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CONDITIONAL OPTIMAL SETS AND THE QUANTIZATION COEFFICIENTS FOR SOME UNIFORM DISTRIBUTIONS

¹EVANS NYANNEY, ²MEGHA PANDEY, AND ³MRINAL KANTI ROYCHOWDHURY

ABSTRACT. Bucklew and Wise (1982) showed that the quantization dimension of an absolutely continuous probability measure on a given Euclidean space is constant and equals the Euclidean dimension of the space, and the quantization coefficient exists as a finite positive number. By giving different examples, in this paper, we have shown that the quantization coefficients for absolutely continuous probability measures defined on the same Euclidean space can be different. We have taken uniform distribution as a prototype of an absolutely continuous probability measure. In addition, we have also calculated the conditional optimal sets of n -points and the n th conditional quantization errors for the uniform distributions in constrained and unconstrained scenarios.

1. INTRODUCTION

The process of transformation of a continuous-valued signal into a discrete-valued one is called ‘quantization’. It has broad applications in engineering and technology. We refer to [GG, GN, Z2] for surveys on the subject and comprehensive lists of references to the literature; see also [AW, GKL, GL1, Z1].

Definition 1.1. Let P be a Borel probability measure on \mathbb{R}^k equipped with a metric d induced by a norm $\|\cdot\|$ on \mathbb{R}^k . Let S be a nonempty closed subset of \mathbb{R}^k . Let $\beta \subset \mathbb{R}^k$ be given with $\text{card}(\beta) = \ell$ for some $\ell \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ with $n \geq \ell$, the n th conditional constrained quantization error for P with respect to the constraint S and the conditional set β , is defined by

$$V_n := V_n(P) = \inf_{\alpha} \left\{ \int \min_{a \in \alpha \cup \beta} d(x, a)^2 dP(x) : \alpha \subseteq S, 0 \leq \text{card}(\alpha) \leq n - \ell \right\}, \quad (1)$$

where $\text{card}(A)$ represents the cardinality of the set A .

Definition 1.2. A set $\alpha \cup \beta$, where $\alpha \subseteq S$ and $P(M(b|\alpha \cup \beta)) > 0$ for $b \in \beta$, for which the infimum in (1) exists and contains no less than ℓ elements, and no more than n elements is called a conditional constrained optimal set of n -points for P with respect to the constraint S and the conditional set β .

We assume that $\int d(x, 0)^2 dP(x) < \infty$ to make sure that the infimum in (1) exists (see [PR1]). For a finite set $\gamma \subset \mathbb{R}^2$ and $a \in \gamma$, by $M(a|\gamma)$ we denote the set of all elements in \mathbb{R}^2 which are nearest to a among all the elements in γ , i.e., $M(a|\gamma) = \{x \in \mathbb{R}^2 : d(x, a) = \min_{b \in \gamma} d(x, b)\}$. $M(a|\gamma)$ is called the Voronoi region in \mathbb{R}^2 generated by $a \in \gamma$.

Write $V_{\infty}(P) := \lim_{n \rightarrow \infty} V_n(P)$. The numbers

$$\underline{D}(P) := \liminf_{n \rightarrow \infty} \frac{2 \log n}{-\log(V_n(P) - V_{\infty}(P))} \text{ and } \overline{D}(P) := \limsup_{n \rightarrow \infty} \frac{2 \log n}{-\log(V_n(P) - V_{\infty}(P))}, \quad (2)$$

are called the *conditional lower* and the *conditional upper constrained quantization dimensions* of the probability measure P , respectively. If $\underline{D}(P) = \overline{D}(P)$, the common value is called the *conditional constrained quantization dimension* of P and is denoted by $D(P)$. For any $\kappa > 0$, the two numbers $\liminf_n n^{\frac{2}{\kappa}}(V_n(P) - V_{\infty}(P))$ and $\limsup_n n^{\frac{2}{\kappa}}(V_n(P) - V_{\infty}(P))$ are, respectively, called the κ -dimensional *conditional lower* and *conditional upper constrained quantization coefficients* for P . If both of them are

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equal, then it is called the κ -dimensional conditional constrained quantization coefficient for P , and is denoted by $\lim_n n^{\frac{2}{\kappa}}(V_n(P) - V_\infty(P))$.

If there is no conditional set, then by the n th constrained quantization error for P with respect to the constraint $S \subseteq \mathbb{R}^k$, it is meant that

$$V_n := V_n(P) = \inf \left\{ \int \min_{a \in \alpha} d(x, a)^2 dP(x) : \alpha \subseteq S \text{ and } 1 \leq \text{card}(\alpha) \leq n \right\}, \quad (3)$$

and then the numbers $D(P)$ and $\lim_n n^{\frac{2}{\kappa}}(V_n(P) - V_\infty(P))$, if they exist, are called the *constrained quantization dimension* and the κ -dimensional conditional constrained quantization coefficient for P , respectively. A set $\alpha \subseteq S$ for which the infimum in (3) exists is called a *constrained optimal set of n -points* for P .

If there is no constraint, i.e., if $S = \mathbb{R}^k$, then by the n th conditional unconstrained quantization error with respect to the conditional set β , it is meant that

$$V_n := V_n(P) = \inf_{\alpha} \left\{ \int \min_{a \in \alpha \cup \beta} d(x, a)^2 dP(x) : \alpha \subseteq \mathbb{R}^k, 0 \leq \text{card}(\alpha) \leq n - \ell \right\}, \quad (4)$$

and then the numbers $D(P)$ and $\lim_n n^{\frac{2}{\kappa}}(V_n(P) - V_\infty(P))$, if they exist, are called the *conditional unconstrained quantization dimension* and the κ -dimensional conditional unconstrained quantization coefficient for P , respectively. A set $\alpha \subseteq S$ for which the infimum in (3) exists is called a *conditional unconstrained optimal set of n -points* for P .

