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# Conditional constrained and unconstrained quantization for probability distributions

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# CONDITIONAL CONSTRAINED AND UNCONSTRAINED QUANTIZATION FOR PROBABILITY DISTRIBUTIONS

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Abstract. In this paper, we present the idea of conditional quantization for a Borel probability measure $P$ on a normed space $\mathbb{R}^k$. We introduce the concept of conditional quantization in both constrained and unconstrained scenarios, along with defining the conditional quantization errors, dimensions, and coefficients in each case. We then calculate these values for specific probability distributions. Additionally, we demonstrate that for a Borel probability measure, the lower and upper quantization dimensions and coefficients do not depend on the conditional set of the conditional quantization in both constrained and unconstrained quantization.



## 1. Introduction

Quantization involves the process of discretizing signals. In the context of probability distributions, quantization refers to finding the best approximation of a $k$-dimensional probability distribution $P$ using a discrete probability distribution with a specified number $n$ of supporting points (referred to as the optimal set of $n$-points). In other words, it aims to find the best approximation of a $k$-dimensional random vector $X$ with distribution $P$ using a random vector $Y$ that has nearly $n$ values in its range.

A plethora of research is given on quantization for probability distributions without using any constraint. The concept of constrained quantization was recently introduced by Pandey and Roychowdhury (see [PR1, PR2, PR3]). This new approach allows us to categorize quantization into two types: unconstrained quantization and constrained quantization. Unconstrained quantization is traditionally known as quantization. For some recent work in the direction of unconstrained quantization, one can see [GL, DFG, DR, GL2, GL3, KNZ, PRRSS, P1, R1, R2, R3]. Quantization theory has broad applications in communications, information theory, signal processing, and data compression (see [GG, GL1, GN, P, Z1, Z2]). This paper deals with conditional quantization in both constrained and unconstrained cases. Conditional quantization also has significant interdisciplinary applications: for example, in radiation therapy of cancer treatment to find the optimal locations of $n$ centers of radiation, where $k$ centers for some $k < n$ of radiation are preselected, the conditional quantization technique can be useful.

Let $P$ be a Borel probability measure on $\mathbb{R}^k$ equipped with a metric $d$ induced by a norm $\|\cdot\|$ on $\mathbb{R}^k$, and $r \in (0, \infty)$. Let $\mathbb{N} := \{1, 2, 3, \cdots\}$ be the set of natural numbers. For a finite set $\gamma \subset \mathbb{R}^k$ and $a \in \gamma$, by $M(a|\gamma)$ we denote the set of all elements in $\mathbb{R}^k$ which are nearest to $a$ among all the elements in $\gamma$, i.e., $M(a|\gamma) = \{x \in \mathbb{R}^k : d(x, a) = \min_{b\in\gamma} d(x, b)\}$. $M(a|\gamma)$ is called the *Voronoi region* in $\mathbb{R}^k$ generated by $a \in \gamma$.

**Definition 1.1.** *Let $\{S_j \subseteq \mathbb{R}^k : j \in \mathbb{N}\}$ be a family of closed sets with $S_1$ nonempty. Let $\beta \subset \mathbb{R}^k$ be given with card$(\beta) = \ell$ for some $\ell \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ with $n \geq \ell$, the nth conditional constrained quantization error for $P$, of order $r$, with respect to the family of constraints $\{S_j \subseteq \mathbb{R}^k : j \in \mathbb{N}\}$ and the set $\beta$, is defined as*

$$V_{n,r} := V_{n,r}(P) = \inf_{\alpha} \Big\{ \int \min_{a\in\alpha\cup\beta} d(x, a)^r dP(x) : \alpha \subseteq \bigcup_{j=1}^{n} S_j,\ 0 \leq card(\alpha) \leq n - \ell \Big\}, \tag{1}$$

*where card$(A)$ represents the cardinality of the set $A$.*

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Notice the error $V_{n,r}$ explicitly depends on $\beta$. For any $\alpha \subseteq \mathbb{R}^k$, the number

$$V_r(P;\alpha) := \int \min_{a\in\alpha} d(x,a)^r dP(x) \tag{2}$$

is called the *distortion error* for $P$, of order $r$, with respect to the set $\alpha$. We assume that $\int d(x,0)^r dP(x) < \infty$ to make sure that the infimum in (1) exists (see [PR1]).

**Definition 1.2.** *A set $\alpha \cup \beta$, where $\alpha \subseteq \bigcup_{j=1}^n S_j$ and $P(M(b|\alpha\cup\beta)) > 0$ for $b \in \beta$, for which the infimum in (1) exists and contains no less than $\ell$ elements, and no more than $n$ elements is called an optimal set of $n$-points for $P$, or more specifically a conditional optimal set of $n$-points for $P$. Elements of an optimal set are called optimal elements.*

**Definition 1.3.** *Instead of the family of constraints $\{S_j \subseteq \mathbb{R}^k : j \in \mathbb{N}\}$ if there is a single constraint $S$, i.e., if $S_j = S$ for all $j \in \mathbb{N}$, then Definition 1.1 reduces to*

$$V_{n,r} := V_{n,r}(P) = \inf_\alpha \Big\{ \int \min_{a\in\alpha\cup\beta} d(x,a)^r dP(x) : \alpha \subseteq S,\ 0 \leq card(\alpha) \leq n-\ell \Big\},$$

*which is called the $n$th conditional constrained quantization error for $P$, of order $r$, with respect to the single constraint $S$ and the set $\beta$.*

Write $V_{\infty,r}(P) := \lim_{n\to\infty} V_{n,r}(P)$. The numbers

$$\underline{D}_r(P) := \liminf_{n\to\infty} \frac{r\log n}{-\log(V_{n,r}(P) - V_{\infty,r}(P))} \text{ and } \overline{D}_r(P) := \limsup_{n\to\infty} \frac{r\log n}{-\log(V_{n,r}(P) - V_{\infty,r}(P))}, \tag{3}$$

are called the *conditional lower* and the *conditional upper constrained quantization dimensions* of the probability measure $P$ of order $r$, respectively. If $\underline{D}_r(P) = \overline{D}_r(P)$, the common value is called the *conditional constrained quantization dimension* of $P$ of order $r$ and is denoted by $D_r(P)$. For any $\kappa > 0$, the two numbers $\liminf_n n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P))$ and $\limsup_n n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P))$ are, respectively, called the *$\kappa$-dimensional conditional lower* and *conditional upper constrained quantization coefficients* for $P$ of order $r$. If both of them are equal, then it is called the *$\kappa$-dimensional conditional constrained quantization coefficient* for $P$ of order $r$.

**Definition 1.4.** *In Definition 1.1 if $S_j = \mathbb{R}^k$ for all $j \in \mathbb{N}$, then for $n \in \mathbb{N}$ with $n \geq \ell$, the $n$th conditional unconstrained quantization error for $P$, of order $r$, with respect to the set $\beta$, is defined as*

$$V_{n,r} := V_{n,r}(P) = \inf_\alpha \Big\{ \int \min_{a\in\alpha\cup\beta} d(x,a)^r dP(x) : \alpha \subseteq \mathbb{R}^k,\ 0 \leq card(\alpha) \leq n-\ell \Big\}.$$

*The corresponding quantization dimension and the $\kappa$-dimensional quantization coefficient, if they exist, are called the conditional unconstrained quantization dimension and the $\kappa$-dimensional conditional unconstrained quantization coefficient for $P$, respectively.*

**Remark 1.5.** The set $\beta$ that occurs in any of the above definitions is referred to as the *conditional set*. Notice that in unconstrained quantization, if $\alpha \cup \beta$ is a conditional optimal set of $n$-points with $\alpha$ nonempty, then all the elements in $\alpha$ are the means, i.e., the conditional expectations in their own Voronoi regions. The elements of the conditional set $\beta$ are not necessarily the means of their own Voronoi regions.

This paper deals with $r = 2$ and $k = 2$, and the metric on $\mathbb{R}^2$ as the Euclidean metric induced by the Euclidean norm $\|\cdot\|$. Thus, instead of writing $V_r(P;\alpha)$ and $V_{n,r} := V_{n,r}(P)$ we will write them as $V(P;\alpha)$ and $V_n := V_n(P)$.

