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CONDITIONAL QUANTIZATION FOR UNIFORM DISTRIBUTIONS ON LINE SEGMENTS AND REGULAR POLYGONS

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Abstract. Quantization for a Borel probability measure refers to the idea of estimating a given probability by a discrete probability with support containing a finite number of elements. If in the quantization some of the elements in the support are preselected, then the quantization is called a conditional quantization. In this paper, we have investigated the conditional quantization for the uniform distributions defined on the unit line segments and m-sided regular polygons, where $m \geq 3$, inscribed in a unit circle.

1. Introduction

The process of transformation of a continuous-valued signal into a discrete-valued one is called 'quantization'. It has broad applications in engineering and technology. We refer to [GG, GN, Z2] for surveys on the subject and comprehensive lists of references to the literature; see also [AW, GKL, GL1, Z1]. For mathematical treatment of quantization one is referred to Graf-Luschgy's book (see [GL1]). Recently, Pandey and Roychowdhury introduced the concepts of constrained quantization and the conditional quantization (see [PR1, PR2, PR4]). A quantization without a constraint is known as an unconstrained quantization, which traditionally in the literature is known as quantization. After the introduction of constrained quantization, and then the conditional quantization, the quantization theory is now much more enriched with huge applications in our real world. For some follow up papers in the direction of constrained quantization, one can see [BCDRV, HNPR, PR3, PR5]). On unconstrained quantization there is a number of papers written by many authors, for example, one can see [DFG, DR, GG, GL, GL1, GL2, GL3, GN, KNZ, P, P1, R1, R2, R3, Z1, Z2]. This paper deals with conditional unconstrained quantization, which traditionally, in the sequel will be refereed to as conditional quantization.

Definition 1.1. *Let P be a Borel probability measure on $\mathbb{R}^2$ equipped with a Euclidean metric d induced by the Euclidean norm $\|\cdot\|$. Let $\beta \subset \mathbb{R}^2$ be given with card$(\beta) = \ell$ for some $\ell \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ with $n \geq \ell$, the nth conditional quantization error for P with respect to the conditional set β, is defined as*

$$V_n := V_n(P) = \inf_{\alpha} \Big\{ \int \min_{a \in \alpha \cup \beta} d(x, a)^2 dP(x) : card(\alpha) \leq n - \ell \Big\}, \tag{1}$$

where card(A) represents the cardinality of the set A.

We assume that $\int d(x, 0)^2 dP(x) < \infty$ to make sure that the infimum in (1) exists (see [PR1]). For a finite set $\gamma \subset \mathbb{R}^2$ and $a \in \gamma$, by $M(a|\gamma)$ we denote the set of all elements in $\mathbb{R}^2$ which are nearest to a among all the elements in γ, i.e., $M(a|\gamma) = \{x \in \mathbb{R}^2 : d(x, a) = \min_{b \in \gamma} d(x, b)\}$. $M(a|\gamma)$ is called the *Voronoi region* in $\mathbb{R}^2$ generated by $a \in \gamma$.

Definition 1.2. *A set $\alpha \cup \beta$, where $P(M(b|\alpha \cup \beta)) > 0$ for $b \in \beta$, for which the infimum in V_n exists and contains no less than ℓ elements, and no more than n elements is called a conditional optimal set of n-points for P with respect to the conditional set β.*

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Write $V_\infty(P) := \lim_{n\to\infty} V_n(P)$. Then, the number

$$D(P) := \lim_{n\to\infty} \frac{2\log n}{-\log(V_n(P) - V_\infty(P))}$$

if it exists, is called the *conditional quantization dimension* of P and is denoted by $D(P)$. The conditional quantization dimension measures the speed at which the specified measure of the conditional quantization error converges as n tends to infinity. For any $\kappa > 0$, the number

$$\lim_n n^{\frac{2}{\kappa}}(V_n(P) - V_\infty(P)),$$

if it exists, is called the *κ-dimensional conditional quantization coefficient* for P.

In this paper, we have investigated the conditional quantization for uniform distributions on the unit line segments and on regular m-sided polygons, where $m \geq 3$, inscribed in a unit circle.

1.3. **Delineation.** In this paper, there are total three sections in addition to the section that contains the basic preliminaries. In Section 3, we have determined the conditional optimal sets of n-points and the nth conditional quantization errors for a uniform distribution with two interior elements as the conditional set for all $n \geq 2$ on a unit line segment. In Section 4, we have calculated the conditional optimal sets of n-points and the nth conditional quantization errors for a uniform distribution with $(k-1)$ interior elements and one boundary element in the conditional set for all $n \geq k$ on a unit line segment. Section 5, deals with a uniform distribution defined on the boundary of a regular m-sided polygon. Let P be a uniform distribution defined on the boundary of a regular m-sided polygon inscribed in a unit circle. In the paper [HMRT], in unconstrained scenario, Hansen et al. in a proposition, first determined the optimal sets of n-means and the nth quantization errors for the probability distribution P when n is of the form $n = mk$ for some $k \in \mathbb{N}$. Then, with the help of the proposition, they have shown that the quantization coefficient for P exists, and equals $\frac{1}{3}m^2 \sin^2\left(\frac{\pi}{m}\right)$, i.e.,

$$\lim_{n\to\infty} n^2 V_n(P) = \frac{1}{3}m^2 \sin^2 \frac{\pi}{m}.$$

After the introduction of conditional quantization, we know that the quantization dimension and the quantization coefficient, in both constrained and unconstrained scenario, for a Borel probability measure do not depend on the conditional set (see [PR4]). Using this scenario, in Section 5, we calculate the quantization coefficient for the uniform distribution P defined on the boundary of the regular m-sided polygon inscribed in the unit circle by calculating the conditional quantization coefficient for P with respect to the conditional set β, which consists of all the vertices of the regular polygon. The significance of our work in Section 5 is that, the work in this section is much more simpler than the work to calculate the quantization coefficient done by Hansen et al. in the paper [HMRT]. In addition, we have also given an explicit formula to calculate the conditional optimal sets of n-points and the nth conditional quantization errors for the uniform distribution P for all $n \geq m$, where m is the number of vertices of the m-sided polygon.

2. Preliminaries

For any two elements (a, b) and (c, d) in $\mathbb{R}^2$, we write

$$\rho((a, b), (c, d)) := (a - c)^2 + (b - d)^2,$$

which gives the squared Euclidean distance between the two elements (a, b) and (c, d). Let p and q be two elements that belong to an optimal set of n-points for some positive integer n, and let e be an element on the boundary of the Voronoi regions of the elements p and q. Since the boundary of the Voronoi regions of any two elements is the perpendicular bisector of the line segment joining the elements, we have

$$\rho(p, e) - \rho(q, e) = 0.$$

We call such an equation a *canonical equation.* Notice that any element $x \in \mathbb{R}$ can be identified as an element $(x, 0) \in \mathbb{R}^2$. Thus,

$$\rho : \mathbb{R} \times \mathbb{R}^2 \to [0, \infty) \text{ such that } \rho(x, (a, b)) = (x - a)^2 + b^2,$$

where $x \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, defines a nonnegative real-valued function on $\mathbb{R} \times \mathbb{R}^2$. On the other hand,

$$\rho : \mathbb{R} \times \mathbb{R} \to [0, \infty) \text{ be such that } \rho(x, y) = x^2 + y^2,$$

where $x, y \in \mathbb{R}$, defines a nonnegative real-valued function on $\mathbb{R} \times \mathbb{R}$.