If there is no constraint and no conditional set, then by the n th unconditional quantization error it is meant that

$$V_n := V_n(P) = \inf \left\{ \int \min_{a \in \alpha} d(x, a)^2 dP(x) : \alpha \subseteq \mathbb{R}^k, 1 \leq \text{card}(\alpha) \leq n \right\}, \quad (5)$$

and then the numbers $D(P)$ and $\lim_n n^{\frac{2}{\kappa}}(V_n(P) - V_\infty(P))$, if they exist, are called the *unconstrained quantization dimension* and the κ -dimensional unconstrained quantization coefficient for P , respectively. A set $\alpha \subseteq S$ for which the infimum in (3) exists is called an *optimal set of n -means* for P . It is known that if the support of P contains infinitely many elements then an optimal set of n -means contains exactly n elements, and $V_\infty = \lim_{n \rightarrow \infty} V_n = 0$.

Constrained quantization and conditional quantization have recently been introduced by Pandey and Roychowdhury (see [PR1, PR2, PR4]). After the introduction of constrained quantization, the quantization theory has now two classifications: constrained quantization and unconstrained quantization. Unconstrained quantization is traditionally known as quantization. Thus, the n th unconditional quantization error, given by (5), will traditionally be referred to as n th quantization error. Likewise, *unconstrained quantization dimension* and the κ -dimensional unconstrained quantization coefficient for P will be referred to as *quantization dimension* and the κ -dimensional quantization coefficient for P , respectively. For some other papers in the direction of constrained quantization and conditional quantization one can see [HNPR, BCDR, BCDRV, OR, PR3, PR5]. For unconstrained quantization, one can see [DFG, DR, GG, GL, GL1, GL2, GL3, GN, KNZ, P, P1, R1, R2, R3, Z1, Z2] and the references therein.

Bucklew and Wise (1982) showed that the quantization dimension of an absolutely continuous probability measure on a given Euclidean space is constant and equals the Euclidean dimension of the space where the support of the probability measure is defined, and the quantization coefficient exists as a finite positive number. By giving different examples, in this paper, we have shown that the quantization coefficients for absolutely continuous probability measures defined in the same Euclidean space can be different. We have taken uniform distribution as a prototype of an absolutely continuous probability measure. To determine the quantization coefficient for a Borel probability measure P , we sometimes have calculated the conditional quantization coefficient for P as the quantization coefficient does not depend on the conditional set. In addition, we have also calculated the conditional optimal sets of n -points and the n th conditional quantization errors with respect to some given conditional sets in constrained and unconstrained scenarios.

1.3. Delineation. In Section 2 we give the basic preliminaries. In Section 3, for a uniform distribution on a line segment taking different conditional sets we have calculated the conditional optimal sets of n -points and the n th conditional quantization errors. Then, for each conditional set we have calculated the quantization coefficient, and see that the quantization coefficient does not depend on the conditional set, but depends on the length of the line segment. In Section 4, we have calculated the conditional optimal sets of n -points, the n th conditional quantization errors, the conditional quantization dimension, and the conditional quantization coefficient, in constrained scenario, for a uniform distribution defined on a circle of radius r with respect to a given conditional set and a constraint. In addition, for the same probability distribution we have investigated the optimal sets of n -means and the n th quantization errors, and the quantization coefficient in unconstrained scenario. From the work in this section, we see that the quantization coefficient for a uniform distribution defined on a circle depends on the radius of the circle. In Section 5, we have calculated the conditional optimal sets of n -points, the n th conditional quantization errors, and the conditional quantization coefficient for a uniform distribution defined on the boundary of a regular polygon which is inscribed in a circle of radius r with respect to a given conditional set. From the work in this section, we see that the quantization coefficient for a uniform distribution defined on the boundary of a regular m -sided polygon depends on both the number of sides of the polygon and the length of the sides.

2. PRELIMINARIES

For any two elements (a, b) and (c, d) in \mathbb{R}^2 , we write

$$\rho((a, b), (c, d)) := (a - c)^2 + (b - d)^2,$$

which gives the squared Euclidean distance between the two elements (a, b) and (c, d) . Let p and q be two elements that belong to an optimal set of n -points for some positive integer n , and let e be an element on the boundary of the Voronoi regions of the elements p and q . Since the boundary of the Voronoi regions of any two elements is the perpendicular bisector of the line segment joining the two elements, we have

$$\rho(p, e) - \rho(q, e) = 0.$$

We call such an equation a *canonical equation*.

Let P be a Borel probability measure on \mathbb{R} which is uniform on its support the closed interval $[a, b]$. Then, the probability density function f for P is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Hence, we have $dP(x) = P(dx) = f(x)dx$ for any $x \in \mathbb{R}$.

Let us now state the following proposition. For the details of the proof see [BCDR].

Proposition 2.1. (see [BCDR]) *Let P be a uniform distribution on the closed interval $[a, b]$ and $c, d \in [a, b]$ be such that $a < c < d < b$. For $n \in \mathbb{N}$ with $n \geq 2$, let α_n be a conditional unconstrained optimal set of n -points for P with respect to the conditional set $\beta = \{c, d\}$ such that α_n contains k elements from the closed interval $[a, c]$, ℓ elements from the closed interval $[c, d]$, and m elements from the closed interval $[d, b]$ for some $k, \ell, m \in \mathbb{N}$ with $k, m \geq 1$ and $\ell \geq 2$. Then, $k + \ell + m = n + 2$,*

$$\begin{aligned} \alpha_n \cap [a, c] &= \left\{ a + \frac{(2j-1)(c-a)}{2k-1} : 1 \leq j \leq k \right\}, \\ \alpha_n \cap [c, d] &= \left\{ c + \frac{j-1}{\ell-1}(d-c) : 1 \leq j \leq \ell \right\}, \text{ and} \\ \alpha_n \cap [d, b] &= \left\{ d + \frac{2(j-1)(b-d)}{2m-1} : 1 \leq j \leq m \right\} \end{aligned}$$

with the conditional unconstrained quantization error

$$V_n := V_{k,\ell,m}(P) = \frac{1}{3(b-a)} \left(\frac{(c-a)^3}{(2k-1)^2} + \frac{1}{4} \frac{(d-c)^3}{(\ell-1)^2} + \frac{(b-d)^3}{(2m-1)^2} \right).$$

Remark 2.2. If nothing is specified, by conditional optimal sets of n -points and the conditional quantization errors, it is meant the conditional unconstrained optimal sets of n -points and the conditional unconstrained quantization errors, respectively.