**Remark 1.6.** Although all the work done in the sequel is for uniform distributions, interested researchers can explore them for any probability distribution.

**Delineation.** In this paper, first we have given the preliminaries. Then, in Section 3, we have investigated the conditional constrained quantization for a uniform distribution on the boundary of a semicircular disc; in Section 4, we have investigated the conditional unconstrained quantization for a uniform distribution on an equilateral triangle; in Section 5, with some examples, we mentioned whether a conditional optimal set of $n$-points always exist or not. In the last section, Section 6, we have proved two theorems: first theorem shows that if $\alpha_n$ are the optimal sets of $n$-means for $P$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} \alpha_n$ is dense in the support of $P$; and in the 2nd theorem we have shown that in both constrained and unconstrained quantization, the lower and upper quantization dimensions, and the lower and upper quantization coefficients for a Borel probability measure do not depend on the conditional set.

## 2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the collection of all real numbers. Recall that the boundary of the Voronoi regions of any two elements $p$ and $q$ in an optimal set is the perpendicular bisector of the line segment joining the elements. Hence, for any element $e$ on the boundary of the Voronoi regions of $p$ and $q$, we have

$$\rho(p, e) - \rho(q, e) = 0, \tag{4}$$

where for any two elements $(a, b)$ and $(c, d)$ in $\mathbb{R}^2$, $\rho((a, b), (c, d))$ represents the squared Euclidean distance between the two elements. Equation (4) is known as a *canonical equation*. Let us now give the following two propositions.

**Proposition 2.1.** *Let $P$ be a uniform distribution on the closed interval $[a, b]$. Let $a \leq c < d \leq b$. Let $\alpha_n$ be an optimal set of $n$-points for $P$ such that $\alpha_n$ contains $m$ elements, for some $m \leq n$, from the closed interval $[c, d]$ including the endpoints $c$ and $d$, then*

$$\alpha_n \cap [c, d] = \{c + \frac{j-1}{m-1}(d-c) : 1 \leq j \leq m\}.$$

*Then, the distortion error contributed by these $m$ elements in the closed interval $[c, d]$ is given by*

$$V(P, \alpha_n \cap [c, d]) := \frac{1}{12}\frac{(d-c)^3}{b-a}\frac{1}{(m-1)^2}.$$

*Proof.* $P$ being a uniform distribution on the closed interval $[a, b]$, its density function is given by $f(x) = \frac{1}{b-a}$ if $x \in [a, b]$, and zero otherwise. Also, notice that the closed interval $[a, b]$ can be represented by

$$[a, b] := \{(t, 0) : a \leq t \leq b\}.$$

Let $c_1, c_2, c_3, \cdots, c_m$ be the $m$ elements that $\alpha_n$ contains from the closed interval $[c, d]$ such that $c = c_1 < c_2 < \cdots < c_m = d$. Since the closed interval $[c, d]$ is a line segment and $P$ is a uniform distribution, we have

$$c_2 - c_1 = c_3 - c_2 = \cdots = c_m - c_{m-1} = \frac{c_m - c_1}{m-1} = \frac{d-c}{m-1}$$

implying

$$c_2 = c_1 + \frac{d-c}{m-1} = c + \frac{d-c}{m-1},$$
$$c_3 = c_2 + \frac{d-c}{m-1} = c + \frac{2(d-c)}{m-1},$$
$$c_4 = c_3 + \frac{d-c}{m-1} = c + \frac{3(d-c)}{m-1},$$

and so on.

Thus, we have $c_j = c + \frac{j-1}{m-1}(d-c)$ for $1 \leq j \leq m$. The distortion error contributed by the $m$ elements in the closed interval $[c, d]$ is given by

$$\begin{aligned} V(P; \alpha_n \cap [c,d]) &= \int_{[c,d]} \min_{x \in \alpha_n \cap [c,d]} \rho((t,0), x)\, dP \\ &= \frac{1}{b-a}\Big(2 \int_{c_1}^{\frac{c_1+c_2}{2}} \rho((t,0),(c_1,0))\, dt + (m-2) \int_{\frac{c_1+c_2}{2}}^{\frac{c_2+c_3}{2}} \rho((t,0),(c_2,0))\, dt\Big) \\ &= \frac{1}{12} \frac{(d-c)^3}{b-a} \frac{1}{(m-1)^2}. \end{aligned}$$

Thus, the proof of the proposition is complete. □

**Proposition 2.2.** *Let $P$ be a uniform distribution on the closed interval $[a, b]$. Let $\alpha_n$ be an optimal set of $n$-points for $P$ such that $\alpha_n$ contains $n$ elements from the closed interval $[a, b]$ including the endpoint $a$. Then,*

$$\alpha_n = \Big\{a + \frac{2(j-1)(b-a)}{2n-1} : 1 \leq j \leq n\Big\}$$

*with the conditional unconstrained quantization error*

$$V_n = \frac{(b-a)^2}{3(2n-1)^2}.$$

*Proof.* As mentioned in Proposition 2.1, here the probability density function is given by $f(x) = \frac{1}{b-a}$ if $x \in [a, b]$, and zero otherwise, and

$$[a, b] := \{(t, 0) : a \leq t \leq b\}.$$

Let $c_1, c_2, c_3, \cdots, c_n$ be the $n$ elements that $\alpha_n$ contains from the closed interval $[a, b]$ including the endpoint $a$, i.e., $c_1 = a$. Let $c_n = d$, where $d < b$. Since the closed interval $[a, d]$ is a line segment and $P$ is a uniform distribution, we have

$$c_2 - c_1 = c_3 - c_2 = \cdots = c_n - c_{n-1} = \frac{c_n - c_1}{n-1} = \frac{d-a}{n-1}$$

implying

$$\begin{aligned} c_2 &= c_1 + \frac{d-a}{n-1} = a + \frac{d-a}{n-1}, \\ c_3 &= c_2 + \frac{d-a}{n-1} = a + \frac{2(d-a)}{n-1}, \\ c_4 &= c_3 + \frac{d-a}{n-1} = a + \frac{3(d-a)}{n-1}, \\ &\text{and so on.} \end{aligned}$$

Thus, we have $c_j = a + \frac{j-1}{n-1}(d-a)$ for $1 \leq j \leq n$. The distortion error contributed by the $n$ elements is given by

$$\begin{aligned} V(P; \alpha_n) &= \int \min_{x \in \alpha_n} \rho((t,0), x)\, dP \\ &= \frac{1}{b-a}\Big( \int_{c_1}^{\frac{c_1+c_2}{2}} \rho((t,0),(c_1,0))\, dt + (n-2) \int_{\frac{c_1+c_2}{2}}^{\frac{c_2+c_3}{2}} \rho((t,0),(c_2,0))\, dt \\ &\quad + \int_{\frac{c_{n-1}+c_n}{2}}^{b} \rho((t,0),(c_n,0))\, dt\Big) \\ &= \frac{\frac{(a-d)^3}{(n-1)^2} - 4(b-d)^3}{12(a-b)}, \end{aligned}$$

the minimum value of which is $\frac{(b-a)^2}{3(2n-1)^2}$ and it occurs when $d = b - \frac{b-a}{2n-1}$. Putting the values of $d$, we have

$$c_j = a + \frac{2(j-1)(b-a)}{2n-1} \text{ for } 1 \leq j \leq n$$

with the conditional unconstrained quantization error

$$V_n = \frac{(b-a)^2}{3(2n-1)^2}.$$

Thus, the proof of the proposition is complete. □

**Proposition 2.3.** *Let $P$ be a uniform distribution on the closed interval $[a, b]$. Let $\alpha_n$ be an optimal set of $n$-points for $P$ such that $\alpha_n$ contains $n$ elements from the closed interval $[a, b]$ including the endpoint $b$. Then,*

$$\alpha_n = \left\{a + \frac{(2j-1)(b-a)}{2n-1} : 1 \leq j \leq n\right\}$$

*with the conditional unconstrained quantization error*

$$V_n = \frac{(b-a)^2}{3(2n-1)^2}.$$

*Proof.* As mentioned in Proposition 2.1, here the probability density function is given by $f(x) = \frac{1}{b-a}$ if $x \in [a, b]$, and zero otherwise, and

$$[a, b] := \{(t, 0) : a \leq t \leq b\}.$$

Let $c_1, c_2, c_3, \cdots, c_n$ be the $n$ elements that $\alpha_n$ contains from the closed interval $[a, b]$ including the endpoint $b$, i.e., $c_n = b$. Let $c_1 = d$, where $a < d$. Since the closed interval $[c_1, b]$ is a line segment and $P$ is a uniform distribution, we have

$$c_2 - c_1 = c_3 - c_2 = \cdots = c_n - c_{n-1} = \frac{c_n - c_1}{n-1} = \frac{b-d}{n-1}$$

implying

$$\begin{aligned} c_2 &= c_1 + \frac{b-a}{n-1} = d + \frac{b-d}{n-1}, \\ c_3 &= c_2 + \frac{b-a}{n-1} = d + \frac{2(b-d)}{n-1}, \\ c_4 &= c_3 + \frac{b-a}{n-1} = d + \frac{3(b-d)}{n-1}, \\ &\text{and so on.} \end{aligned}$$

