∞ vectors with compact support form a dense subspace of $\mathcal{H}(M)$. For t sufficiently small $U_t(f\widetilde{d\mu}) = f \circ \varphi_t\ d(\widetilde{\mu \circ \varphi_t})$ is a well-defined element of $\mathcal{H}(M)$. So we define

$$\tilde{X}(f\ \widetilde{d\mu}) = \frac{1}{i}\lim_{t\to\infty}\frac{1}{t}[U_t(f\widetilde{d\mu}) - f\widetilde{d\mu}] = -i\left(X\, f + \frac{1}{2}(\mathrm{div}_\mu X)\, f\right) d\mu. \tag{5.1}$$

The operator $\tilde{X}(f\ \widetilde{d\mu})$ in (5.1) is constructed by intrinsic methods and the right side depends only on the equivalence class (f, μ). The operator $\tilde{X}$ is symmetric, since if $\xi = f\widetilde{d\mu}$ and $\eta = g\ \widetilde{d\nu}$

$$(U_t\xi,\, U_t\eta) = \int (f \circ \varphi_t)(\widetilde{\bar{g} \circ \varphi_t})d\mu \circ \varphi_t = \int f\bar{g}\ d\mu = (\xi,\, \eta) \tag{5.2}$$

so by differentiation of (5.2) with $t = 0$ we have $(\tilde{X}\ \xi,\, \eta) = (\xi,\, \tilde{X}\eta) = 0$.

If the classical momentum function $P(X)$ on T^*M which is associated to the vector field X on M by $P(X)(\alpha_x) = \alpha_x(X(m))$, $x \in M$. In this event, operator $\tilde{X}$ can be called the corresponding quantum momentum observable. If f is a C^∞ function on M, the corresponding quantum position operator observable is $Q_f\,(g\widetilde{d\mu}) = f \cdot g\widetilde{d\mu}$, that is, multiplication by f.

Theorem 5.1. *Let* Q *be a finite-dimensional manifold with intrinsic Hilbert space* $\mathcal{H}(Q)$ *and* X, Y *smooth vector fields and* f, g *smooth functions on* Q. (*i*) $-i[\tilde{X},\, \tilde{Y}] = -\widetilde{[X,\, Y]}$ (*ii*) $-i[Q_f,\, Q_g] = 0$ (*iii*) $-i[Q_f,\, \tilde{X}] = Q_{X(f)}$.

Proof. To prove (i) select a $\xi = \varphi\ \tilde{d\mu}$ which satisfies (5.1). Since vector fields are derivations, compute

$$[\tilde{X}, \tilde{Y}] = \{X(Y\varphi + \frac{1}{2}(\mathrm{div}_\mu Y)\varphi\}\ \tilde{d\mu} + \frac{1}{2}\ (\mathrm{div}_\mu X(Y\varphi + \frac{1}{2}(\mathrm{div}_\mu Y)\varphi)\tilde{d\mu}\}$$
$$-\{Y(X\varphi + \frac{1}{2}(\mathrm{div}_\mu X)\varphi)\widetilde{d\mu} + \frac{1}{2}(\mathrm{div}_\mu Y)\{X\varphi + \frac{1}{2}(\mathrm{div}_\mu X)\varphi\}\widetilde{d\mu}\}$$
$$=-\{(XY - YX)\varphi + \frac{1}{2}(X\mathrm{div}_\mu Y - Y\mathrm{div}_\mu X)\varphi\}\widetilde{d\mu}. \tag{5.3}$$