The following theorem is known.

Theorem 2.3. (see [PR4]) *In both constrained and unconstrained quantization, the lower and upper quantization dimensions and the lower and upper quantization coefficients for a Borel probability measure do not depend on the conditional set.*

Remark 2.4. Given that the underlying spaces for all considered probability measures P in this work are one dimensional, their quantization dimensions are given by $D(P) = 1$ (see [BW]). Hence, in the sequel we are mostly interested to calculate the quantization coefficients for different uniform distributions, though in some cases we have also calculated the conditional optimal sets of n -points and the n th conditional quantization errors in constrained and unconstrained scenarios.

In the following sections we give the main results of the paper.

3. CONDITIONAL OPTIMAL SETS OF n -POINTS AND THE QUANTIZATION COEFFICIENTS FOR UNIFORM DISTRIBUTIONS ON LINE SEGMENTS

Without any loss of generality we can assume the line segment as a closed interval $[a, b]$, where $0 < a < b < +\infty$. Let P be the uniform distribution defined on the closed interval $[a, b]$. Then, the probability density function f for P is given by (6). Let us now give the following theorem.

Theorem 3.1. *Let P be the uniform distribution on the line segment joining a and b , where $a, b \in \mathbb{R}$ with $a < b$. Then we have:*

(i) *the conditional optimal set of n -points with respect to the conditional set $\beta := \{a, b\}$ is*

$$\left\{ a + \frac{j-1}{\ell-1}(b-a) : 1 \leq j \leq n \right\} \text{ with conditional quantization error } V_n = \frac{(b-a)^2}{12(n-1)^2};$$

(ii) *the conditional optimal set of n -points with respect to the conditional set $\beta := \{a\}$ is*

$$\left\{ a + \frac{2(j-1)(b-a)}{2n-1} : 1 \leq j \leq n \right\} \text{ with conditional quantization error } V_n = \frac{(b-a)^2}{3(2n-1)^2};$$

(iii) *the conditional optimal set of n -points with respect to the conditional set $\beta := \{b\}$ is*

$$\left\{ a + \frac{(2j-1)(b-a)}{2n-1} : 1 \leq j \leq n \right\} \text{ with conditional quantization error } V_n = \frac{(b-a)^2}{3(2n-1)^2}.$$

Proof. The proof of the theorem follows by Proposition 2.1 upon substitutions $c = a$ and $d = b$, $c = d = a$, and $c = d = b$, respectively. \square

Theorem 3.2. *Let P be the uniform distribution on the line segment joining a and b , where $a, b \in \mathbb{R}$ with $a < b$. Then, the conditional quantization coefficients for P with respect to the conditional sets $\{a, b\}$, $\{a\}$, and $\{b\}$ exist as finite positive numbers and each equals $\frac{(a-b)^2}{12}$.*

Proof. By Theorem 3.1(i), we obtain the n th conditional quantization error for the uniform distribution P with respect to the conditional set $\beta := \{a, b\}$ as

$$V_n = \frac{(b-a)^2}{12(n-1)^2}.$$

Then,

$$V_\infty = \lim_{n \rightarrow \infty} V_n = 0 \text{ yielding } \lim_{n \rightarrow \infty} n^2(V_n - V_\infty) = \frac{(a-b)^2}{12}.$$

Similarly, if the conditional set is $\beta := \{a\}$ or $\beta := \{b\}$, by (ii) and (iii) in Theorem 3.1, we obtain $\lim_{n \rightarrow \infty} n^2(V_n - V_\infty) = \frac{(a-b)^2}{12}$. Thus, the proof of the theorem is complete. \square

Remark 3.3. By Theorem 2.3 and Theorem 3.2, we see that the quantization coefficient for the uniform distribution P on a line segment depends on the length of the line segment, and does not depend on the conditional sets.

4. OPTIMAL SETS OF n -POINTS AND THE QUANTIZATION COEFFICIENTS FOR THE UNIFORM DISTRIBUTIONS ON THE CIRCLES

In this section, we have two subsections. In the first subsection, we calculate the conditional constrained optimal sets of n -points and the n th conditional constrained quantization errors, the conditional constrained quantization dimension, and the conditional constrained quantization coefficient for a uniform distribution P defined on a circle of radius r with respect to a given conditional set and a constraint. In the second subsection, for the same probability distribution, we investigate the optimal sets of n -means and the n th quantization errors, and the quantization coefficient in unconstrained scenario.

Let L be the circle of radius r . Without any loss of generality, we can take the equation of the circle as $x_1^2 + x_2^2 = r^2$, i.e., the parametric equations of the circle is given by $L := \{(x_1, x_2) : x_1 = r \cos \theta, x_2 = r \sin \theta \text{ for } 0 \leq \theta \leq 2\pi\}$. Notice that any point on the circle can be given by $(r \cos \theta, r \sin \theta)$, which will be identified as θ , where $0 \leq \theta \leq 2\pi$. Let the positive direction of the x_1 -axis cut the circle at the point A , i.e., A is represented by the parametric value $\theta = 0$. Let s be the distance of a point on L along the arc starting from the point A in the counterclockwise direction. Then,

$$ds = \sqrt{\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dx_2}{d\theta}\right)^2} d\theta = r d\theta.$$