Let P be a Borel probability measure on $\mathbb{R}$ which is uniform on its support the closed interval $[a, b]$. Then, the probability density function f for P is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have $dP(x) = P(dx) = f(x)dx$ for any $x \in \mathbb{R}$.

Notation 2.1. Let α be a discrete set. Then, for a Borel probability measure μ and a set A, by $V(\mu; \{\alpha, A\})$, it is meant the distortion error for μ with respect to the set α over the set A, i.e.,

$$V(\mu; \{\alpha, A\}) := \int_A \min_{a \in \alpha} d(x, a)^2 \, d\mu(x). \tag{2}$$

The following proposition is a generalized version of Proposition 2.1, Proposition 2.2 and Proposition 2.3 that appear in [PR4]. For the readers' convenience, we give the complete proof here.

Proposition 2.2. *Let P be a uniform distribution on the closed interval $[a, b]$ and $c, d \in [a, b]$ be such that $a < c < d < b$. For $n \in \mathbb{N}$ with $n \geq 2$, let α_n be a conditional unconstrained optimal set of n-points for P with respect to the conditional set $\beta = \{c, d\}$ such that α_n contains k elements from the closed interval $[a, c]$, ℓ elements from the closed interval $[c, d]$, and m elements from the closed interval $[d, b]$ for some $k, \ell, m \geq 1$. Then, $k + \ell + m = n + 2$,*

$$\alpha_n \cap [a, c] = \Big\{a + \frac{(2j-1)(c-a)}{2k-1} : 1 \leq j \leq k\Big\},$$
$$\alpha_n \cap [c, d] = \Big\{c + \frac{j-1}{\ell-1}(d-c) : 1 \leq j \leq \ell\Big\}, \text{ and}$$
$$\alpha_n \cap [d, b] = \Big\{d + \frac{2(j-1)(b-d)}{2m-1} : 1 \leq j \leq m\Big\}$$

with the conditional unconstrained quantization error

$$V_n := V_{k,\ell,m}(P) = \frac{1}{3(b-a)}\Big(\frac{(c-a)^3}{(2k-1)^2} + \frac{1}{4}\frac{(d-c)^3}{(\ell-1)^2} + \frac{(b-d)^3}{(2m-1)^2}\Big).$$

Proof. Notice that the element c in the conditional set β is common to both the intervals $[a, c]$ and $[c, d]$, the element d in the conditional set β is common to both the intervals $[c, d]$ and $[d, b]$, and so c and d are counted two times. Hence, $k + \ell + m = n + 2$. We have

$$[a, b] := \{t : a \leq t \leq b\}.$$

Let α_n be a conditional unconstrained optimal set of n-points such that

$$\text{card}(\alpha_n \cap [a, c]) = k, \ \text{card}(\alpha_n \cap [c, d]) = \ell, \text{ and } \text{card}(\alpha_n \cap [d, b]) = m, \text{ where } k, m \geq 1 \text{ and } \ell \geq 2.$$

Then, we can write

$$\alpha_n \cap [a, c] = \{a_1, a_2, \cdots, a_k\}, \ \alpha_n \cap [c, d] = \{c_1, c_2, \cdots, c_\ell\} \text{ and } \alpha_n \cap [d, b] = \{d_1, d_2, \cdots, d_m\},$$

such that

$$a < a_1 < a_2 < \cdots < a_k = c = c_1 < c_2 < \cdots < c_\ell = d = d_1 < d_2 < \cdots < d_m < b.$$

Since the closed intervals $[a, c]$ is a line segments and P is a uniform distribution, we have

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} = \frac{a_k - a_1}{k-1} = \frac{c - a_1}{k-1}$$

implying

$$\begin{aligned} a_2 &= a_1 + \frac{c-a_1}{k-1} = a_1 + \frac{c-a_1}{k-1}, \\ a_3 &= a_2 + \frac{c-a_1}{k-1} = a_1 + 2\frac{c-a_1}{k-1}, \\ a_4 &= a_3 + \frac{c-a_1}{k-1} = a_1 + 3\frac{c-a_1}{k-1}, \\ &\text{and so on.} \end{aligned}$$

Thus, we have $a_j = a_1 + (j-1)\frac{c-a_1}{k-1}$ for $1 \leq j \leq k$. The distortion error due to the elements $\alpha_n \cap [a, c]$ is given by

$$\begin{aligned} V(P; \{\alpha_n \cap [a,c], [a,c]\}) &= \int_{[a,c]} \min_{x \in \alpha_n \cap [a,c]} \rho(t, x)\, dP \\ &= \frac{1}{b-a}\Big(\int_a^{\frac{a_1+a_2}{2}} \rho(t, a_1)\, dt + (k-2)\int_{\frac{a_1+a_2}{2}}^{\frac{a_2+a_3}{2}} \rho(t, a_2)\, dt + \int_{\frac{a_{k-1}+a_k}{2}}^{a_k} \rho(t, a_k)\, dt\Big) \\ &= \frac{4a^3(k-1)^2 - 3a_1\left(4a^2(k-1)^2 - c^2\right) + 3a_1^2\left(4a(k-1)^2 - c\right) - c^3 + a_1^3\left(-4k^2 + 8k - 3\right)}{12(k-1)^2(a-b)}, \end{aligned}$$

the minimum value of which is $\frac{(c-a)^3}{3(b-a)(2k-1)^2}$ and it occurs when $a_1 = a + \frac{c-a}{2k-1}$. Putting the values of a_1, we have

$$a_j = a + \frac{(2j-1)(c-a)}{2k-1} \text{ for } 1 \leq j \leq k \text{ with } V(P; \{\alpha_n \cap [a,c], [a,c]\}) = \frac{(c-a)^3}{3(b-a)(2k-1)^2}.$$

Since the closed interval $[c, d]$ is a line segment and P is a uniform distribution, we have

$$c_2 - c_1 = c_3 - c_2 = \cdots = c_\ell - c_{\ell-1} = \frac{c_\ell - c_1}{\ell - 1} = \frac{d-c}{\ell-1}$$

implying

$$\begin{aligned} c_2 &= c_1 + \frac{d-c}{\ell-1} = c + \frac{d-c}{\ell-1}, \\ c_3 &= c_2 + \frac{d-c}{\ell-1} = c + \frac{2(d-c)}{\ell-1}, \\ c_4 &= c_3 + \frac{d-c}{\ell-1} = c + \frac{3(d-c)}{\ell-1}, \\ &\text{and so on.} \end{aligned}$$