With respect to the volume form Ω_μ we compute $(\mathrm{div}_\mu[X, Y])\Omega_\mu = L_{[X,Y]}\Omega_\mu = L_X L_Y \Omega_\mu - L_Y L_X \Omega_\mu = L_X(\mathrm{div}_\mu Y\ \Omega_\mu) - L_Y(\mathrm{div}_\mu X\ \Omega_\mu) = (X\mathrm{div}_\mu\ Y - Y\mathrm{div}_\mu X)\ \Omega_\mu = (X\mathrm{div}_\mu Y - Y\mathrm{div}_\mu X)\Omega_\mu$. Substitute this into (5.3) and the result follows

$$-i[\tilde{X}, \tilde{Y}]\xi = i([X, Y]\ \varphi + \frac{1}{2}\ \mathrm{div}_\mu[X, Y]\varphi)\widetilde{d\mu} = -\widetilde{[X, Y]}\varphi\xi. \tag{5.4}$$

The second (ii) follows immediately from

$$[Q_f, Q_g]\xi = (Q_f Q_g - Q_g Q_f)\varphi\tilde{d\mu} = (f\ g\varphi - g\ f\varphi)\tilde{d\mu} = 0. \tag{5.5}$$

Finally for the third property

$$[Q_f, \tilde{X}_f]\xi = -i(Q_f(X_g\varphi + \frac{1}{2}(\mathrm{div}_\mu X)\varphi) - (X(f\ \varphi) + \frac{1}{2}(\mathrm{div}_\mu X)f\ \varphi)\widetilde{d\mu}$$
$$=-i\ (fX\varphi + \frac{1}{2}f\ \mathrm{div}_\mu X)\varphi) - f\ X\varphi - \varphi X\ f - \frac{1}{2}f(\mathrm{div}_\mu X)\varphi)\widetilde{d\mu} = i\ X(f)\ \varphi\widetilde{d\mu} = -\frac{1}{i}\ Q_{X(f)}\xi.$$

□

The associated divergence operation on vector fields is the divergence relative to metric g so $L_X\ \Omega = \mathrm{div}_\mu\ X\ \Omega_\mu$. If $\varphi \in C^\infty(M)$ then $d\varphi$ is a one-form so $g^{-1}\ (d\varphi) = \mathrm{grad}\ \varphi$ is a vector field, the gradient of φ relative to g. The Laplace-Beltrami operator is defined as $\Delta_g\ \varphi = \mathrm{div}\,\mathrm{grad}\ \varphi$ where g refers to the metric here and it is a symmetric operator. Accordingly, using the canonical identification of $L^2(Q, \Omega)$ with $\mathcal{H}(Q)$, operator Δ_g can be regarded as a symmetric operator with dense domain. In fact, Δ_g is essentially self-adjoint if Q is complete with respect to g. This allows us to propose one definition of the Hamiltonian operator. If the classical potential energy is given by a function V on Q, the Hamiltonian operator can be defined on $\mathcal{H}(Q)$ as

$$H = -\frac{1}{2}\Delta_g + Q_V. \tag{5.6}$$

A nontrivial example which has an important applications in the construction of the WKB approximation in quantum mechanics is presented. To begin with, let X be an integral manifold of an integrable distribution, x any point in X, and γ a closed curve in X originating at x. For each linear frame $b \in B$, let $g(b, \gamma)$ be the element of the holonomy group at b of the canonical flat connection in $B\backslash X$ which corresponds to the horizontal lift of γ starting from b. Let m_γ be defined up to an integer by $\chi(g(b, \gamma)) = \exp(-2\pi i\ m_\gamma)$. It depends only on the homotopy class of γ in X and χ: $ML(n, C) \to C$ is the unique holomorphic square root of the complex character of $ML(n, C)$ satisfying $\chi(1) = 1$.

Lemma 5.1. *Let* γ: $[0, 1] \to X$ *be a closed curve and for each* $t \in [0, 1]$, $v(t)$ *is the tangent vector to the curve at* $\gamma(t)$. *Let* τ *be a half-form such that* $\nabla_{\gamma(t)}\tau + 2\pi\ i\ f(t)\ \tau = 0$ *for some function* $f(t)$ *on* $[0, 1]$. *Then* τ *vanishes identically on* γ *unless*

$$\int_0^1 f(t)\ dt - m_\gamma \tag{5.7}$$

is an integer.