Then, the probability density function (pdf) $f(x_1, x_2)$ for P is given by

$$f(x_1, x_2) = \begin{cases} \frac{1}{2\pi r} & \text{if } (x_1, x_2) \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have $dP(s) = P(ds) = f(x_1, x_2)ds = \frac{1}{2\pi}d\theta$. Moreover, we know that if $\hat{\theta}$ radians is the central angle subtended by an arc of length S of the circle, then $S = r\hat{\theta}$, and

$$P(S) = \int_S dP(s) = \frac{1}{2\pi} \int_S d\theta = \frac{\hat{\theta}}{2\pi}.$$

4.1. Conditional quantization in constrained scenario. In this subsection to investigate the conditional quantization in constrained scenario for the uniform distribution P on the circle L , we take the circle L as the constraint and the set $\{(r, 0)\}$ as the conditional set. Let us define a function

$$T : L \rightarrow [0, 2\pi r] \text{ such that } T(\theta) := T((r \cos \theta, r \sin \theta)) = r\theta,$$

where $0 \leq \theta \leq 2\pi$. Then, notice that $T : L \setminus \{(r, 0)\} \rightarrow (0, 2\pi r)$ is a bijective function. Let Q be the image measure of P under the function T , i.e., $Q = TP$ such that for any Borel subset $A \subseteq [0, 2\pi r]$, we have

$$Q(A) = TP(A) = P(T^{-1}(A)).$$

Lemma 4.1.1. *The image measure Q is a uniform distribution on $[0, 2\pi r]$.*

Proof. Since P is a uniform distribution on L , we can assume that P is also a uniform distribution on $L \setminus \{(r, 0)\}$, as the deletion, or addition, of a finite number of points from, or with, the support of a continuous probability measure does not change the distribution. Take any $[c, d] \subseteq (0, 2\pi r)$, where $0 < c < d < 2\pi r$. Since T is a bijection, there exist θ_1, θ_2 , where $0 < \theta_1 < \theta_2 < 2\pi$, such that $T(\theta_1) = r\theta_1 = c$ and $T(\theta_2) = r\theta_2 = d$. Then,

$$Q([c, d]) = P(T^{-1}([c, d])) = P(\{(r \cos \theta, r \sin \theta) : \theta_1 \leq \theta \leq \theta_2\}) = \frac{\theta_2 - \theta_1}{2\pi} = \frac{r\theta_2 - r\theta_1}{2\pi r} = \frac{d - c}{2\pi r}.$$

Notice $Q([c, d]) = \lambda([c, d])$, where λ is the normalized Lebesgue measure on $(0, 2\pi r)$. Hence, we can conclude that Q is a uniform distribution on $(0, 2\pi r)$, i.e., Q is a uniform distribution on $[0, 2\pi r]$. Thus, the lemma is obtained. \square

Notation 4.1.2. For any two elements $c, d \in [0, 2\pi r]$, by $\ell(\{c, d\})$ it is meant the distance between the two elements c, d , i.e., $\ell(\{c, d\}) = d - c$. Similarly, for any two elements $\theta_1, \theta_2 \in L$ with $\theta_1 < \theta_2$, by $\ell(\{\theta_1, \theta_2\})$ it is meant the arc distance between the two elements θ_1 and θ_2 , i.e., the length of the arc on L subtended by the angle $\theta_2 - \theta_1$, i.e., $\ell(\{\theta_1, \theta_2\}) = r(\theta_2 - \theta_1)$.

Lemma 4.1.3. *The function $T : L \setminus \{(r, 0)\} \rightarrow (0, 2\pi r)$ preserves the distance.*

Proof. Take any $c, d \in (0, 2\pi r)$ such that $0 < c < d < 2\pi r$. The lemma will be proved if we can prove that $\ell(\{c, d\}) = \ell(T^{-1}(\{c, d\}))$. Since $T : L \setminus \{(r, 0)\} \rightarrow (0, 2\pi r)$ is a bijection, there exist $\theta_1, \theta_2 \in L \setminus \{(r, 0)\}$ such that $T(\theta_1) = r\theta_1 = c$ and $T(\theta_2) = r\theta_2 = d$. Then,

$$\ell(T^{-1}(\{c, d\})) = \ell(\{\theta_1, \theta_2\}) = r(\theta_2 - \theta_1) = d - c = \ell(\{c, d\}).$$

Thus, the lemma is yielded. \square

Lemma 4.1.4. *The conditional unconstrained optimal set for the uniform distribution Q with respect to the conditional set $\{0, 2\pi r\}$ is given by*

$$\left\{ \frac{(j-1)2\pi r}{n-1} : 1 \leq j \leq n \right\} \text{ with conditional unconstrained quantization error } V_n(Q) = \frac{\pi^2 r^2}{3(n-1)^2}.$$

Proof. The lemma is followed by Proposition 2.1 upon substitution $a = c = 0$ and $b = d = 2\pi r$. \square

Remark 4.1.5. Let $\{a_1, a_2, \dots, a_n\}$ be a conditional unconstrained optimal set of n -points for Q with respect to the conditional set $\{0, 2\pi r\}$, where $a_1 = 0$ and $a_n = 2\pi r$. Since both P and Q are uniform distributions, and Q is the image measure of P under the function T , and $T : L \setminus \{(r, 0)\} \rightarrow (0, 2\pi r)$ preserves the distance, we can say that the set $T^{-1}(\{a_1, a_2, \dots, a_n\})$, i.e., the set $\{T^{-1}(a_j) : 1 \leq j \leq n-1\}$ forms a conditional constrained optimal set of $(n-1)$ -points for P with respect to the conditional set $\{(r, 0)\}$ and the constraint L , as $T^{-1}(0) = T^{-1}(2\pi r) = (r, 0)$.

Let us now prove the following theorems, which give the main results in this subsection.