Thus, we have $c_j = c + \frac{j-1}{\ell-1}(d-c)$ for $1 \leq j \leq \ell$. The distortion error contributed by the ℓ elements in the closed interval $[c, d]$ is given by

$$\begin{aligned} V(P; \{\alpha_n \cap [c,d], [c,d]\}) &= \int_{[c,d]} \min_{x \in \alpha_n \cap [c,d]} \rho((t,0), x)\, dP \\ &= \frac{1}{b-a}\Big(2\int_{c_1}^{\frac{c_1+c_2}{2}} \rho((t,0),(c_1,0))\, dt + (\ell-2)\int_{\frac{c_1+c_2}{2}}^{\frac{c_2+c_3}{2}} \rho((t,0),(c_2,0))\, dt\Big) \\ &= \frac{1}{12}\frac{(d-c)^3}{b-a}\frac{1}{(\ell-1)^2}. \end{aligned}$$

Again, the closed interval $[d, b]$ is a line segment and P is a uniform distribution, we have

$$d_2 - d_1 = d_3 - d_2 = \cdots = d_m - d_{m-1} = \frac{d_m - d_1}{m-1} = \frac{d_m - d}{m-1}$$

implying

$$\begin{aligned} d_2 &= d_1 + \frac{d_m - d}{m-1} = d + \frac{d_m - d}{m-1}, \\ d_3 &= d_2 + \frac{d_m - d}{m-1} = d + 2\frac{d_m - d}{m-1}, \\ d_4 &= d_3 + \frac{d_m - d}{m-1} = d + 3\frac{d_m - d}{m-1}, \\ &\text{and so on.} \end{aligned}$$

Thus, we have $d_j = d + (j-1)\frac{d_m - d}{m-1}$ for $1 \leq j \leq m$. The distortion error contributed by the m elements is given by

$$\begin{aligned} &V(P; \{\alpha_n \cap [d, b], [d, b]\}) = \int_{[d,b]} \min_{x \in \alpha_n \cap [d,b]} \rho(t, x)\, dP \\ &= \frac{1}{b-a}\Big(\int_{d_1}^{\frac{d_1+d_2}{2}} \rho(t, d_1)\, dt + (m-2)\int_{\frac{d_1+d_2}{2}}^{\frac{d_2+d_3}{2}} \rho(t, d_2)\, dt + \int_{\frac{d_{m-1}+d_m}{2}}^{b} \rho(t, d_m)\, dt\Big) \\ &= \frac{-4b^3(m-1)^2 + 3d_m\left(4b^2(m-1)^2 - d^2\right) - 3d_m^2\left(4b(m-1)^2 - d\right) + d^3 + \left(4m^2 - 8m + 3\right) d_m^3}{12(m-1)^2(a-b)} \end{aligned}$$

the minimum value of which is $\frac{(b-d)^3}{3(b-a)(2m-1)^2}$ and it occurs when $d_m = d + \frac{2(m-1)(b-d)}{2m-1}$. Putting the values of d_m, we have

$$d_j = d + \frac{2(j-1)(b-d)}{2m-1} \text{ for } 1 \leq j \leq m \text{ with } V(P; \{\alpha_n \cap [d, b], [d, b]\}) = \frac{(b-d)^3}{3(b-a)(2m-1)^2}.$$

Since $a_j = a + \frac{(2j-1)(c-a)}{2k-1}$ for $1 \leq j \leq k$, $c_j = c + \frac{j-1}{\ell-1}(d-c)$ for $1 \leq j \leq \ell$, and $d_j = d + \frac{2(j-1)(b-c)}{2m-1}$ for $1 \leq j \leq m$, and

$$V_n := V_{k,\ell,m} = V(P; \{\alpha_n \cap [a, c], [a, c]\}) + V(P; \{\alpha_n \cap [c, d], [c, d]\}) + V(P; \{\alpha_n \cap [d, b], [d, b]\}),$$

the proposition is yielded. □

In the following sections, we give the main results of the paper.

3. Conditional optimal sets of n-points and the conditional quantization errors with two interior elements in the conditional set for all $n \geq 2$ on a unit line segment

In this section, for the uniform distribution P on the line segment $[0, 1]$ with respect to the conditional set $\beta := \{\frac{1}{4}, \frac{2}{4}\}$, we calculate the conditional optimal sets of n-points and the nth conditional quantization errors for all $n \in \mathbb{N}$ with $n \geq 2$. Let α_n be a conditional optimal set of n-points with the nth conditional quantization error V_n for all $n \in \mathbb{N}$. Let $\text{card}(\alpha_n \cap [0, \frac{1}{4}]) = k$, $\text{card}(\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]) = \ell$, and $\text{card}(\alpha_n \cap [\frac{1}{2}, 1]) = m$. Then, $k, m \geq 1$, and $\ell \geq 2$. By Proposition 2.2, we know that

$$\begin{aligned} \alpha_n \cap [0, \frac{1}{4}] &= \Big\{\frac{2j-1}{4(2k-1)} : 1 \leq j \leq k\Big\}, \\ \alpha_n \cap [\frac{1}{4}, \frac{1}{2}] &= \Big\{\frac{1}{4} + \frac{j-1}{4(\ell-1)} : 1 \leq j \leq \ell\Big\}, \text{ and} \\ \alpha_n \cap [\frac{1}{2}, 1] &= \Big\{\frac{1}{2} + \frac{j-1}{2m-1} : 1 \leq j \leq m\Big\}. \end{aligned} \tag{3}$$

Notice that $\alpha_n = (\alpha_n \cap [0, \frac{1}{4}]) \cup (\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]) \cup (\alpha_n \cap [\frac{1}{2}, 1])$ with the nth conditional quantization error

$$V_n := V_{k,\ell,m}(P) = \frac{1}{3}\Big(\frac{1}{64(2k-1)^2} + \frac{1}{256(\ell-1)^2} + \frac{1}{8(2m-1)^2}\Big). \tag{4}$$

Proposition 3.1. *The optimal set of two-points is the set $\beta = \{\frac{1}{4}, \frac{1}{2}\}$ with $V_2 = 0.0481771$.*

Proof. By definition, the conditional optimal set of two-points is the conditional set β itself, and the corresponding conditional quantization error is given by

$$V_2 = V_{1,2,1} = \frac{37}{768} = 0.0481771.$$

Thus, the proposition is yielded. $\square$

Proposition 3.2. *The conditional optimal set of three-points is the set $\alpha_3 = \{\frac{1}{4}, \frac{1}{2}, \frac{5}{6}\}$ with $V_3 =$ 0.00651042.*