Thus, we have $c_j = d + \frac{j-1}{n-1}(b-d)$ for $1 \leq j \leq n$. The distortion error contributed by the $n$ elements is given by

$$\begin{aligned} V(P; \alpha_n) &= \int \min_{x \in \alpha_n} \rho((t, 0), x)\, dP \\ &= \frac{1}{b-a}\Big(\int_a^{\frac{c_1+c_2}{2}} \rho((t, 0), (c_1, 0))\, dt + (n-2) \int_{\frac{c_1+c_2}{2}}^{\frac{c_2+c_3}{2}} \rho((t, 0), (c_2, 0))\, dt \\ &\quad + \int_{\frac{c_{n-1}+c_n}{2}}^{c_n} \rho((t, 0), (c_n, 0))\, dt\Big) \\ &= \frac{4(a-d)^3 + \frac{(d-b)^3}{(n-1)^2}}{12(a-b)}, \end{aligned}$$

the minimum value of which is $\frac{(b-a)^2}{3(2n-1)^2}$ and it occurs when $d = a + \frac{b-a}{2n-1}$. Putting the values of $d$, we have

$$c_j = a + \frac{(2j-1)(b-a)}{2n-1} \text{ for } 1 \leq j \leq n$$

with the conditional unconstrained quantization error

$$V_n = \frac{(b-a)^2}{3(2n-1)^2}.$$

Thus, the proof of the proposition is complete. □

**Remark 2.4.** Although a detailed proof is given, the elements in $\alpha_n$ in Proposition 2.3 can be obtained from the elements in $\alpha_n$ given in Proposition 2.2 by translating the elements to the right in the amount of $\frac{b-a}{2n-1}$.

## 3. Conditional constrained quantization for a uniform distribution on the boundary of a semicircular disc

Let $L$ be the boundary of the semicircular disc $x_1^2 + x_2^2 = 1$, where $x_2 \geq 0$. Let the base of the semicircular disc be $AOB$, where $A$ and $B$ have the coordinates $(-1,0)$ and $(1,0)$, and $O$ is the origin $(0,0)$. Let $s$ represent the distance of any point on $L$ from the origin tracing along the boundary $L$ in the counterclockwise direction. Notice that $L = L_1 \cup L_2$, where

$$L_1 = \{(x_1, x_2) : x_1 = t,\ x_2 = 0 \text{ for } -1 \leq t \leq 1\}, \text{ and}$$
$$L_2 = \{(x_1, x_2) : x_1 = \cos t,\ x_2 = \sin t \text{ for } 0 \leq t \leq \pi\}.$$

Let $P$ be the uniform distribution on the boundary of the semicircular disc. Then, the probability density function for $P$ is given by

$$f(x_1, x_2) = \begin{cases} \frac{1}{2+\pi} & \text{if } (x_1, x_2) \in L, \\ 0 & \text{otherwise,} \end{cases}$$

On both $L_1$ and $L_2$, we have $ds = \sqrt{(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2}\, dt = dt$ yielding $dP(s) = P(ds) = f(x_1, x_2)ds = f(x_1, x_2)dt$. In the definition of conditional constrained quantization error, let $\beta := \{(-1,0), (1,0)\}$, and let $L$ be the single constraint. Upon the given set $\beta$, the $n$th conditional constrained quantization errors are defined for all $n \geq 2$. Notice that the boundary $L$ has 'maximum symmetry' with respect to the $y$-axis. By the maximum symmetry of $L$ with respect to the $y$-axis, it is meant that if two regions of the same geometrical shapes are equidistant and are on opposite sides from the line, then they have the same probability. In finding the conditional optimal sets of $n$-points, we will use this property.

Notice that the conditional optimal set of two-points is the set $\beta$ itself. In the sequel of this section, we investigate the conditional optimal sets of $n$-points for $n \geq 3$. Let us now give the following proposition, which plays a vital role in this section.

**Proposition 3.1.** *Let $\alpha_n$ be a conditional optimal set of $n$-points for $P$ for some $n \geq 3$. Let card$(\alpha_n \cap L_1) = n_1$ and card$(\alpha_n \cap L_2) = n_2$ with corresponding conditional quantization error $V_n := V_{n_1,n_2}(P)$ for some $n_1, n_2 \geq 2$. Then,*

$$\alpha_n \cap L_1 = \{(-1 + \frac{2(j-1)}{n_1 - 1}, 0) : 1 \leq j \leq n_1\}, \text{ and } \alpha_n \cap L_2 = \{(\cos\frac{(j-1)\pi}{n_2 - 1}, \sin\frac{(j-1)\pi}{n_2 - 1}) : 1 \leq j \leq n_2\},$$

*with*

$$V_{n_1,n_2}(P) = \frac{2}{3(2+\pi)}\Big(\frac{1}{(n_1-1)^2} - 6(n_2-1)\sin\Big(\frac{\pi}{2(n_2-1)}\Big) + 3\pi\Big).$$

*Proof.* By the hypothesis, card$(\alpha_n \cap L_1) = n_1$ and card$(\alpha_n \cap L_2) = n_2$ for some $n_1, n_2 \geq 2$. The proof of

$$\alpha_n \cap L_1 = \{(-1 + \frac{2(j-1)}{n_1 - 1}, 0) : 1 \leq j \leq n_1\}$$

directly follows from Proposition 2.1. Moreover, as shown in Proposition 2.1, the distortion error for these $n_1$ elements is obtained as

$$V(P;\alpha_n\cap L_1)=\frac{2}{3(2+\pi)}\frac{1}{(n_1-1)^2}.$$

Let $\alpha_n\cap L_2=\{(\cos\theta_j,\sin\theta_j):1\leq j\leq n_2\}$, where $\theta_1=0$, and $\theta_{n_2}=\pi$. $L_2$ being a circular arc and $P$ is a uniform distribution, we have

$$\theta_2-\theta_1=\theta_3-\theta_2=\cdots=\theta_{n_2}-\theta_{n_2-1}=\frac{\theta_{n_2}-\theta_1}{n_2-1}=\frac{\pi}{n_2-1}.$$

Thus, proceeding in a similar way as Proposition 2.1, we have $\theta_j=\frac{(j-1)\pi}{n_2-1}$ for $1\leq j\leq n_2$ yielding

$$\alpha_n\cap L_2=\{(\cos\frac{(j-1)\pi}{n_2-1},\sin\frac{(j-1)\pi}{n_2-1}):1\leq j\leq n_2\}.$$

The distortion error due to these $n_2$ elements is obtained as

$$\begin{aligned}V(P;\alpha_n\cap L_2)&=\frac{1}{2+\pi}\Big(2\int_0^{\frac{\theta_1+\theta_2}{2}}\rho((\cos t,\sin t),(1,0))\,dt+(n_2-2)\int_{\frac{\theta_1+\theta_2}{2}}^{\frac{\theta_2+\theta_3}{2}}\rho((\cos t,\sin t),(\cos\theta_2,\sin\theta_2))\,dt\Big)\\&=\frac{2}{3(2+\pi)}\Big(-6(n_2-1)\sin\Big(\frac{\pi}{2(n_2-1)}\Big)+3\pi\Big).\end{aligned}$$

Thus, we have

$$\begin{aligned}V_n=V_{n_1,n_2}(P)&=V(P;\alpha_n\cap L_1)+V(P;\alpha_n\cap L_2)\\&=\frac{2}{3(2+\pi)}\Big(\frac{1}{(n_1-1)^2}-6(n_2-1)\sin\Big(\frac{\pi}{2(n_2-1)}\Big)+3\pi\Big).\end{aligned}$$

Hence, the proof of the proposition is complete. □

**Remark 3.2.** Proposition 3.1 plays a significant role in calculating the optimal sets of $n$-points for the probability distribution $P$ on the boundary of the semicircular disc, as shown in the following two propositions.