Theorem 4.1.6. *Let P be the uniform distribution on the circle of radius r with center $(0, 0)$. Then, the set $\{(r \cos \frac{(j-1)2\pi}{n}, r \sin \frac{(j-1)2\pi}{n}) : 1 \leq j \leq n\}$ forms a conditional constrained optimal set of n -points with respect to the conditional set $\{(r, 0)\}$ and the constraint L with conditional constrained quantization error*

$$V_n = 2r^2 \left(1 - \frac{n}{\pi} \sin \frac{\pi}{n}\right).$$

Proof. By Lemma 4.1.4, we know that the set $\{\frac{(j-1)2\pi r}{n-1} : 1 \leq j \leq n\}$ forms a conditional unconstrained optimal set of n -points for Q with respect to the conditional set $\{0, 2\pi r\}$, where $n \geq 2$. Hence, by Remark 5.3, the set $\{T^{-1}(\frac{(j-1)2\pi r}{n-1}) : 1 \leq j \leq n-1\}$ forms a conditional constrained optimal set of $(n-1)$ -points for P with respect to the conditional set $\{(0, \pi)\}$ and the constraint L , where $(n-1) \geq 1$. Now, notice that

$$\begin{aligned} \{T^{-1}(\frac{(j-1)2\pi r}{n-1}) : 1 \leq j \leq n-1\} &= \{\frac{(j-1)2\pi}{n-1} : 1 \leq j \leq n-1\} \\ &= \{(\cos \frac{(j-1)2\pi}{n-1}, \sin \frac{(j-1)2\pi}{n-1}) : 1 \leq j \leq n-1\}. \end{aligned}$$

Hence, replacing n by $n+1$, we deduce that the set $\{(r \cos \frac{(j-1)2\pi}{n}, r \sin \frac{(j-1)2\pi}{n}) : 1 \leq j \leq n\}$ forms a conditional constrained optimal set of n -points with respect to the conditional set $\{(0, \pi)\}$ and the constraint L . Due to rotational symmetry, we obtain the conditional constrained quantization error as

$$\begin{aligned} V_n &= n(\text{distortion error contributed by the element } (r, 0)) \\ &= \frac{n}{2\pi} \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \rho((r \cos \theta, r \sin \theta), (r, 0)) d\theta \\ &= 2r^2 \left(1 - \frac{n}{\pi} \sin \frac{\pi}{n}\right). \end{aligned}$$

Thus, the proof of the theorem is complete. \square

Theorem 4.1.7. *Let P be the uniform distribution on the circle of radius r with center $(0, 0)$. Then, with respect to the conditional set $\{(r, 0)\}$ and the constraint L , the conditional constrained quantization dimension $D(P)$ exists and equals one, and the conditional constrained quantization coefficient for P exists as a finite positive number and equals $\frac{\pi^2 r^2}{3}$, i.e., $\lim_{n \rightarrow \infty} n^2(V_n - V_\infty) = \frac{\pi^2 r^2}{3}$.*

Proof. By Theorem 4.1.6, we obtain the n th conditional constrained quantization error for the uniform distribution P with respect to the conditional set $\{(r, 0)\}$ and the constraint L as $V_n = 2r^2(1 - \frac{n}{\pi} \sin \frac{\pi}{n})$. Then, $V_\infty = \lim_{n \rightarrow \infty} V_n = 0$. Hence,

$$D(P) = \lim_{n \rightarrow \infty} \frac{2 \log n}{-\log(V_n - V_\infty)} = 1 \text{ and } \lim_{n \rightarrow \infty} n^2(V_n - V_\infty) = \frac{\pi^2 r^2}{3}.$$

Thus, the proof of the theorem is complete. \square

Remark 4.1.8. By Theorem 2.3, we know that quantization dimension and the quantization coefficient in constrained and unconstrained cases do not depend on the conditional set. Thus, by Theorem 4.1.7, we see that constrained quantization dimension of the uniform distribution P with respect to the constraint L equals one, which is the dimension of the underlying space where the support of the probability measure is defined, and does not depend on the radius r of the circle. This fact is not true, in general, in constrained quantization, for example, one can see [PR1, PR3]. However, we see that the constrained quantization coefficient for the uniform distribution P with respect to the constraint L depends on the radius r of the circle.

4.2. Quantization in unconstrained scenario. In this subsection, we investigate the optimal sets of n -means, n th quantization errors, and the quantization coefficient for the uniform distribution P when there is no constraint and no conditional set. The following lemma is a generalized version of a similar theorem that appears in [RR].

Theorem 4.2.1. *Let α_n be an optimal set of n -means for the uniform distribution P on the circle $x_1^2 + x_2^2 = r^2$ for $n \in \mathbb{N}$. Then,*

$$\alpha_n := \left\{ \frac{nr}{\pi} \left(\sin\left(\frac{\pi}{n}\right) \cdot \cos\left((2j-1)\frac{\pi}{n}\right), \sin\left(\frac{\pi}{n}\right) \cdot \sin\left((2j-1)\frac{\pi}{n}\right) \right) : j = 1, 2, \dots, n \right\}$$

forms an optimal set of n -means, and the corresponding quantization error is given by $r^2(1 - \frac{n^2}{\pi^2} \sin^2 \frac{\pi}{n})$.

Proof. Let $\alpha_n := \{a_1, a_2, \dots, a_n\}$ be an optimal set of n -means for P . Let the boundaries of the Voronoi regions of a_k intersect L at the points given by the parameters θ_{k-1} and θ_k such that $\theta_{k-1} < \theta_k$ for $1 \leq k \leq n$. Without any loss of generality, we can assume that $\theta_0 = 0$ and $\theta_n = 2\pi$. Since, P is a uniform distribution and the circle L is rotationally symmetric, without going into the much details of calculations, we see that

$$\theta_1 - \theta_0 = \theta_2 - \theta_1 = \theta_3 - \theta_2 = \dots = \theta_n - \theta_{n-1} = \frac{\theta_n - \theta_0}{n} = \frac{2\pi}{n}.$$

implying $\theta_k = \frac{2\pi k}{n}$ for $0 \leq k \leq n$. It is well-known that in unconstrained quantization, the elements in an optimal set are the conditional expectations in their own Voronoi regions. Hence, for $1 \leq k \leq n$, we have

$$a_k = \frac{2\pi}{\theta_k - \theta_{k-1}} \int_{\theta_{k-1}}^{\theta_k} \frac{1}{2\pi} (r \cos \theta, r \sin \theta) d\theta = \frac{r}{\theta_k - \theta_{k-1}} (\sin \theta_k - \sin \theta_{k-1}, \cos \theta_{k-1} - \cos \theta_k),$$

yielding

$$a_k = \frac{nr}{\pi} \left(\sin\left(\frac{\pi}{n}\right) \cdot \cos\left((2k-1)\frac{\pi}{n}\right), \sin\left(\frac{\pi}{n}\right) \cdot \sin\left((2k-1)\frac{\pi}{n}\right) \right).$$