Proof. By Equation (4), we see that

$$V_{2,2,1} = 0.0435475,\ V_{1,3,1} = 0.0472005, \text{ and } V_{1,2,2} = 0.01114.$$

Since $V_{1,2,2}$ is minimum among all the above possible errors, we can deduce that $k = 1$, $\ell = 2$, and $m = 2$. Hence, by (3), we obtain the conditional optimal set of three-points as $\alpha_3 = \{\frac{1}{4}, \frac{1}{2}, \frac{5}{6}\}$ with $V_3 = 0.01114$. $\square$

Proposition 3.3. *The conditional optimal set of four-points is the set $\alpha_4 = \{\frac{1}{12}, \frac{1}{4}, \frac{1}{2}, \frac{5}{6}\}$ with $V_4 =$ 0.00651042.*

Proof. Considering all possible errors $V_{i,j,k}$ we see that it is minimum when $i = 2$, $j = 2$ and $k = 2$. Hence, using (3) and (4), we deduce that $\alpha_4 = \{\frac{1}{12}, \frac{1}{4}, \frac{1}{2}, \frac{5}{6}\}$ with $V_4 = 0.00651042$. $\square$

Proceeding in the similar way as the previous propositions, we can deduce the following two propositions:

Proposition 3.4. *The conditional optimal set of five-points is the set $\alpha_5 = \{\frac{1}{12}, \frac{1}{4}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}\}$ with $V_5 =$ 0.00354745.*

Proposition 3.5. *The conditional optimal set of six-points is the set $\alpha_6 = \{\frac{1}{12}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}\}$ with $V_6 = 0.00257089$.*

Lemma 3.6. *Let $n \in \mathbb{N}$ be such that $n = 4x + 2$ for some $x \in \mathbb{N}$. Let card$(\alpha_n \cap [0, \frac{1}{4}]) = k$, card$(\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]) = \ell$, and card$(\alpha_n \cap [\frac{1}{2}, 1]) = m$. Then, $(k-1) : (\ell-2) : (m-1) = 1 : 1 : 2$.*

Proof. Let $n = 4x + 2$ for some $x \in \mathbb{N}$, and k, ℓ, m be the positive integers as defined in the hypothesis. Since $m = n + 2 - k - \ell = 4x + 4 - k - \ell$, by (4), we have

$$V_{k,\ell,m} = \frac{1}{768}\left(\frac{32}{(-2k - 2L + 8x + 7)^2} + \frac{4}{(1-2k)^2} + \frac{1}{(\ell-1)^2}\right),$$

which is minimum if $k = x + 1$ and $\ell = x + 2$. Then, $m = 2x + 1$. Thus, we see that $(k-1) : (\ell-2) : (m-1) = 1 : 1 : 2$, which is the lemma. $\square$

As a consequence of Lemma 3.6, we deduce the following corollary.

Corollary 3.7. Let α_n be a conditional optimal set of n-points with card$(\alpha_n \cap [0, \frac{1}{4}]) = k$, card$(\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]) = \ell$, and card$(\alpha_n \cap [\frac{1}{2}, 1]) = m$. Then, for $n \geq 6$, we have $k, m \geq 1$ and $\ell \geq 2$.

Let us now give the following theorem, which is the main theorem in this section.

Theorem 3.8. *For $n \in \mathbb{N}$ with $n \geq 6$, let α_n be a conditional optimal set of n-points for P. Let card$(\alpha_n \cap [0, \frac{1}{4}]) = k$, card$(\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]) = \ell$, and card$(\alpha_n \cap [\frac{1}{2}, 1]) = m$. For some $x \in \mathbb{N}$ if $n = 4x + 2$, then $(k, \ell, m) = (x+1, x+2, 2x+1)$; if $n = 4x + 3$, then $(k, \ell, m) = (x+1, x+2, 2x+2)$; if $n = 4x + 4$, then $(k, \ell, m) = (x+2, x+2, 2x+2)$; if $n = 4x + 5$, then $(k, \ell, m) = (x+2, x+2, 2x+3)$.*

Proof. By Lemma 3.6, it is known that if $n = 4x + 2$, then $(k, \ell, m) = (x + 1, x + 2, 2x + 1)$. Using the similar technique that is used in Lemma 3.6, we can show that if $n = 4x + 3$, then $(k, \ell, m) = (x + 1, x + 2, 2x + 2)$; if $n = 4x + 4$, then $(k, \ell, m) = (x + 2, x + 2, 2x + 2)$; if $n = 4x + 5$, then $(k, \ell, m) = (x + 2, x + 2, 2x + 3)$. Thus, the proof of the theorem is complete. $\square$

Note 3.9. By Theorem 3.8, for any given $n \geq 6$, we can easily calculate the values of (k, ℓ, m). Since the values of (k, ℓ, m) depend on n, writing $(k, \ell, m) := (k(n), \ell(n), m(n))$, we have

$$\Big\{(k(n), \ell(n), m(n)\Big\}_{n=6}^{\infty}$$
$$= \Big\{(2, 3, 3), (2, 3, 4), (3, 3, 4), (3, 3, 5), (3, 4, 5), (3, 4, 6), (4, 4, 6), (4, 4, 7), (4, 5, 7), (4, 5, 8), \cdots\Big\}.$$

Notice that if $n = 4x + 2$ for $x \in \mathbb{N}$, then we have

$$\Big\{(k(4x + 2) - 1, \ell(4x + 2) - 2, m(4x + 2) - 1\Big\}_{x=1}^{\infty} = \Big\{(1, 1, 2), (2, 2, 4), (3, 3, 6), (4, 4, 8), \cdots\Big\}$$

implying

$$\Big\{(k(4x + 2) - 1, \ell(4x + 2) - 2, m(4x + 2) - 1\Big\}_{x=1}^{\infty} = \Big\{x(1, 1, 2) : x \in \mathbb{N}\Big\}.$$

3.10. **Conditional optimal sets of n-points and the nth conditional quantization errors.** Let $n \geq 6$ be a positive integer. To determine the optimal sets of n-points and the nth conditional quantization errors, first using Theorem 3.8, we determine the corresponding values of $k, \ell,$ and m. Once k, ℓ, m are known, by using (3), we calculate the sets $\alpha_n \cap [0, \frac{1}{4}]$, $\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]$, and $\alpha_n \cap [\frac{1}{2}, 1]$. Then, α_n is given by

$$\alpha_n = (\alpha_n \cap [0, \frac{1}{4}]) \bigcup (\alpha_n \cap [\frac{1}{4}, \frac{1}{2}]) \bigcup (\alpha_n \cap [\frac{1}{2}, 1]),$$

and the nth conditional quantization error is obtained by using the formula (4). $\square$

Example 3.11. Let $n = 59$, then as $n = 4 \times 14 + 3 = 4x + 3$, where $x = 14$, by Theorem 3.8, we have $(k, \ell, m) = (x + 1, x + 2, 2x + 2) = (15, 16, 30)$. Hence, by (3) and (4), we have the nth conditional optimal set of n-points, for $n = 56$ as

$$\alpha_{59} = \Big\{\frac{1}{116}(2j - 1) : 1 \leq j \leq 15\Big\} \bigcup \Big\{\frac{j - 1}{60} + \frac{1}{4} : 1 \leq j \leq 16\Big\} \bigcup \Big\{\frac{j - 1}{59} + \frac{1}{2} : 1 \leq j \leq 30\Big\}$$

with nth conditional quantization error $V_{59} = V_{15,16,30} = \frac{12115621}{505875628800}$.