**Proposition 3.3.** *The conditional optimal set of three-points is given by* $\{(1,0),(0,1),(-1,0)\}$ *with conditional constrained quantization error* $V_3=\frac{2}{2+\pi}(-2\sqrt{2}+\frac{1}{3}+\pi)$.

*Proof.* Let $\alpha$ be an optimal set of three-points. By the hypothesis $(1,0),(-1,0)\in\alpha$. Due to maximum symmetry, we can assume that either $(0,0)$ or $(0,1)\in\alpha$. If $(0,0)\in\alpha$, then as $n_1=3$ and $n_2=2$, using Proposition 3.1, we have the distortion error as $V_{3,2}(P)=0.476477$. If $(0,1)\in\alpha$, then as $n_1=2$ and $n_2=3$, we have $V_{2,3}(P)=0.251478$. Since $V_{2,3}(P)<V_{3,2}(P)$, the conditional optimal set of three-points is given by $\{(1,0),(0,1),(-1,0)\}$ with conditional constrained quantization error $V_3=\frac{2}{2+\pi}(-2\sqrt{2}+\frac{1}{3}+\pi)$. Thus, the proof of the proposition is complete (see Figure 1). □

**Proposition 3.4.** *The conditional optimal set of four-points is given by* $\{(0,0),(1,0),(0,1),(-1,0)\}$ *with conditional constrained quantization error* $V_4=\frac{-24\sqrt{2}+12\pi+1}{12+6\pi}$.

*Proof.* Let $\alpha$ be an optimal set of four-points. By the hypothesis $(1,0),(-1,0)\in\alpha$. Due to maximum symmetry, we can assume that the other two elements in $\alpha$ are on the axis of symmetry, or they are symmetrically located on $L$. Thus, the following cases can occur:

*Case 1. The two elements in* $\alpha\setminus\{(1,0),(-1,0)\}$ *are on the axis of symmetry.*

In this case, we can assume that $(0,0),(0,1)\in\alpha$. Then, as $n_1=3$ and $n_2=3$, the distortion error is $V_{3,3}(P)=0.154232$.

*Case 2. The two elements in* $\alpha\setminus\{(1,0),(-1,0)\}$ *are symmetrically located on* $L$.

In this case, the two elements are either symmetrically located on $L_1$ or on $L_2$. If they are symmetrically located on $L_1$, then the distortion error is $V_{4,2}(P)=0.458469$. If they are symmetrically located on $L_2$, then the distortion error is $V_{2,4}(P)=0.184739$.

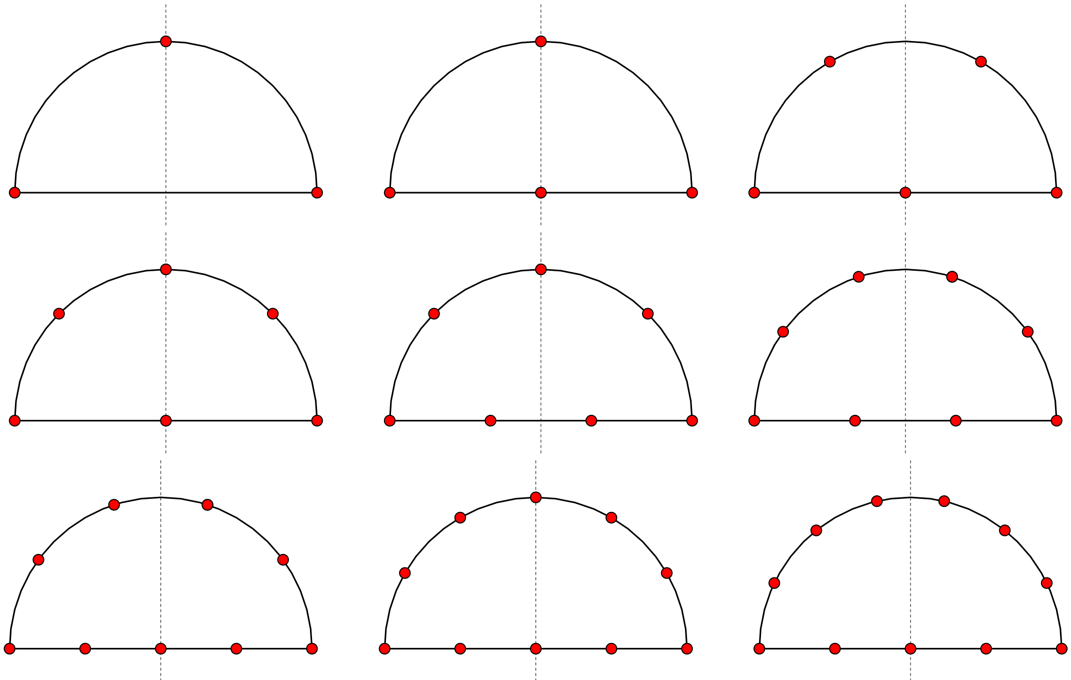

Figure 1. Optimal configuration of $n$-points for $3 \leq n \leq 11$.

Considering all the above possible distortion errors, we see that the distortion error is minimal when $n_1 = 3$ and $n_2 = 3$. Thus, $\{(0,0),(1,0),(0,1),(-1,0)\}$ forms the conditional optimal set of four-points with conditional constrained quantization error $V_4 = \frac{-24\sqrt{2}+12\pi+1}{12+6\pi}$, which is the proposition (see Figure 1). □

Let us now give the following theorem, which gives the main result in this section.

**Theorem 3.5.** *Let $\alpha_n$ be a conditional optimal set of $n$-points for $P$ for some $n \geq 3$. Let card$(\alpha_n \cap L_1) = n_1$ be known. Then,*

$$\alpha_n = \{(-1+\frac{2(j-1)}{n_1-1}, 0) : 1 \leq j \leq n_1\} \bigcup \{(\cos\frac{(j-1)\pi}{n-n_1+1}, \sin\frac{(j-1)\pi}{n-n_1+1}) : 2 \leq j \leq n-n_1+1\}$$

*with the conditional constrained quantization error*

$$V_n = \frac{2}{3(2+\pi)}\Big(\frac{1}{(n_1-1)^2} - 6(n-n_1+1)\sin(\frac{\pi}{2n-2n_1+2}) + 3\pi\Big).$$

*Proof.* Let $\alpha_n$ be a conditional optimal set of $n$-points for $P$. Let card$(\alpha_n \cap L_1) = n_1$ and card$(\alpha_n \cap L_2) = n_2$. Notice that $\alpha_n \cap L_1 \cap L_2 = \{(-1,0),(0,1)\}$, and hence $n_1 + n_2 = n + 2$ yielding $n_2 = n - n_1 + 2$. Thus, if $n_1$ is known, one can easily calculate $n_2$, and then the conditional optimal set $\alpha_n$ of $n$-points and the corresponding conditional constrained quantization error can be deduced by Proposition 3.1, and they are given by

$$\alpha_n = \{(-1+\frac{2(j-1)}{n_1-1}, 0) : 1 \leq j \leq n_1\} \bigcup \{(\cos\frac{(j-1)\pi}{n_2-1}, \sin\frac{(j-1)\pi}{n_2-1}) : 2 \leq j \leq n-n_1+1\}$$

with $V_n = V_{n_1, n-n_1+2} = \frac{2}{3(2+\pi)}\Big(\frac{1}{(n_1-1)^2} - 6(n-n_1+1)\sin(\frac{\pi}{2n-2n_1+2}) + 3\pi\Big)$. Thus, the proof of the theorem is complete. □

For a given positive integer $n \geq 3$, to determine the positive integer $n_1$ as mentioned in Theorem 3.5, we proceed as follows:

**Definition 3.6.** *Define the sequence $\{a(n)\}$ such that $a(n) = \lfloor \frac{5(n+4)}{13} \rfloor$ for $n \geq 1$, i.e.,*

$$\{a(n)\}_{n=1}^{\infty} = \{1,2,2,3,3,3,4,4,5,5,5,6,6,6,7,7,8,8,8,9,9,10,10,10,11,11,11,12,12,13,13,13,\cdots\},$$

*where $\lfloor x \rfloor$ represents the greatest integer not exceeding $x$.*

The following algorithm helps us to determine the exact value of $n_1$ mentioned in Theorem 3.5.