Let V_n be the n th quantization error. Due to symmetry, the distortion errors contributed by a_k in their own Voronoi regions are equal for all $1 \leq k \leq n$. Again, notice that $\theta_0 = 0$, $\theta_1 = \frac{2\pi}{n} - \theta_0 = \frac{2\pi}{n}$, and $a_1 = \frac{r}{\theta_1}(\sin \theta_1, 1 - \cos \theta_1)$. Hence,

$$V_n = \frac{n}{2\pi} \int_{\theta_0}^{\theta_1} \rho((r \cos \theta, r \sin \theta), a_1) d\theta = r^2 \left(1 - \frac{n^2}{\pi^2} \sin^2 \frac{\pi}{n}\right).$$

Thus, the proof of the theorem is complete. \square

Theorem 4.2.2. *Let P be the uniform distribution on the circle of radius r with center $(0, 0)$. Then, the quantization coefficient for P exists as a finite positive number and equals $\frac{\pi^2 r^2}{3}$.*

Proof. By Theorem 4.2.1, we have $V_n = r^2 \left(1 - \frac{n^2}{\pi^2} \sin^2 \frac{\pi}{n}\right)$, and hence $\lim_{n \rightarrow \infty} n^2 V_n = \frac{\pi^2 r^2}{3}$, which is the theorem. \square

Remark 4.2.3. Theorem 4.2.2 implies that the quantization coefficient for a uniform distribution on a circle of radius r , though exists as a finite positive number, depends on the radius r of the circle.

5. CONDITIONAL OPTIMAL SETS AND THE QUANTIZATION COEFFICIENTS FOR THE UNIFORM DISTRIBUTIONS ON THE BOUNDARIES OF THE REGULAR POLYGONS

In this section, for a uniform distribution defined on the boundary of a regular m -sided polygon, we calculate the conditional optimal sets of n -points and the n th conditional quantization errors, and the conditional quantization coefficient taking the conditional set as the set of all vertices of the polygon.

Let P be the uniform distribution defined on the boundary L of a regular m -sided polygon given by $A_1 A_2 \cdots A_m$ for some $m \geq 3$. Without any loss of generality we can assume that the polygon is inscribed in the circle $x^2 + y^2 = r^2$ which has center $O(0, 0)$ and radius r with the Cartesian coordinates of the vertex A_1 as $(r, 0)$. Let θ be the central angle subtended by each side of the polygon, and let θ_j be the polar angles of the vertices A_j . Then, we have $\theta = \frac{2\pi}{m}$ and $\theta_j = (j-1)\frac{2\pi}{m}$. Then, the polar coordinates of the vertices A_j are given by $(r \cos \theta_j, r \sin \theta_j)$. Hence, if ℓ is the length of each of the sides $A_j A_{j+1}$ for $1 \leq j \leq m$, where the vertex A_{m+1} is identified as the vertex A_1 , then we have

$$\ell = \text{length of } A_j A_{j+1} = \sqrt{\rho((r \cos \theta_j, r \sin \theta_j), (r \cos \theta_{j+1}, r \sin \theta_{j+1}))} = 2r \sin \frac{\pi}{m}.$$

The probability density function (pdf) f for the uniform distribution P is given by $f(x, y) = \frac{1}{m\ell}$ for all $(x, y) \in A_1 A_2 \cdots A_m$, and zero otherwise. Moreover, we can write

$$L = \bigcup_{j=1}^m L_j, \text{ where } L_j \text{ represents the side } A_j A_{j+1} \text{ for } 1 \leq j \leq m.$$

Notice that

$$A_j A_{j+1} = \{(1-t)(r \cos \theta_j, r \sin \theta_j) + t(r \cos \theta_{j+1}, r \sin \theta_{j+1}) : 0 \leq t \leq 1\}.$$

for $1 \leq j \leq m$. Write

$$a_j := -\sec \frac{\pi}{m} \left(j \sin \frac{2\pi(j-1)}{m} - (j-1) \sin \frac{2\pi j}{m} \right), \quad (7)$$

$$b_j := -2 \sin \frac{\pi}{m} \csc \frac{2\pi}{m} \left((j-1) \cos \frac{2\pi j}{m} - j \cos \frac{2\pi(j-1)}{m} \right), \quad (8)$$

$$c_j := 2(j-1)r \sin \frac{\pi}{m}. \quad (9)$$

Let us consider the affine transformation

$$T : L \rightarrow [0, 2mr \sin \frac{\pi}{m}] \text{ such that } T(x, y) = a_j x + b_j y \text{ if } (x, y) \in A_j A_{j+1},$$

where a_j and b_j are given by (7) and (8) for all $1 \leq j \leq m$. Then, notice that $T : L \setminus \{(r, 0)\} \rightarrow (0, 2mr \sin \frac{\pi}{m})$ is a bijective function. Let Q be the image measure of P under the function T , i.e., $Q = TP$ such that for any Borel subset $A \subseteq [0, 2mr \sin \frac{\pi}{m}]$, we have

$$Q(A) = TP(A) = P(T^{-1}(A)).$$

By the distance between any two elements in $[0, 2mr \sin \frac{\pi}{m}]$, it is meant the Euclidean distance between the two elements. On the other hand, by the distance between any two elements on L , it is meant the Euclidean distance between the two elements along the polygonal arc L in the counterclockwise direction. Let T_j be the restriction of the mapping T to the set $A_j A_{j+1}$, i.e., $T_j = T|_{A_j A_{j+1}}$ for $1 \leq j \leq m$. Notice that each T_j is a bijective function.