Theorem 3.12. *The conditional quantization dimension $D(P)$ of the probability measure P exists, and $D(P) = 1$.*

Proof. For any $n \in \mathbb{N}$ with $n \geq 6$, there exists a positive integer x depending on n such that $4x + 2 \leq n \leq 4(x + 1) + 2$. Then, $V_{x+2,x+3,2x+3} \leq V_n \leq V_{x+1,x+2,2x+1}$. By (4), we see that $V_{x+2,x+3,2x+3} \to 0$ and $V_{x+1,x+2,2x+1} \to 0$ as $n \to \infty$, and so by squeeze theorem, $V_n \to 0$ as $n \to \infty$, i.e., $V_\infty = 0$. We can take n large enough so that $V_{x+1,x+2,2x+1} - V_\infty < 1$. Then,

$$0 < -\log(V_{x+1,x+2,2x+1} - V_\infty) \leq -\log(V_n - V_\infty) \leq -\log(V_{x+2,x+3,2x+3} - V_\infty)$$

yielding

$$\frac{2\log(4x + 2)}{-\log(V_{x+2,x+3,2x+3} - V_\infty)} \leq \frac{2\log n}{-\log(V_n - V_\infty)} \leq \frac{2\log(4x + 6)}{-\log(V_{x+1,x+2,2x+1} - V_\infty)}.$$

Notice that

$$\lim_{n\to\infty} \frac{2\log(4x + 2)}{-\log(V_{x+2,x+3,2x+3} - V_\infty)} = 1, \text{ and } \lim_{n\to\infty} \frac{2\log(4x + 6)}{-\log(V_{x+1,x+2,2x+1} - V_\infty)} = 1.$$

Hence, $\lim_{n\to\infty} \frac{2\log n}{-\log(V_n - V_\infty)} = 1$, i.e., the conditional quantization dimension $D(P)$ of the probability measure P exists and $D(P) = 1$. Thus, the proof of the theorem is complete. $\square$

Theorem 3.13. *The $D(P)$-dimensional quantization coefficient for P exists as a finite positive number and equals $\frac{1}{12}$.*

Proof. For any $n \in \mathbb{N}$ with $n \geq 6$, there exists a positive integer x depending on n such that $4x+2 \leq n \leq 4(x+1)+2$. Then, $V_{x+2,x+3,2x+3} \leq V_n \leq V_{x+1,x+2,2x+1}$ and $V_\infty = 0$. Since

$$\lim_{n\to\infty} n^2(V_n - V_\infty) \geq \lim_{n\to\infty}(4x+2)^2(V_{x+2,x+3,2x+3} - V_\infty) = \frac{1}{12}, \text{ and}$$
$$\lim_{n\to\infty} n^2(V_n - V_\infty) \leq \lim_{n\to\infty}(4x+6)^2(V_{x+1,x+2,2x+1} - V_\infty) = \frac{1}{12},$$

by squeeze theorem, we have $\lim\limits_{n\to\infty} n^2(V_n - V_\infty) = \frac{1}{12}$, which is the theorem. □

4. Conditional optimal sets of n-points and the nth conditional quantization errors with $(k-1)$ interior elements and one boundary element in the conditional set for all $n \geq k$ on a unit line segment

In this section, for the uniform distribution P on the line segment $[0,1]$ with respect to the conditional set $\beta := \{\frac{1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}, \frac{k}{k}\}$, we calculate the conditional optimal sets of n-points and the nth conditional quantization errors for all $n \in \mathbb{N}$ with $n \geq k$. Let α_n be a conditional optimal set of n-points with the nth conditional quantization error V_n, where $n \in \mathbb{N}$ with $n \geq k$. Write

$$J_{k,j} := [\frac{j-1}{k}, \frac{j}{k}] \text{ and } \text{card}(\alpha_n \cap J_{k,j}) = n_j \text{ for } 1 \leq j \leq k. \tag{5}$$

Notice that n_j satisfies: $n_1 \geq 1$, $n_j \geq 2$ for $2 \leq j \leq k$. By Proposition 2.2, we know that

$$\alpha_n \cap J_{k,1} = \Big\{\frac{2j-1}{k(2n_1-1)} : 1 \leq j \leq n_1\Big\} \text{ with } V(P;\{\alpha_n \cap J_{k,1}, J_{k,1}\}) = \frac{1}{3k^3(2n_1-1)^2}, \tag{6}$$

and

$$\alpha_n \cap J_{k,j} = \Big\{\frac{j-1}{k} + \frac{q-1}{k(n_j-1)} : 1 \leq q \leq n_j\Big\} \text{ with } V(P;\{\alpha_n \cap J_{k,j}, J_{k,j}\}) = \frac{1}{12k^3(n_j-1)^2} \tag{7}$$

for $2 \leq j \leq k$. Notice that

$$\alpha_n = \bigcup_{j=1}^k \alpha_n \cap J_{k,j} \text{ with } V_n := V_{n_1,n_2,\cdots,n_k}(P) = \sum_{j=1}^k V(P;\{\alpha_n \cap J_{k,j}, J_{k,j}\}). \tag{8}$$

Proposition 4.1. *The optimal set of k-points is the set $\beta = \{\frac{j}{k} : 1 \leq j \leq k\}$ with $V_k = \frac{k+3}{12k^3}$.*

Proof. By definition, the conditional optimal set of k-points is the conditional set β itself, and the corresponding conditional quantization error is given by

$$V_k = \sum_{j=1}^k V(P;\{\alpha_n \cap J_{k,j}, J_{k,j}\}) = V(P;\{\{\frac{1}{k}\}, J_{k,1}\}) + (k-1)V(P;\{\{\frac{1}{k}, \frac{2}{k}\}, J_{k,2}\}) = \frac{k+3}{12k^3}.$$

Thus, the proposition is yielded. □

Lemma 4.2. *Let $n \in \mathbb{N}$ be such $n \geq k$. Let n_j be the positive integers as defined by (5). Then, for $2 \leq i < j \leq k$, $|n_i - n_j| = 0$ or 1.*