3.7. **Algorithm.** Let $n \geq 3$, and let $V(n, n_1) := V_{n_1, n-n_1+2}$, as given by Theorem 3.5, denote the distortion error if an optimal set $\alpha_n$ contains $n_1$ elements from the base of the semicircular disc. Let $\{a(n)\}$ be the sequence defined by Definition 3.6. Then, the algorithm runs as follows:

$(i)$ Write $n_1 := a(n)$.
$(ii)$ If $n_1 = 2$ go to step $(v)$, else step $(iii)$.
$(iii)$ If $V(n, n_1 - 1) < V(n, n_1)$ replace $n_1$ by $n_1 - 1$ and go to step $(ii)$, else step $(iv)$.
$(iv)$ If $V(n, n_1 + 1) < V(n, n_1)$ replace $n_1$ by $n_1 + 1$ and return, else step $(v)$.
$(v)$ End.

When the algorithm ends, then the value of $n_1$, obtained, is the exact value of $n_1$ that an optimal set $\alpha_n$ contains from the base of the semicircular disc.

**Remark 3.8.** If $n = 50$, then $a(n) = 20$, and by the algorithm we also obtain $n_1 = 20$; if $n = 80$, then $a(n) = 32$, and by the algorithm we also obtain $n_1 = 32$. If $n = 1200$, then $a(n) = 463$, and by the algorithm, we obtain $n_1 = 468$; if $n = 2000$, then $a(n) = 770$, and by the algorithm, we obtain $n_1 = 779$; and if $n = 3000$, then $a(n) = 1155$, and by the algorithm, we obtain $n_1 = 1168$. Thus, we see that with the help of the sequence and the algorithm, one can easily determine the exact value of $n_1$ for any positive integer $n \geq 3$.

## 4. Conditional unconstrained quantization for a uniform distribution on an equilateral triangle

Let $\triangle OAB$ be an equilateral triangle with vertices $O(0,0), A(1,0), B(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $L_1, L_2, L_3$ be the sides $OA$, $AB$ and $BO$, respectively. Let $P$ be the uniform distribution defined on the equilateral triangle $\triangle$ formed by the sides $L_1, L_2, L_3$. Let $s$ represent the distance of any point on $\triangle$ from the origin tracing along the boundary of the triangle in the counterclockwise direction. Then, the points $O, A, B$ are, respectively, represented by $s = 0, s = 1, s = 2$. The probability density function (pdf) $f$ of the uniform distribution $P$ is given by $f(s) := f(x_1, x_2) = \frac{1}{3}$ for all $(x_1, x_2) \in L_1 \cup L_2 \cup L_3$, and zero otherwise. The sides $L_1, L_2, L_3$ are represented by the parametric equations as follows:

$$L_1 = \{(t, 0)\}, \quad L_2 = \Big\{\Big(1 - \frac{t}{2}, \frac{\sqrt{3}t}{2}\Big)\Big\}, \quad L_3 = \Big\{\Big(\frac{1-t}{2}, \frac{\sqrt{3}(1-t)}{2}\Big)\Big\},$$

where $0 \leq t \leq 1$. Again, $dP(s) = P(ds) = f(x_1, x_2)ds$. On each $L_j$ for $1 \leq j \leq 3$, we have $(ds)^2 = (dx_1)^2 + (dx_2)^2 = (dt)^2$ yielding $ds = dt$. In the definition of conditional unconstrained quantization error, let $\beta := \{(0,0), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$. Let $L := L_1 \cup L_2 \cup L_3$ which is the support of $P$. Upon the given set $\beta$, the $n$th conditional unconstrained quantization errors are defined for all $n \geq 3$. Notice that the conditional optimal set of three-points is the set $\beta$ itself. In the sequel of this section, we investigate the conditional optimal sets of $n$-points for $n \geq 4$.

Consider the following two affine transformations:

$$T_1(x, y) = (-\frac{1}{2}x + 1, \frac{\sqrt{3}}{2}x) \text{ and } T_2(x, y) = (-\frac{1}{2}x + \frac{1}{2}, -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}). \tag{5}$$

Let us now give the following proposition, which plays a vital role in this section. In this regard, one can also see [RR].

**Proposition 4.1.** *Let $\alpha_n$ be a conditional optimal set of $n$-points for $P$ for some $n \geq 4$. Let $card(\alpha_n \cap L_1) = n_1$, $card(\alpha_n \cap L_2) = n_2$ and $card(\alpha_n \cap L_3) = n_3$ with corresponding conditional quantization error*

*$V_n := V_{n_1,n_2,n_3}(P)$ for some $n_1, n_2, n_3 \geq 2$. Then,*

$$\alpha_n \cap L_1 = \Big\{(\frac{j-1}{n_1-1}, 0) : 1 \leq j \leq n_1\Big\},$$
$$\alpha_n \cap L_2 = \Big\{T_1(\frac{j-1}{n_2-1}, 0) : 1 \leq j \leq n_2\Big\}, \text{ and}$$
$$\alpha_n \cap L_3 = \Big\{T_2(\frac{j-1}{n_3-1}, 0) : 1 \leq j \leq n_3\Big\},$$

*with*

$$V_{n_1,n_2,n_3}(P) = \frac{1}{36}\Big(\frac{1}{(n_1-1)^2} + \frac{1}{(n_2-1)^2} + \frac{1}{(n_3-1)^2}\Big).$$

*Proof.* By the hypothesis, $\text{card}(\alpha_n \cap L_1) = n_1$ for some $n_1 \geq 2$. Let

$$\alpha_n \cap L_1 = \{(a_j, 0) : 1 \leq j \leq n_1\}, \tag{6}$$

By Proposition 2.1, we have $a_j = \frac{j-1}{n_1-1}$ for $1 \leq j \leq n_1$ implying

$$\alpha_n \cap L_1 = \{(\frac{j-1}{n_1-1}, 0) : 1 \leq j \leq n_1\}.$$

Given $\text{card}(\alpha_n \cap L_2) = n_2$. If $\text{card}(\alpha_n \cap L_1) = n_2$, then as before we have

$$\alpha_n \cap L_1 = \{(\frac{j-1}{n_2-1}, 0) : 1 \leq j \leq n_2\}.$$

Hence, $T_2$ being an affine transformation such that $T_2(L_1) = L_2$, we have

$$\alpha_n \cap L_2 = \{T_1(\frac{j-1}{n_2-1}, 0) : 1 \leq j \leq n_2\}.$$

Similarly, we have

$$\alpha_n \cap L_3 = \{T_2(\frac{j-1}{n_3-1}, 0) : 1 \leq j \leq n_3\}.$$

To find the quantization error, we proceed as follows. Let $V(P; \alpha_n \cap L_j)$ denote the distortion error contributed by the elements in $\alpha_n \cap L_j$ for $j = 1, 2, 3$. Then,

$$V_{n_1,n_2,n_3}(P) = V(P; \alpha_n \cap L_1) + V(P; \alpha_n \cap L_2) + V(P; \alpha_n \cap L_3).$$

By Proposition 2.1, we have

$$V(P; \alpha_n \cap L_1) = \frac{1}{36\,(n_1-1)^2}.$$

Due to rotational symmetry, we have

$$V(P; \alpha_n \cap L_2) = \frac{1}{36\,(n_2-1)^2} \text{ and } V(P; \alpha_n \cap L_3) = \frac{1}{36\,(n_3-1)^2}.$$

Hence,

$$V_{n_1,n_2,n_3}(P) = \frac{1}{36}\Big(\frac{1}{(n_1-1)^2} + \frac{1}{(n_2-1)^2} + \frac{1}{(n_3-1)^2}\Big).$$