Proof. Choose any non-zero half-form τ_0 at $x = \gamma(0)$ and denote by τ_t the half-form at $\gamma(t)$ obtained from τ_0 by parallel transport along γ. Then for such $b \in B_x$

$$\tau_1(b) = \tau_0(b\ g(b, \gamma)) = \chi(g(b, \gamma))^{-1}\tau_0(b) = e^{2\pi i m_\gamma}\ \tau_0(b).$$

The restriction of τ to γ can be represented at $\gamma(t)$ as $h(t)\tau_t$ and, since τ is a single-valued function $h(0)\ \tau_0 = h(1)\tau_1$, hence $h(0) = \exp(2\pi\ i\ m_\gamma)$ The differential equation

$$\nabla_{v(t)}\tau + 2\pi\ i\ f(t)\tau = 0 \tag{5.8}$$

yields the equation

$$h'(t) + 2\pi\, if(t)\, h(t) = 0. \tag{5.9}$$

If h does not vanish identically, integration of (5.9) gives

$$\log h(1) - \log h(0) + 2\pi\, i \int_0^1 f(t)\, dt = 2\pi\, i\, \mu, \tag{5.10}$$

In (5.10), μ is an integer. Hence $\int_0^1 f(t)\, dt - m_\gamma$ is an integer or τ restricted to γ vanishes identically. □

Suppose that ω is exact so $\omega = d\zeta$. The space of wave functions consists of half-forms τ such that $\nabla_v \tau + 2\pi\, i\langle v, \zeta\rangle\, \tau = 0$, then from lemma 5.1, each wave function vanishes on all closed curves γ contained in integral manifolds of D for which the condition $\int_\gamma \zeta - m_\gamma$ is equal to an integer is not satisfied. This statement can be regarded as a corollary to the lemma and is essentially the Bohr–Sommerfeld condition.

6. Integrable equations potential functions and lax

There is a nice way to bring integrability into the approach that has been outlined so far [20]. This will serve as a natural choice for the potential function θ. It is the case that a set of one forms that make up a closed differential ideal can be defined which is related to a specific system of interest. Conservation laws can be associated with this system correspond to the existence of exact two forms contained in the ring of these α_i. Suppose we can find a set of functions f_i such that the two-form

$$\beta = \sum_i f_i\, \alpha_i \tag{6.1}$$

satisfies $d\beta = 0$ which is the condition for existence. This is the integrability condition for the existence of a one-form ω such that

$$\beta = d\omega. \tag{6.2}$$

This implies that $d\,\beta = 0$ by the usual identity for the double exterior derivative of any differential form. The associated conservation laws result by applying Stokes' theorem

$$\oint_{M_1} \omega = \int_{M_2} d\omega. \tag{6.3}$$

This is written for the case of any simply-connected two-dimensional manifold M_2 with closed one-dimensional M_1. The notation implies that ω and $d\omega$ are evaluated on their respective manifolds. If for M_2 we take a solution manifold S_2 which annuls the α_i and if S_1 is any closed curve in S_2, then

$$d\,\tilde{\omega} = \tilde{\beta}, \qquad \oint_{S_1} \tilde{\omega} = 0. \tag{6.4}$$

It has become clear that the equations of motion of nonrelativistic quantum mechanics are of the form

$$i\hbar\, \dot{\hat{f}} = [\hat{f}, \hat{H}]$$

where $\hat{H}$ is a fixed operator associated to a given system, and $\hat{f}$ is an arbitrary operator. This result is quite analogous to the form of the Lax equation in integrable systems $\dot{L} = [W, L]$ where L is a fixed operator and W is a variable operator. Lax equations have many properties. The most important of these is they possess nontrivial Hamiltonian structures, imply an infinite number of conservation laws and are often bi-hamiltonian systems. In fact, the Schrödinger, Pauli and Dirac equations are noncanonical bi-hamiltonian systems with an infinite number of conservation laws. This follows from the fact they are associated with hermitian operators. Thus Schrödinger equations can be represented in the form,