Proof. Recall that for $2 \leq i < j \leq k$, $n_i + n_j \geq 4$. Let us first assume that $n_i + n_j$ is an even number, i.e., $n_i + n_j = 2m$ for some $m \geq 2$. Then,

$$V(P;\{\alpha_n \cap J_{k,i}, J_{k,i}\}) + V(P;\{\alpha_n \cap J_{k,j}, J_{k,j}\}) = \frac{1}{12k^3}\Big(\frac{1}{(n_i-1)^2} + \frac{1}{(n_j-1)^2}\Big).$$

By routine, we see that the above expression is minimum if $n_1 = n_2 = m$. Similarly, if $n_i + n_j = 2m+1$ for some $m \geq 2$, then we see that the above expression is minimum if $(n_i, n_j) = (m, m+1)$, or $(n_i, n_j) = (m+1, m)$. This yields the fact that for $2 \leq i < j \leq k$, $|n_i - n_j| = 0$ or 1, which is the lemma. □

Lemma 4.3. *Let $n \in \mathbb{N}$ be such $n \geq k$. Let n_j be the positive integers as defined by (5). Then, for $2 \leq j \leq k$, $|n_1 - n_j| = 0$ or 1 with $n_1 \leq n_j$.*

Proof. Recall that $n_1 \geq 1$ and for $2 \leq j \leq k$, we have $n_j \geq 2$. Let us first assume that $n_1 + n_j$ is an even number, i.e., $n_1 + n_j = 2m$, i.e., $n_j = 2m - n_1$ for some $m \in \mathbb{N}$ with $m \geq 2$. Then,

$$V(P; \{\alpha_n \cap J_{k,1}, J_{k,1}\}) + V(P; \{\alpha_n \cap J_{k,j}, J_{k,j}\}) = \frac{1}{3k^3}\Big(\frac{1}{(2n_1 - 1)^2} + \frac{1}{4(2m - n_1 - 1)^2}\Big).$$

By routine, we see that the above expression is minimum if $n_1 = n_2 = m$. Similarly, if $n_1 + n_j = 2m + 1$ for some $m \in \mathbb{N}$, then we see that the above expression is minimum if $(n_1, n_j) = (m, m + 1)$. Thus, for $2 \leq j \leq k$, we have $|n_1 - n_j| = 0$ or 1 with $n_1 \leq n_j$, which is the lemma. $\square$

Let us now give the following theorem, which is the main theorem is this section. This theorem helps us to determine the conditional optimal sets of n-points and the nth conditional quantization errors for all $n \in \mathbb{N}$ with $n \geq k$.

Theorem 4.4. *For $n \geq k$, let α_n be a conditional optimal set of n-points such that $n = mk + \ell$ some $\ell, m \in \mathbb{N}$ and $0 \leq \ell < k$. Then,*

(i) if $\ell = 0$, then card$(\alpha_n \cap J_{k,1}) = m$ and card$(\alpha_n \cap J_{k,j}) = m + 1$ for $2 \leq j \leq k$;

(ii) if $1 \leq \ell < k$, then card$(\alpha_n \cap J_{k,1}) = m + 1$ and card$(\alpha_n \cap J_{k,j}) = m + 2$ for $j \in \{j_1, j_2, \cdots, j_{\ell-1}\}$, and card$(\alpha_n \cap J_{k,j}) = m + 1$ for $j \in \{2, 3, \cdots, k\} \setminus \{j_1, j_2, \cdots, j_{\ell-1}\}$, where $\{j_1, j_2, \cdots, j_{\ell-1}\}$ is any subset of $\ell - 1$ elements of the set $\{2, 3, \cdots, k\}$.

Proof. The proof follows as a consequence of Lemma 4.2 and Lemma 4.3. $\square$

Remark 4.5. Notice that in (i) of Theorem 4.4, we have $\sum_{j=1}^k \text{card}(\alpha_n \cap J_{k,j}) = mk + (k - 1)$, on the other hand, in (ii) of Theorem 4.4, we have $\sum_{j=1}^k \text{card}(\alpha_n \cap J_{k,j}) = mk + \ell + (k - 1)$, i.e., in the sum an extra term $(k - 1)$ occurs. This happens because in the conditional optimal set of n-points, $(k - 1)$ elements from the conditional set are counted two times.

4.6. **conditional optimal sets of n-points and the nth conditional quantization errors.** Let $n \geq k$ be a positive integer. To determine the optimal sets of n-points and the nth conditional quantization errors, first using Theorem 4.4, we determine the values of n_j, where $n_j = \text{card}(\alpha_n \cap J_{k,j})$. Once n_j are known by using the formulae given in (6) and (7), we calculate the sets $\alpha_n \cap J_{k,j}$ and the corresponding distortion errors $V(P; \{\alpha_n \cap J_{k,j}, J_{k,j}\})$ for all $1 \leq j \leq k$. Then, using the expressions in (8), we obtain the conditional optimal set α_n and the corresponding nth conditional quantization error V_n. As an illustration, see Example 4.7 given below. $\square$

Example 4.7. Let P be the uniform distribution on the closed interval $[0, 1]$. Choose $k = 5$, i.e., the conditional set is $\beta := \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$. Then, the optimal set of n-points for any $n \geq 5$ exists. Notice that by Proposition 4.1, the conditional optimal set of five-points is the conditional set β with the conditional quantization error

$$V_5 = \frac{k + 3}{12k^3} = \frac{2}{375}.$$

To determine a conditional optimal set of n-points, for some n, $n = 19$ say, we proceed as follows:

We have $n = 19 = 3 \times 5 + 4$, i.e., we have $m = 3$ and $\ell = 4$. Recall Theorem 4.4 (ii). Let card$(\alpha_n \cap J_{k,j}) = n_j$ for $1 \leq j \leq 5$. Choose any $\{j_1, j_2, j_3\} \subseteq \{2, 3, 4, 5\}$. Let $\{j_1, j_2, j_3\} = \{2, 4, 5\}$. Then, $\{2, 3, 4, 5\} \setminus \{j_1, j_2, j_3\} = \{3\}$ yielding $n_1 = 4$, $n_2 = n_4 = n_5 = 5$, and $n_3 = 4$. Then, using (6)

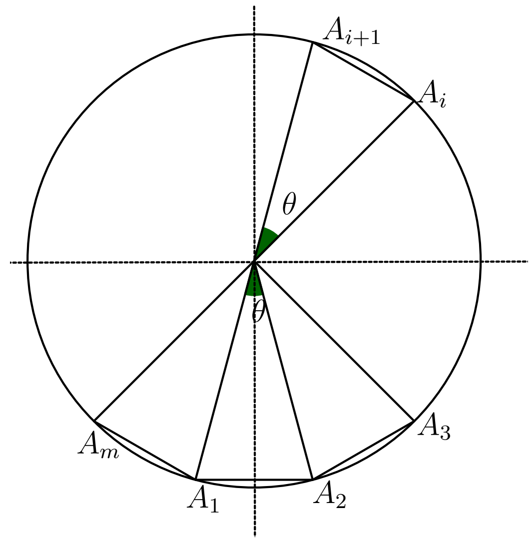

FIGURE 1. The regular m-sided polygon inscribed in a unit circle.