Thus, the proof of the proposition is complete. □

**Note 4.2.** Let $\alpha_n(L_j)$ be the set consisting of all the elements in $\alpha_n \cap L_j$ except the right endpoint for each $j = 1, 2, 3$. Then,

$$\alpha_n(L_1) = \Big\{(\frac{j-1}{n_1-1}, 0) : 1 \leq j \leq n_1 - 1\Big\},$$
$$\alpha_n(L_2) = \Big\{T_1(\frac{j-1}{n_2-1}, 0) : 1 \leq j \leq n_2 - 1\Big\}, \text{ and}$$
$$\alpha_n(L_3) = \Big\{T_2(\frac{j-1}{n_3-1}, 0) : 1 \leq j \leq n_3 - 1\Big\}$$

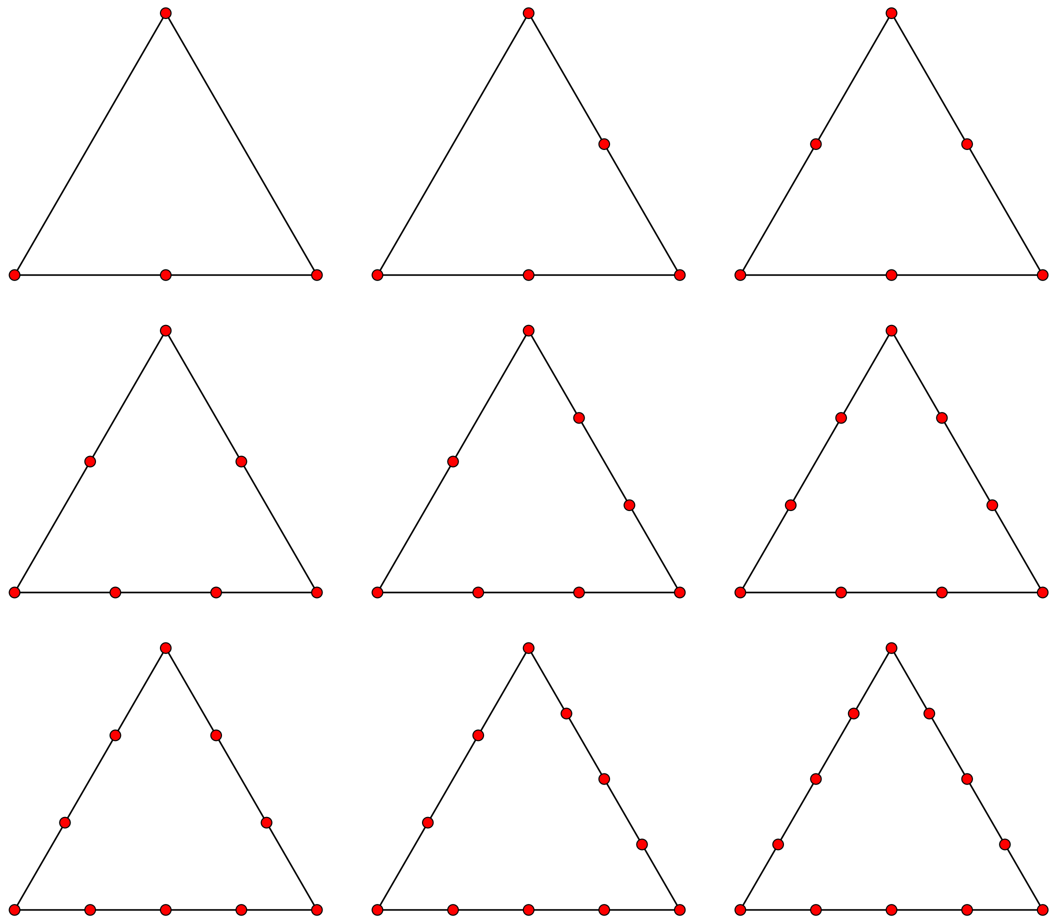

FIGURE 2. Optimal configuration of $n$-points for $4 \leq n \leq 12$.

with $\text{card}(\alpha_n(L_1)) = n_1 - 1$, $\text{card}(\alpha_n(L_2)) = n_2 - 1$, and $\text{card}(\alpha_n(L_3)) = n_3 - 1$. Notice that the sets $\alpha_n(L_j)$ are disjoints and $(n_1 - 1) + (n_2 - 1) + (n_3 - 1) = n$.

**Lemma 4.3.** *Let $n_j \in \mathbb{N}$ for $j = 1, 2, 3$ be the numbers as defined in Proposition 4.1. Then, $|n_i - n_j| = 0$, or $1$ for $1 \leq i \neq j \leq 3$.*

*Proof.* We first show that $|n_1 - n_2| = 0$, or 1. Write $m := (n_1 - 1) + (n_2 - 1)$. Now, the distortion error contributed by the $m$ elements in $\alpha_1(L_1) \cup \alpha_n(L_2)$ is given by

$$\frac{1}{36}\Big(\frac{1}{(n_1-1)^2} + \frac{1}{(n_2-1)^2}\Big).$$

The above expression is minimum if $n_1 - 1 \approx \frac{m}{2}$ and $n_2 - 1 \approx \frac{m}{2}$. Thus, we see that if $m = 2k$ for some positive integer $k$, then $n_1 - 1 = n_2 - 1 = k$, and if $m = 2k + 1$ for some positive integer $k$, then either $(n_1 - 1 = k + 1$ and $n_2 - 1 = k)$ or $(n_1 - 1 = k$ and $n_2 - 1 = k + 1)$ which yields the fact that $|n_1 - n_2| = 0$, or 1. Similarly, we can show that $|n_i - n_j| = 0$, or 1 for any $1 \leq i \neq j \leq 3$. Thus, the proof of the lemma is complete. □

Let us now give the following theorem, which gives the main result in this section.

**Theorem 4.4.** *Let $\alpha_n$ be a conditional optimal set of $n$-points for $P$ for some $n \geq 4$. Then,*

$$\alpha_n = \Big\{(\frac{j-1}{n_1-1}, 0) : 1 \leq j \leq n_1 - 1\Big\} \bigcup \Big\{T_1(\frac{j-1}{n_2-1}, 0) : 1 \leq j \leq n_2 - 1\Big\}$$
$$\bigcup \Big\{T_2(\frac{j-1}{n_3-1}, 0) : 1 \leq j \leq n_3 - 1\Big\}$$

*with the conditional unconstrained quantization error*

$$V_n = \frac{1}{36}\Big(\frac{1}{(n_1-1)^2}+\frac{1}{(n_2-1)^2}+\frac{1}{(n_3-1)^2}\Big),$$

*where $n_1, n_2, n_3$ are given as follows: For some $k \in \mathbb{N}$, if $n = 3k$, then $n_1 - 1 = n_2 - 1 = n_3 - 1 = k$; if $n = 3k+1$, then $n_1 - 1 = k+1$ and $n_2 - 1 = n_3 - 1 = k$; and if $n = 3k+2$, then $n_1 - 1 = n_2 - 1 = k+1$ and $n_3 - 1 = k$.*

*Proof.* Let $\alpha_n(L_j)$ be the sets defined by Note 4.2 for $j = 1, 2, 3$. Notice that for $j = 1, 2, 3$, the sets $\alpha_n(L_j)$ are nonempty, and so we can find three positive integers $n_j \geq 2$ as defined in Proposition 4.1 such that $\text{card}(\alpha_n(L_j)) = n_j - 1$. Since

$$\alpha_n = \bigcup_{j=1}^{3} \alpha_n(L_j),$$

the expression for $\alpha_n$ follows by Note 4.2. The expression for the conditional constrained quantization error $V_n$ follows from Proposition 4.1. By Lemma 4.3, it follows that for some $k \in \mathbb{N}$, if $n = 3k$, then $n_1 - 1 = n_2 - 1 = n_3 - 1 = k$; if $n = 3k + 1$, then $n_1 - 1 = k + 1$ and $n_2 - 1 = n_3 - 1 = k$; and if $n = 3k+2$, then $n_1 - 1 = n_2 - 1 = k+1$ and $n_3 - 1 = k$. Thus, the proof of the theorem is complete. ☐

Using Theorem 4.4, the following example can be obtained.