$$i\dot{\psi} = \mathcal{L}^r\, \psi, \qquad -i\dot{\bar{\psi}} = \bar{\mathcal{L}}^r\, \bar{\psi} \tag{6.5}$$

for a column vector ψ: $R^M \to C^R$ and its complex conjugate $\bar{\psi}$ with $\mathcal{L}$ a hermitian operator $\bar{\mathcal{L}} = \mathcal{L}^\dagger$, where †stands for adjoint. In addition, t denotes transpose and in general r is a natural number which is usually $r = 1$. Let us define

$$H_n = \bar{\psi}^t \mathcal{L}^n\, \psi. \tag{6.6}$$

The H_n in (6.6) are in fact conservation laws of system (6.5). They are conserved under differentiation with respect to the evolution parameter t. Using (6.5) and $\bar{\mathcal{L}}^\dagger = \mathcal{L}$, it follows that

$$\dot{H}_n = \dot{\bar{\psi}}^t \mathcal{L}^n\, \psi - \bar{\psi}^t\, \mathcal{L}^n\, \dot{\psi} = i\bar{\mathcal{L}}^r\, \bar{\psi}^t \mathcal{L}^n\, \psi - i\bar{\psi}^t \mathcal{L}^n\, (\mathcal{L}^r \psi)$$

$$=i(\bar{\psi}^t(\bar{\mathcal{L}}^r)^\dagger \mathcal{L}^n \psi - \bar{\psi}^t \mathcal{L}^{n+r}\psi) = 0. \tag{6.7}$$

An example of a particular $\mathcal{L}$ is given. Let e be the charge, c the speed of light, $\mathbf{A} = (A_1,\ldots,A_M)$ and φ_s the time-independent vector and scalar potentials, respectively. Then $\mathcal{L}$ for a static electromagnetic field is

$$\mathcal{L} = -\frac{\hbar}{2m}\Delta + \hbar^{-1}\left(\frac{e^2}{2mc^2}A^2 + e\varphi_s\right) + \frac{i\,e}{2mc}[2\,\mathcal{A}\cdot\boldsymbol{\nabla} + \mathrm{div}(\mathbf{A})]. \tag{6.8}$$

7. Extension to infinitely many degrees of freedom

The concepts of symplectic geometry and geometric quantization apply to relativistic systems with infinitely many degrees of freedom. The dynamical behavior is determined by the Lagrangian density which is a coordinate dependent function

$$L = L(\varphi^\alpha,\, \nabla_b\,\varphi^\beta,\, q^a). \tag{7.1}$$

So L is a map that assigns a real number to each event at which the fields and first covariant derivatives have been specified. The dynamical theory can be obtained in roughly two equivalent ways. The action integral can then be given so one then sets the first variation to zero. Alternatively one can violate covariance by introducing a time coordinate and considering the integral of L over a hypersurface. The values of the fields at each point in space are the configuration coordinates and their time derivatives are the velocity coordinates.

Fix the fields φ^α and let X be a second section of bundle E over a manifold Q which vanishes outside a compact subset D_X of Q. Then φ is a solution of the field equations if and only if

$$\frac{d}{dt}\left[\int L(\varphi + tX,\, \nabla\varphi + t\nabla X,\, q)\,dv_\omega\right]_{t=0} = 0. \tag{7.2}$$

for every choice of X so

$$\int_{D_X}\left(\frac{\partial L}{\partial\varphi^a}X^a + \frac{\partial L}{\partial\varphi_a^\alpha}\nabla_a X^\alpha\right)dv_\omega = \int_{D_X}\left[\frac{\partial L}{\partial\varphi^\alpha} - \nabla_a\left(\frac{\partial L}{\partial\varphi_a^\alpha}\right)\right]X^\alpha dv_\omega = 0. \tag{7.3}$$