and (7), we have

$$\alpha_n \cap J_{k,1} = \Big\{\frac{2j-1}{35} : 1 \le j \le 4\Big\} = \Big\{\frac{1}{35}, \frac{3}{35}, \frac{1}{7}, \frac{1}{5}\Big\} \text{ with } V(P; \{\alpha_n \cap J_{k,1}, J_{k,1}\}) = \frac{1}{18375},$$
$$\alpha_n \cap J_{k,2} = \Big\{\frac{1}{5} + \frac{q-1}{20} : 1 \le q \le 5\Big\} = \Big\{\frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}\Big\} \text{ with } V(P; \{\alpha_n \cap J_{k,2}, J_{k,2}\}) = \frac{1}{24000},$$
$$\alpha_n \cap J_{k,3} = \Big\{\frac{2}{5} + \frac{q-1}{15} : 1 \le q \le 4\Big\} = \Big\{\frac{2}{5}, \frac{7}{15}, \frac{8}{15}, \frac{3}{5}\Big\} \text{ with } V(P; \{\alpha_n \cap J_{k,3}, J_{k,3}\}) = \frac{1}{13500},$$
$$\alpha_n \cap J_{k,4} = \Big\{\frac{3}{5} + \frac{q-1}{20} : 1 \le q \le 5\Big\} = \Big\{\frac{3}{5}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}\Big\} \text{ with } V(P; \{\alpha_n \cap J_{k,4}, J_{k,4}\}) = \frac{1}{24000},$$
$$\alpha_n \cap J_{k,5} = \Big\{\frac{4}{5} + \frac{q-1}{20} : 1 \le q \le 5\Big\} = \Big\{\frac{4}{5}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, 1\Big\} \text{ with } V(P; \{\alpha_n \cap J_{k,5}, J_{k,5}\}) = \frac{1}{24000}.$$

Hence, using the expressions in (8), we obtain

$$\alpha_n = \Big\{\frac{1}{35}, \frac{3}{35}, \frac{1}{7}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}, \frac{7}{15}, \frac{8}{15}, \frac{3}{5}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, 1\Big\} \text{ with}$$
$$V_n = \sum_{j=1}^{5} V(P; \{\alpha_n \cap J_{k,j}, J_{k,j}\}) = \frac{2683}{10584000}.$$

5. Conditional quantization for uniform distributions on the boundaries of regular polygons inscribed in a unit circle

Let the equation of the unit circle be $x_1^2 + x_2^2 = 1$. Let $A_1A_2 \cdots A_m$ be a regular m-sided polygon for some $m \ge 3$ inscribed in the circle as shown in Figure 1. Let ℓ be the length of each side. Then the length of the boundary of the polygon is given by ℓm. Let P be the uniform distribution defined on the boundary of the polygon. Then, the probability density function (pdf) f for the uniform distribution P is given by $f(x_1, x_2) = \frac{1}{m\ell}$ for all $(x_1, x_2) \in A_1A_2 \cdots A_m$, and zero otherwise. Let θ be the central angle subtended by each side of the polygon. Then, we know $\theta = \frac{2\pi}{m}$. Let the polar angles of the vertices A_j of the polygon be given by θ_j, where $1 \le j \le m$. Without any loss of generality, due to rotational symmetry, we can always assume that the side A_1A_2 of the polygon is parallel to the x_1-axis, as shown in Figure 1. Then, we have

$$\theta_1 = \frac{3\pi}{2} - \frac{\theta}{2} = \frac{3\pi}{2} - \frac{\pi}{m} \text{ and } \theta_j = \theta_1 + (j-1)\frac{2\pi}{m} \text{ for } 2 \le j \le m.$$

Let β be the set of all vertices of the polygon, i.e.,

$$\beta := \{(\cos\theta_j, \sin\theta_j) : 1 \leq j \leq m\}.$$

Notice that the Cartesian coordinates of the vertices A_1 and A_2 are given by, respectively, $(-\sin\frac{\pi}{m}, -\cos\frac{\pi}{m})$ and $(\sin\frac{\pi}{m}, -\cos\frac{\pi}{m})$. Hence,

$$A_1A_2 = \{(t, -\cos\frac{\pi}{m}) : -\sin\frac{\pi}{m} \leq t \leq \sin\frac{\pi}{m}\}.$$

Moreover, the length ℓ of each side is given by $\ell = 2\sin\frac{\pi}{m}$. Let α_n be a conditional optimal set of n-points for P with respect to the conditional set β, i.e., α_n exists for all $n \geq m$. Let

$$\text{card}(\alpha_n \cap A_iA_{i+1}) = n_i \text{ where } 1 \leq i \leq m \text{ and } A_{m+1} \text{ is identified as } A_1. \tag{9}$$

Then, notice that

$$n_i \geq 2 \text{ for all } 1 \leq i \leq m \text{ and } n_1 + n_2 + \cdots + n_m = n + m,$$

as each of the vertices are counted two times.

Proposition 5.1. *Let P be the uniform distribution defined on the boundary of the regular m-sided polygon inscribed in the unit circle. Let card$(\alpha_n \cap A_1A_2) = n_1$. Then,*

$$\alpha_n \cap A_1A_2 = \Big\{\Big(-\sin\frac{\pi}{m} + \frac{2(j-1)\sin\frac{\pi}{m}}{n_1 - 1}, -\cos\frac{\pi}{m}\Big) : 1 \leq j \leq n_1\Big\} \tag{10}$$

with the corresponding distortion error

$$V(P; \{\alpha_n \cap A_1A_2, A_1A_2\}) = \frac{\sin^2\frac{\pi}{m}}{3m(n_1-1)^2}. \tag{11}$$

Proof. Notice that the line segment A_1A_2 is parallel to the x_1-axis and lies on the line $x_2 = -\cos\frac{\pi}{m}$. Hence, replacing c by $(-\sin\frac{\pi}{m}, -\cos\frac{\pi}{m})$ and d by $(\sin\frac{\pi}{m}, -\cos\frac{\pi}{m})$, by Proposition 2.2, we obtain

$$\alpha_n \cap A_1A_2 = \Big\{\Big(c_j, -\cos\frac{\pi}{m}\Big) : 1 \leq j \leq n_1\Big\}, \text{ where } c_j = -\sin\frac{\pi}{m} + \frac{2(j-1)\sin\frac{\pi}{m}}{m}.$$

Recall $\ell = 2\sin\frac{\pi}{m}$. Hence,

$$\begin{aligned}V(P; \{\alpha_n \cap A_1A_2, A_1A_2\}) &= \frac{1}{m\ell}\Big(2\int_{c_1}^{\frac{1}{2}(c_1+c_2)} \rho((t, -\cos\frac{\pi}{m}), (c_1, -\cos\frac{\pi}{m}))\,dt \\ &\qquad + (n_1 - 2)\int_{\frac{1}{2}(c_1+c_2)}^{\frac{1}{2}(c_2+c_3)} \rho((t, -\cos\frac{\pi}{m}), (c_2, -\cos\frac{\pi}{m}))\,dt\Big) \\ &= \frac{\sin^2\frac{\pi}{m}}{3m(n_1-1)^2},\end{aligned}$$

which yields the proposition. $\square$

The following lemma, which is similar to Lemma 4.2, is also true here.