**Example 4.5.** A conditional optimal set of four-points is given by $\{(0,0), (\frac{1}{2}, 1), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ with conditional constrained quantization error $V_4 = \frac{1}{16}$; a conditional optimal set of five-points is given by $\{(0,0), (\frac{1}{2}, 1), (1,0), (\frac{1}{4}, \frac{\sqrt{3}}{4}), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ with conditional constrained quantization error $V_5 = \frac{1}{24}$; and so on (see Figure 2).

**Theorem 4.6.** *The conditional unconstrained quantization dimension $D(P)$ of the probability measure $P$ exists, and $D(P) = 1$, and the $D(P)$-dimensional unconstrained quantization coefficient for $P$ exists as a finite positive number and equals $\frac{3}{4}$.*

*Proof.* For $n \in \mathbb{N}$ with $n \geq 4$, let $\ell(n)$ be the unique natural number such that $3\ell(n) \leq n < 3(\ell(n) + 1)$. Then, $V_{3(\ell(n)+1)} \leq V_n \leq V_{3\ell(n)}$. We can take $n$ large enough so that $V_{3\ell(n)} < 1$. Then,

$$0 < -\log V_{3\ell(n)} \leq -\log V_n \leq -\log V_{3(\ell(n)+1)}$$

yielding

$$\frac{2\log 3\ell(n)}{-\log V_{3(\ell(n)+1)}} \leq \frac{2\log n}{-\log V_n} \leq \frac{2\log 3(\ell(n)+1)}{-\log V_{3\ell(n)}}.$$

Notice that

$$\lim_{n\to\infty} \frac{2\log 3\ell(n)}{-\log V_{3(\ell(n)+1)}} = \lim_{n\to\infty} \frac{2\log 3\ell(n)}{-\log \frac{1}{12(\ell(n)+1)^2}} = 1, \text{ and}$$

$$\lim_{n\to\infty} \frac{2\log 3(\ell(n)+1)}{-\log V_{3\ell(n)}} = \lim_{n\to\infty} \frac{2\log 3(\ell(n)+1)}{-\log \frac{1}{12(\ell(n))^2}} = 1.$$

Hence, $\lim_{n\to\infty} \frac{2\log n}{-\log V_n} = 1$, i.e., the conditional unconstrained quantization dimension $D(P)$ of the probability measure $P$ exists and $D(P) = 1$. Since

$$\lim_{n\to\infty} n^2 V_n \geq \lim_{n\to\infty} (3\ell(n))^2 V_{3(\ell(n)+1)} = \lim_{n\to\infty} (3\ell(n))^2 \frac{1}{12(\ell(n)+1)^2} = \frac{3}{4}, \text{ and}$$

$$\lim_{n\to\infty} n^2 V_n \leq \lim_{n\to\infty} (3(\ell(n)+1))^2 V_{3\ell(n)} = \lim_{n\to\infty} (3(\ell(n)+1))^2 \frac{1}{12(\ell(n))^2} = \frac{3}{4},$$

by the squeeze theorem, we have $\lim_{n\to\infty} n^2 V_n = \frac{3}{4}$, i.e., the $D(P)$-dimensional unconstrained quantization coefficient for $P$ exists as a finite positive number and equals $\frac{3}{4}$. Thus, the proof of the theorem is complete. ☐

## 5. Observations

To distinguish, let $V_{c,n}(P)$ denote the $n$th conditional quantization error for $n$-points, and $V_n(P)$ denote the $n$th unconstrained quantization error for $n$-means for a Borel probability measure $P$. Also, to distinguish we denote the conditional quantization dimension by $D_c(P)$ and the unconstrained quantization dimension by $D(P)$, respectively.

In the following remarks, we give some observations.

**Remark 5.1.** Conditional optimal sets of $n$-points for a Borel probability measure can exist with $V_{c,n} > V_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$, where $\ell$ is the number of elements in the conditional set $\beta$. In this regard, we give the following example: Let $P$ be a Borel probability measure on $\mathbb{R}$, which is uniform on its support $[0, 1]$, and let $\beta := \{0\}$ be the conditional set. Then, by Proposition 2.2, we see that the conditional optimal sets of $n$-points for all $n \geq 1$ are given by

$$\Big\{\frac{2(j-1)}{2n-1} : 1 \leq j \leq n\Big\} \text{ with } V_{c,n}(P) = \frac{1}{3(2n-1)^2}.$$

On the other hand, $P$ is a uniform distribution, the optimal sets of $n$-means are given by

$$\Big\{\frac{2j-1}{2n} : 1 \leq j \leq n\Big\} \text{ with } V_n(P) = \frac{1}{12n^2}.$$

Thus, we see that for all $n \geq 1$, the conditional optimal sets of $n$-points exist in this case and

$$V_{c,n} = \frac{1}{3(2n-1)^2} = \frac{1}{3(n+n-1)^2} > \frac{1}{3(n+n)^2} = \frac{1}{12n^2} > V_n.$$

Moreover, we see that $D_c(P) = D(P) = 1$.

**Remark 5.2.** In unconstrained quantization, for a Borel probability measure with infinite support, an optimal set of $n$-means contains exactly $n$ elements. It is not true in constrained quantization (see [PR1]). In constrained quantization for a Borel probability measure with infinite support, an optimal set of $n$-points for all $n$ may not contain exactly $n$ elements. Let $k$ be the largest positive integer for which an optimal set of $k$-points contains exactly $k$ elements. In this case, there is no conditional optimal set of $n$-points for any $n \geq (k+1)$.

**Remark 5.3.** Optimal sets of $n$-points for all positive integers can exist, but a conditional optimal set may not exist for all $n$. In this regard, we give the following example:

Let $P$ be a probability measure on $\mathbb{R}^2$ with its support the closed interval $\{(t, 0) : 0 \leq t \leq 1\}$, and let $P$ be uniform on its support. Let $\beta := \{(0, \frac{1}{100})\}$ be the conditional set. Let $\alpha := \{(0, \frac{1}{100})\} \cup \{(t_j, 0) : 1 \leq j \leq n\}$ be a conditional optimal set of $(n+1)$-points such that $t_1 < t_2 < \cdots < t_n$. Let the boundary of the Voronoi regions of $(0, \frac{1}{100})$ and $(t_1, 0)$ intersect the support of $P$ at the point $(d, 0)$. Clearly $0 \leq d < \frac{t_1}{2}$. Also, notice that $t_j = d + \frac{(2j-1)(1-d)}{2n}$ for $j = 1, 2, \cdots n$. Then, the distortion error is given by

$$\begin{aligned} V(P;\alpha) &= \text{distortion error contributed by } (0, \frac{1}{100}) + \text{distortion error contributed by all } t_j \\ &= \int_0^d \rho((t,0),(0,\frac{1}{100}))\,dt + \frac{(1-d)^3}{12n^2} \\ &= \frac{d^3}{3} + \frac{d}{10000} + \frac{(1-d)^3}{12n^2} \end{aligned}$$

the minimum value of which is

$$\frac{n\left(4n\left(n\sqrt{10001-4n^2}+249925\right)-10001\sqrt{10001-4n^2}\right)+250075}{750000\left(1-4n^2\right)^2}$$

and it occurs when $d = \frac{50-n\sqrt{10001-4n^2}}{50-200n^2}$. Notice that $d$ is a decreasing sequence of real numbers for $n = 1, 2, \cdots, 50$, and becomes imaginary if $n \geq 51$. In fact, we see that $d > 0$ if $1 \leq n < 50$, and $d = 0$ if $n = 50$. Hence, we can say that the conditional optimal sets of $n$-points exist for $1 \leq n \leq 49$, and it does not exist if $n \geq 50$. But, notice that the optimal sets of $n$-points for all $n \geq 50$ still exist which are given by $\{(\frac{2j-1}{2n}, 0) : 1 \leq j \leq n\}$ with quantization error $\frac{1}{12n^2}$.

**Remark 5.4.** In Section 4, we have seen that for a uniform distribution defined on the equilateral triangle, the conditional unconstrained quantization dimension is one, and the conditional unconstrained quantization coefficient is $\frac{3}{4}$. These values are the same as the unconstrained quantization dimension and the unconstrained quantization coefficient for the uniform distribution on the equilateral triangle (see [RR]). In the next section, we give a general proof to show that the lower and upper quantization dimensions and the lower and upper quantization coefficients for a Borel probability measure do not depend on the underlying conditional set.