In (7.3), $\varphi_a^\alpha = \nabla_a\,\varphi^\alpha$ and X^α vanishes on the boundary ∂D_X. The field equations are

$$\nabla_a\left(\frac{\partial L}{\partial\varphi_a^\alpha}\right) - \frac{\partial L}{\partial\varphi^\alpha} = 0. \tag{7.4}$$

Let M be the set of solutions of (7.4) which can be regarded as an infinite-dimensional manifold. The tangent space at $\varphi \in M$ is the space of solutions of (7.5) linearized about φ. Thus a tangent vector $X \in T_pM$ is a section of E such that $\varphi + t\,X$ is also a solution to

$$\frac{\partial^2 L}{\partial\varphi^\beta\partial\varphi^\alpha}X^\beta + \frac{\partial^2 L}{\partial\varphi_b^\beta\partial\varphi^\alpha}\nabla_b X^\beta = \nabla_a\left(\frac{\partial^2 L}{\partial\varphi^\beta\partial\varphi_a^\alpha}X^\beta + \frac{\partial^2 L}{\partial\varphi_b^\beta\partial\varphi_a^\alpha}\nabla_b X^\beta\right) \tag{7.5}$$

Choose a spacelike hypersurface $\Sigma \subset Q$ where Σ is a hyperplane on Minkowski space, and a Cauchy surface in curved space-time. Let ϑ be a one-form on M defined by

$$X\rfloor\,\vartheta = \int_\Sigma X^\alpha\frac{\partial L}{\partial\varphi_a^\alpha}\,d\sigma_a. \tag{7.6}$$

Although ϑ depends on the choice of Σ, the exterior derivative $d\,\vartheta$ is independent of Σ. Suppose X, Y are maps that assign solutions X_α, Y_α of the linearized equations to each $\varphi \in M$. The Lie bracket $[X, Y]$ is the linearized solution given at φ

$$[X,\, Y]^\alpha = \frac{d}{dt}(Y^\alpha_{\varphi+tX} - X^\alpha_{\varphi+tY})\Big]_{t=0}.$$

The derivative of $Y\rfloor\,\vartheta$ along X is given by contracting

$$\left[Y_\varphi^\alpha\frac{d}{dt}\left(\frac{\partial L}{\partial\varphi_a^\alpha}\right)_{\varphi+tX} + \left(\frac{\partial L}{\partial\varphi_a^\alpha}\right)_\varphi\frac{d}{dt}Y^\alpha_{\varphi+tX}\right]_{t=0},$$

with $d\,\sigma_\alpha$, so integrating over Σ

$$\omega(X,\, Y) = \frac{1}{2}(X(Y\Big|\vartheta) - Y(X\Big|\vartheta) - [X,\, Y]\Big|\vartheta) = \int_\Sigma \omega^a\,d\sigma_a, \tag{7.7}$$

where

$$2\omega^a = \frac{\partial^2 L}{\partial\varphi^\beta \partial\varphi_a^\alpha}(X^\beta Y^\alpha - Y^\beta X^\alpha) + \frac{\partial^2 L}{\partial\varphi_b^\beta \partial\varphi_a^\alpha}(Y^\alpha \nabla_b X^\beta - X^\alpha \nabla_b Y^\beta). \tag{7.8}$$

However, (7.8) implies that $\nabla_a \omega^a = 0$. As long as the linearized fields fall off sufficiently fast at infinity, $\omega(X, Y)$ is independent of Σ, so ω is a natural closed 2-form on M but not necessarily nondegenerate.