Lemma 5.1. *The function $T : L \setminus \{(r, 0)\} \rightarrow (0, 2mr \sin \frac{\pi}{m})$ preserves the distance.*

Proof. Notice that

$$[0, 2mr \sin \frac{\pi}{m}] = \bigcup_{j=1}^m [c_j, c_{j+1}] \text{ and } T(A_j A_{j+1}) = T_j(A_j A_{j+1}) = [c_j, c_{j+1}],$$

where c_j are given by (9) for all $1 \leq j \leq m$. Since the length of $A_j A_{j+1}$ equals the length of the closed interval $[c_j, c_{j+1}]$, and T_j is a bijection, we can say that $T : L \setminus \{(r, 0)\} \rightarrow (0, 2mr \sin \frac{\pi}{m})$ preserves the distance. Thus, the lemma is yielded. \square

The following lemma which is similar to Lemma 4.1.1 is also true here.

Lemma 5.2. *The image measure Q is a uniform distribution on $[0, 2mr \sin \frac{\pi}{m}]$.*

Remark 5.3. Let $\{a_1, a_2, \dots, a_n\}$ be a conditional unconstrained optimal set of n -points for Q with respect to the conditional set $\{c_j : 1 \leq j \leq m+1\}$ such that $a_1 < a_2 < \dots < a_n$. Then, by the definition of conditional set, we have $n \geq m+1$. Moreover, notice that $a_1 = c_1 = 0$ and $a_n = c_{m+1} = 2mr \sin \frac{\pi}{m}$. Since both P and Q are uniform distributions, and Q is the image measure of P under the function T , and $T : L \setminus \{(r, 0)\} \rightarrow (0, 2mr \sin \frac{\pi}{m})$ preserves the distance, we can say that the set $T^{-1}(\{a_1, a_2, \dots, a_n\})$, i.e., the set $\{T^{-1}(a_j) : 1 \leq j \leq n-1\}$ forms a conditional unconstrained optimal set of $(n-1)$ -points for P with respect to the conditional set $\{T^{-1}(c_j) : 1 \leq j \leq m\}$, i.e., with respect to the conditional set $\{A_j : 1 \leq j \leq m\}$.

Lemma 5.4. *Let γ_n be a conditional optimal set of n -points for the uniform distribution Q with respect to the conditional set $\{c_j : 1 \leq j \leq m+1\}$ for any $n \geq m+1$. Let $n_j = \text{card}(\gamma_n \cap [c_j, c_{j+1}])$ for $1 \leq j \leq m$. Then, $n_j \geq 2$ and $n_1 + n_2 + \dots + n_m = n + m - 1$, and $|n_i - n_j| = 0$ or 1 for all $1 \leq i \neq j \leq m$.*

Proof. Let γ_n be a conditional optimal set of n -points and n_j be the positive integers as defined in the hypothesis. Notice that each of the sets $\gamma_n \cap [c_j, c_{j+1}]$ always contains the end elements c_j, c_{j+1} for $1 \leq j \leq m$, where $n \geq m+1$. Moreover, except the two elements c_1 and c_{m+1} all the end elements c_j are counted two times. Hence, $n_j \geq 2$ and $n_1 + n_2 + \dots + n_m = n + (m+1-2) = n + m - 1$. Since Q is a uniform distribution and the lengths of the intervals $[c_j, c_{j+1}]$ for $1 \leq j \leq m$ are all equal, the proof of $|n_i - n_j| = 0$ or 1 for all $1 \leq i \neq j \leq m$ is routine. Thus, the lemma is yielded. \square

Let us now give the following proposition.

Proposition 5.5. *Let γ_n be a conditional optimal set of n -points and $V_n(Q)$ be the n th conditional quantization error for the uniform distribution Q with respect to the conditional set $\{c_j : 1 \leq j \leq m+1\}$ for any $n \geq m+1$. Let $n = mk + 1 + q$, where $0 \leq q < m$. Then, if $q = 0$, we have*

$$\gamma_n = \bigcup_{j=1}^m \{c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k} : 1 \leq i \leq k+1\} \text{ with } V_n(Q) = \frac{r^2 \sin^2 \frac{\pi}{m}}{3k^2}.$$

On the other hand, if $0 < q < m$, then there are mC_q possible sets γ_n , one such set is given by

$$\gamma_n = \left(\bigcup_{j=1}^q \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k+1} : 1 \leq i \leq k+2 \right\} \right) \bigcup \left(\bigcup_{j=q+1}^m \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k} : 1 \leq i \leq k+1 \right\} \right)$$

$$\text{with } V_n(Q) = \frac{r^2 q \sin^2 \frac{\pi}{m}}{3m(k+1)^2} + \frac{r^2(m-q) \sin^2 \frac{\pi}{m}}{3mk^2}.$$

Proof. First assume that $q = 0$, then we have $n = mk + 1$, i.e., γ_n contains $k - 1$ elements from each of the intervals $[c_j, c_{j+1}]$ except the boundary elements c_j and c_{j+1} . Hence, if $n_j = \text{card}(\gamma_n \cap [c_j, c_{j+1}])$, we have $n_j = k - 1 + 2 = k + 1$. Hence, by (i) of Theorem 3.1, we have

$$\gamma_n = \bigcup_{j=1}^m \left\{ c_j + \frac{i-1}{k} (c_{j+1} - c_j) : 1 \leq i \leq k+1 \right\} = \bigcup_{j=1}^m \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k} : 1 \leq i \leq k+1 \right\}$$

$$\text{with } V_n = \sum_{j=1}^m \frac{(c_{j+1} - c_j)^2}{12mk^2} = \sum_{j=1}^m \frac{r^2 \sin^2 \frac{\pi}{m}}{3mk^2} = \frac{r^2 \sin^2 \frac{\pi}{m}}{3k^2}.$$