## 6. Some important properties

In this section, in the first theorem, we show that in unconstrained quantization, the union of the optimal sets of $n$-means is dense in the support of $P$. In the next theorem, we show that in both constrained and unconstrained quantization, the lower and upper quantization dimensions and the lower and upper quantization coefficients for a Borel probability measure do not depend on the conditional set of the conditional quantization.

**Theorem 6.1.** *Let $P$ be a Borel probability measure on $\mathbb{R}^k$. Let $\alpha_n$ be the optimal sets of $n$-means for $P$ for all $n \in \mathbb{N}$. Then, $\bigcup\limits_{n=1}^{\infty} \alpha_n$ is dense in the support of $P$.*

*Proof.* Let $x \in \text{Supp}(P)$. Our aim is to show that for each $\epsilon > 0$, $B(x, \epsilon) \cap \bigcup\limits_{n=1}^{\infty} \alpha_n \neq \emptyset$. We prove it by contradiction. Let there exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap \bigcup_{n=1}^{\infty} \alpha_n = \emptyset$. Then,

$$V_{n,r}(P) = \int d(x, \alpha_n)^r dP(x) \geq \epsilon^r P(B(x, \epsilon) \cap \text{Supp}(P)).$$

We claim that $P(B(x, \epsilon) \cap \text{Supp}(P)) > 0$. Assume that $P(B(x, \epsilon) \cap \text{Supp}(P)) = 0$. This implies that $x \notin \text{Supp}(P)$, which is a contradiction. Thus $P(B(x, \epsilon) \cap \text{Supp}(P)) > 0$. Therefore, we get

$$\lim_{n \to \infty} V_{n,r}(P) \geq \epsilon^r P(B(x, \epsilon) \cap \text{Supp}(P)) > 0,$$

which contradicts the fact that $\lim_{n\to\infty} V_{n,r}(P) = 0$. This implies that for each $\epsilon > 0$, we have $B(x, \epsilon) \cap \bigcup\limits_{n=1}^{\infty} \alpha_n \neq \emptyset$. Thus, $\bigcup\limits_{n=1}^{\infty} \alpha_n$ is dense in the support of $P$. This completes the proof. ☐

**Theorem 6.2.** *In both constrained and unconstrained quantization, the lower and upper quantization dimensions and the lower and upper quantization coefficients for a Borel probability measure do not depend on the conditional set.*

*Proof.* Since the unconstrained quantization is a special case of constrained quantization, we give the proof of the theorem for constrained cases only. Let $\beta \subset \mathbb{R}^k$ be a conditional set with $\text{card}(\beta) = \ell$ for some $\ell \in \mathbb{N}$. Let $V_{n,r}(P)$ and $V_{c,n,r}(P)$ denote the $n$th constrained and the $n$th conditional constrained quantization errors, respectively. Take $n > \ell$. Then, one can easily see that

$$V_{n,r}(P) \leq V_{c,n,r}(P) \leq V_{n-\ell,r}(P). \tag{7}$$

By the previous inequalities and application of the squeeze theorem, we have $V_{\infty,r}(P) = V_{c,\infty,r}(P)$. Thus, by (7), we have

$$V_{n,r}(P) - V_{\infty,r}(P) \leq V_{c,n,r}(P) - V_{c,\infty,r}(P) \leq V_{n-\ell,r}(P) - V_{\infty,r}(P). \tag{8}$$

Again, by the previous inequalities and application of the squeeze theorem, we obtain that the lower and upper constrained quantization coefficients are the same in both constrained and conditional constrained cases. Choose $n \in \mathbb{N}$ large enough such that $V_{n-\ell,r}(P) - V_{\infty,r}(P) < 1$. Then, by Equation (8), we get

$$\frac{r \log n}{-\log(V_{n,r}(P) - V_{\infty,r}(P))} \leq \frac{r \log n}{-\log(V_{c,n,r}(P) - V_{c,\infty,r}(P))} \leq \frac{r \log n}{-\log(V_{n-\ell,r}(P) - V_{\infty,r}(P))}.$$

Thus, by the squeeze theorem, we see that the lower and upper constrained quantization dimensions are the same in both constrained and conditional constrained cases. This completes the proof of the theorem. □

## References


[DFG] Q. Du, V. Faber and M. Gunzburger, *Centroidal Voronoi Tessellations: Applications and Algorithms*, SIAM Review, Vol. 41, No. 4 (1999), pp. 637-676.

[DR] C.P. Dettmann and M.K. Roychowdhury, *Quantization for uniform distributions on equilateral triangles*, Real Analysis Exchange, Vol. 42(1), 2017, pp. 149-166.

[GG] A. Gersho and R.M. Gray, *Vector quantization and signal compression*, Kluwer Academy publishers: Boston, 1992.

[GL] S. Graf and H. Luschgy, *Foundations of quantization for probability distributions*, Lecture Notes in Mathematics 1730, Springer, Berlin, 2000.

[GL1] A. György and T. Linder, *On the structure of optimal entropy-constrained scalar quantizers*, IEEE transactions on information theory, vol. 48, no. 2, February 2002.

[GL2] S. Graf and H. Luschgy, *The Quantization of the Cantor Distribution*, Math. Nachr., 183 (1997), 113-133.

[GL3] S. Graf and H. Luschgy, *Quantization for probability measures with respect to the geometric mean error*, Math. Proc. Camb. Phil. Soc. (2004), 136, 687-717.

[GN] R.M. Gray and D.L. Neuhoff, *Quantization*, IEEE Transactions on Information Theory, Vol. 44, No. 6, October 1998, 2325-2383.

[KNZ] M. Kesseböhmer, A. Niemann and S. Zhu, *Quantization dimensions of compactly supported probability measures via Rényi dimensions*, Trans. Amer. Math. Soc. (2023).

[P] D. Pollard, *Quantization and the Method of k-Means*, IEEE Transactions on Information Theory, 28 (1982), 199-205.

[P1] K. Pötzelberger, *The quantization dimension of distributions*, Math. Proc. Cambridge Philos. Soc., 131 (2001), 507-519.

[PR1] M. Pandey and M.K. Roychowdhury, *Constrained quantization for probability distributions*, arXiv:2305.11110 [math.PR].

[PR2] M. Pandey and M.K. Roychowdhury, *Constrained quantization for the Cantor distribution*, arXiv:2306.16653 [math.DS].

[PR3] M. Pandey and M.K. Roychowdhury, *Constrained quantization for a uniform distribution with respect to a family of constraints*, arXiv:2306.16653 [math.PR].

[PRRSS] G. Pena, H. Rodrigo, M.K. Roychowdhury, J. Sifuentes, and E. Suazo, *Quantization for uniform distributions on hexagonal, semicircular, and elliptical curves*, Journal of Optimization Theory and Applications, (2021) 188: 113-142.

[RR] J. Rosenblatt and M.K. Roychowdhury, *Uniform distributions on curves and quantization*, Commun. Korean Math. Soc. 38 (2023), No. 2, pp. 431-450.

[R1] M.K. Roychowdhury, *Quantization and centroidal Voronoi tessellations for probability measures on dyadic Cantor sets*, Journal of Fractal Geometry, 4 (2017), 127-146.

[R2] M.K. Roychowdhury, *Least upper bound of the exact formula for optimal quantization of some uniform Cantor distributions*, Discrete and Continuous Dynamical Systems- Series A, Volume 38, Number 9, September 2018, pp. 4555-4570.

[R3] M.K. Roychowdhury, *Optimal quantization for the Cantor distribution generated by infinite similitudes*, Israel Journal of Mathematics 231 (2019), 437-466.

[Z1] P.L. Zador, *Asymptotic Quantization Error of Continuous Signals and the Quantization Dimension*, IEEE Transactions on Information Theory, 28 (1982), 139-149.

[Z2] R. Zam, *Lattice Coding for Signals and Networks: A Structured Coding Approach to Quantization, Modulation, and Multiuser Information Theory*, Cambridge University Press, 2014.



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