As an example, consider the case of a complex scalar field. Suppose φ is a complex function on Q and

$$L = \frac{1}{2}(\hbar^2 \nabla_a \varphi \nabla^a \bar{\varphi} - m^2 \varphi\bar{\varphi}). \tag{7.9}$$

From (7.4) the field equations are given by

$$\hbar^2 \Box \varphi + m^2 \varphi = 0. \tag{7.10}$$

The symplectic form is

$$\omega(\varphi, \bar{\phi}) = -\frac{\hbar^2}{4}\int_\Sigma (\varphi\nabla_a \phi + \bar{\varphi}\nabla_a\phi - \phi\nabla_a\bar{\varphi} - \bar{\phi}\nabla_a\varphi)d\sigma^a. \tag{7.11}$$

8. Conclusions

A number of topics which concern quantization and geometric quantization in particular have been considered. Some of the results do involve integrability and its appearance in quantum mechanics. Theorems related to integrability and quantization are introduced. There is a deep geometric structure to quantum mechanics, well documented here, as well as how it relates to integrability. Geometric quantization is an attempt to pass from classical mechanics in a canonical fashion making use of the geometrical ideas that classical mechanics comes with and those that can be canonically linked to a classical system. Two nontrivial physical applications, the first having implications for establishing the WKB approximation and the second related to Lax pairs and integrability in quantum mechanics have been introduced. The subject of integrable potential functions in a previous section deserves further development in the future.

Data availability statement

No data in paper The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Paul Bracken https://orcid.org/0000-0003-1409-7272

References

[1] Abraham R and Marsden J (ed) 1978 *Foundations of Mechanics* (Benjamin-Cummings)
[2] Nomizu K and Kobayashi S (ed) 1963 *Foundations of Differential Geometry* (Wiley)
[3] Flander H (ed) 1963 *Differential Forms* (Academic)
[4] Hirsch M W and Smale S (ed) 1974 *Differential Equations, Dynamical Systems and Linear Algebra* (Academic)
[5] Hörmander L (ed) 1963 *Linear Differential Operators* (Springer)
[6] Woodhouse N M (ed) 1980 *Geometric Quantization* (Oxford University Press)
[7] Reed M and Simon B (ed) 1974 *Methods of Modern Mathematical Physics* (Academic Press)
[8] Souriau J-M 1966 *Commun. Math. Physics* **1** 374–98
[9] Souriau J-M (ed) 1970 *Structure des Sytèmes Dynamique* (Dunod)
[10] Kostant B (ed) 1970 Quantization and unitarity representations *Lecture Notes in Math* vol 170 (North Holland: Springer-Verlag)
[11] Berry M V 1983 Semiclassical mechanics of regular and irregular motion *Chaotic Behavior in Deterministic Systems, Les Houches Lectures XXXIV* ed G Ioose, R Hellman and R Stora (North Holland: Elsevier) pp 171–271
[12] Berry M V 1985 Semiclassical theory of spectral rigidity *Proc. R. Soc. London, Ser.* A **400** 229–51
[13] Berry M V 1991 Some quantum-to-classical asymptotics *Les Houches Lectures LII* ed M Giannoni, A Voros and J Zinn-Justin (Elsevier)
[14] Bracken P (ed) 2020 Dynamical symmetries and quantum chaos *Recent Advances in Chaos Theory* ed P Bracken (Intech Open Publishers)
[15] Zheng W and Feng D 1995 Quantum nonintegrability in finite systems *Phys. Rep.* **252** 1–100
[16] Marsden J, Montgomery R and Ratiu T (ed) 1990 Reduction, symmetry and phase in mechanics *Memoirs of the AMS* **vol 88**
[17] Eckhardt B 1988 Quantum mechanics of classically non-integrable systems *Phys. Rep.* **163** 205–97

[18] Sniatyzki J 1983 Constraints and quantization *Non-linear Partial Differential Operators and Quantization Procedures, Lecture Notes in Mathematics* vol 1037 (Springer-Verlag)
[19] Blattner R J (ed) 1983 Lecture notes in mathematics *On Geometric Quantization* vol 1037 (Springer-Verlag)
[20] Bracken P 2013 Geometric approaches to produce prolongations for nonlinear partial differential equations *Int. J. Geom. Methods in Modern Physics* **10** 1350002