On the other hand, if $0 < q < m$, then due to Lemma 5.4, we can assume that γ_n contains k elements from each of the first q intervals except the boundary elements, and γ_n contains $(k-1)$ elements from each of the remaining $m-q$ intervals except the boundary elements implying $n_1 = n_2 = \dots = n_q = k+2$ and $n_{q+1} = n_{q+2} = \dots = n_m = k+1$. Hence, the expressions for γ_n and the corresponding n th conditional quantization errors are obtained by (i) of Theorem 3.1 as

$$\gamma_n = \left(\bigcup_{j=1}^q \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k+1} : 1 \leq i \leq k+2 \right\} \right) \bigcup \left(\bigcup_{j=q+1}^m \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k} : 1 \leq i \leq k+1 \right\} \right)$$

with

$$V_n = \sum_{j=1}^q \frac{(c_{j+1} - c_j)^2}{12m(k+1)^2} + \sum_{j=q+1}^m \frac{(c_{j+1} - c_j)^2}{12mk^2} = \sum_{j=1}^q \frac{r^2 \sin^2 \frac{\pi}{m}}{3m(k+1)^2} + \sum_{j=q+1}^m \frac{r^2 \sin^2 \frac{\pi}{m}}{3mk^2}$$

yielding

$$V_n = \frac{r^2 q \sin^2 \frac{\pi}{m}}{3m(k+1)^2} + \frac{r^2(m-q) \sin^2 \frac{\pi}{m}}{3mk^2}.$$

Notice that if $n = mk + q$, the optimal set γ_n can be constructed in mC_q ways. Thus, the proof of the proposition is complete. \square

The following two theorems give the main results in this section.

Theorem 5.6. Let α_n be a conditional optimal set of n -points and $V_n(P)$ be the n th conditional quantization error for the uniform distribution P with respect to the conditional set $\{A_j : 1 \leq j \leq m\}$ for any $n \geq m$. Let $n = mk + q$, where $0 \leq q < m$. Then, if $q = 0$, we have

$$\alpha_n = \bigcup_{j=1}^m T_j^{-1} \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k} : 1 \leq i \leq k+1 \right\} \text{ with } V_n(P) = \frac{r^2 \sin^2 \frac{\pi}{m}}{3k^2}.$$

On the other hand, if $0 < q < m$, then there are mC_q possible sets α_n , one such set is given by

$$\alpha_n = \left(\bigcup_{j=1}^q T_j^{-1} \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k+1} : 1 \leq i \leq k+2 \right\} \right) \bigcup \left(\bigcup_{j=q+1}^m T_j^{-1} \left\{ c_j + \frac{2(i-1)r \sin \frac{\pi}{m}}{k} : 1 \leq i \leq k+1 \right\} \right)$$

$$\text{with } V_n(P) = \frac{r^2 q \sin^2 \frac{\pi}{m}}{3m(k+1)^2} + \frac{r^2(m-q) \sin^2 \frac{\pi}{m}}{3mk^2}.$$

Proof. For $n = mk + 1 + q$, let γ_n be a conditional optimal set of n -points for Q as given by Proposition 5.5. By Remark 5.3 and Proposition 5.5, it follows that the set α_n given by the statement of the theorem forms a conditional optimal set of n -points for P for $n = mk + q$. Notice that the $(mk + q)$ th-conditional quantization error with respect to the uniform distribution P remains same as the $(mk + 1 + q)$ th-conditional quantization error with respect to the uniform distribution Q . This happens due to the fact that the bijective functions T_j preserve the distance as well as the collinearity of the elements in each interval $[c_j, c_{j+1}]$. \square

Theorem 5.7. *Let P be the uniform distribution defined on the boundary of a regular m -sided polygon inscribed in a circle of radius r with center $(0, 0)$. Then, the conditional quantization coefficient for P exists as a finite positive number and equals $\frac{1}{3}m^2r^2\sin^2(\frac{\pi}{m})$, i.e., $\lim_{n \rightarrow \infty} n^2(V_n(P) - V_\infty(P)) = \frac{1}{3}m^2r^2\sin^2\frac{\pi}{m}$.*

Proof. Let $n \in \mathbb{N}$ be such that $n \geq m$. Then, there exists a unique positive integer k such that $n = mk + q$ for some $0 \leq q < m$. Then, by Theorem 5.6, we have

$$V_n(P) = \frac{r^2q\sin^2\frac{\pi}{m}}{3m(k+1)^2} + \frac{r^2(m-q)\sin^2\frac{\pi}{m}}{3mk^2} = \frac{m^2r^2\sin^2\left(\frac{\pi}{m}\right)(m^2 + 2mn - 3mq + n^2 - 4nq + 3q^2)}{3(n-q)^2(m+n-q)^2}.$$

Then, we see that $V_\infty(P) = \lim_{n \rightarrow \infty} V_n(P) = 0$. In fact, we have

$$\lim_{n \rightarrow \infty} n^2(V_n(P) - V_\infty(P)) = \frac{1}{3}m^2r^2\sin^2\frac{\pi}{m}.$$

Thus, the proof of the theorem is complete. \square

Remark 5.8. By Theorem 2.3 and Theorem 5.7, we can say that the quantization coefficient for the uniform distribution P defined on the boundary of a regular m -sided polygon is $\frac{1}{3}m^2r^2\sin^2\frac{\pi}{m}$, which is a finite positive number, but it is not a constant as it depends on both m and r , where m is the number of sides of the polygon and r is the radius of the circle in which the polygon is inscribed, and this lead us to conclude that the quantization coefficient for a uniform distribution defined on the boundary of a regular m -sided polygon depends on both the number of sides of the polygon and the length of the sides.

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