Lemma 5.2. *Let $n \in \mathbb{N}$ be such $n \geq m$. Let n_i be the positive integers as defined by (9). Then, for $1 \leq i \neq j \leq m$, $|n_i - n_j| = 0$ or 1.*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformations such that for all $(x, y) \in \mathbb{R}^2$, we have

$$T(x, y) = (ax + by, cx + dy),$$

where

$$a = \frac{1}{2}\Big(\sin\frac{3\pi}{m}\csc\frac{\pi}{m} - 1\Big),\ b = \frac{1}{2}\Big(-\sin\frac{3\pi}{m}\sec\frac{\pi}{m} - \tan\frac{\pi}{m}\Big),$$
$$c = \frac{1}{2}\Big(\cot\frac{\pi}{m} - \cos\frac{3\pi}{m}\csc\frac{\pi}{m}\Big),\ \text{and } d = \frac{1}{2}\Big(\cos\frac{3\pi}{m}\sec\frac{\pi}{m} + 1\Big).$$

Also, for any $j \in \mathbb{N}$, by T^j it is meant the composition mapping $T^j = T \circ T \circ T \circ \cdots j$-times. If $j = 0$, i.e., by T^0 it is meant the identity mapping on $\mathbb{R}^2$. Then, notice that

$$T^{i-1}(A_1A_2) = A_iA_{i+1} \text{ for } 1 \leq i \leq m, \text{ where } A_{m+1} \text{ is identified as } A_1.$$

Let us now give the following theorem, which is the main theorem is this section. This theorem helps us to determine the conditional optimal sets of n-points and the nth conditional quantization errors for all $n \in \mathbb{N}$ with $n \geq m$.

Theorem 5.3. *For $n \geq m$, let α_n be a conditional optimal set of n-points such that $n = mk + \ell$ some $k, \ell \in \mathbb{N}$ and $0 \leq \ell < m$. Then, identifying A_{m+1} by A_1, we have*

(i) if $\ell = 0$, then card$(\alpha_n \cap A_iA_{i+1}) = k + 1$ for $1 \leq i \leq m$;

(ii) if $1 \leq \ell < m$, then card$(\alpha_n \cap A_iA_{i+1}) = k + 2$ for $i \in \{i_1, i_2, \cdots, i_\ell\}$ and card$(\alpha_n \cap A_iA_{i+1}) = k + 1$ for $i \in \{1, 2, \cdots, m\} \setminus \{i_1, i_2, \cdots, i_\ell\}$, where $\{i_1, i_2, \cdots, i_\ell\}$ is any subset of ℓ elements of the set $\{1, 2, \cdots, m\}$.

Proof. The proof follows as a consequence of Lemma 5.2. □

5.4. **Conditional optimal sets of n-points and the nth conditional quantization errors.** Let $n \geq m$ be a positive integer. To determine the conditional optimal sets of n-points and the nth conditional quantization errors, first using Theorem 5.3, we determine the values of n_i, where $n_i =$ card$(\alpha_n \cap A_iA_{i+1})$ and A_{m+1} is identified as A_1. Recall Proposition 5.1. For each n_i assume that card$(\alpha_n \cap A_1A_2) = n_i$, and calculate $\alpha_n \cap A_1A_2$ and $V(P; \{\alpha_n \cap A_1A_2, A_1A_2\})$, denote them by $\alpha_n \cap A_1A_2(n_i)$ and $V(P; \{\alpha_n \cap A_1A_2, A_1A_2\})(n_i)$, respectively. Now, recall the affine transformation. Since the affine transformation, considered in this section, preserves the length, the distortion errors do not change under the affine transformation. Hence, for each n_i, we obtain $\alpha_n \cap A_iA_{i+1}$ and $V(P; \{\alpha_n \cap A_iA_{i+1}, A_iA_{i+1}\})$ as follows:

$$\alpha_n \cap A_iA_{i+1} = T^{i-1}\Big(\alpha_n \cap A_1A_2(n_i)\Big), \text{ and}$$
$$V(P; \{\alpha_n \cap A_iA_{i+1}, A_iA_{i+1}\}) = V(P; \{\alpha_n \cap A_1A_2, A_1A_2\})(n_i).$$

Once $\alpha_n \cap A_iA_{i+1}$ and $V(P; \{\alpha_n \cap A_iA_{i+1}, A_iA_{i+1}\})$ are obtained, we calculate the conditional optimal sets α_n and the nth conditional quantization errors using the following formulae:

$$\alpha_n = \bigcup_{i=1}^{m} T^{i-1}\Big(\alpha_n \cap A_1A_2(n_i)\Big) = \bigcup_{i=1}^{m} T^{i-1}\Big\{\Big(-\sin\frac{\pi}{m} + \frac{2(j-1)\sin\frac{\pi}{m}}{n_i - 1}, -\cos\frac{\pi}{m}\Big) : 1 \leq j \leq n_i\Big\}$$

and

$$V_n = \sum_{i=1}^{m} V(P; \{\alpha_n \cap A_1A_2, A_1A_2\})(n_i) = \sum_{i=1}^{m} \frac{\sin^2\frac{\pi}{m}}{3m(n_i - 1)^2}.$$

□

Remark 5.5. Since the conditional quantization dimension is same as the quantization dimension (see [PR4]), and it is well-known that the quantization dimension of an absolutely continuous probability measure equals the Euclidean dimension of the underlying space, we can assume that the conditional quantization dimension of P is one, i.e., $D(P) = 1$.

Let us now give the following proposition.

Proposition 5.6. *Let α_n be an optimal set of n-points for P such that $n = mk$, where $k \in \mathbb{N}$. Then,*

$$V_n = \frac{1}{3k^2}\sin^2\frac{\pi}{m}.$$

Proof. Let $n = mk$ for some $k \in \mathbb{N}$. Let n_i be the positive integers as defined by (9). Then, by Lemma 5.2, we can say that

$$n_1 = n_2 = \cdots = n_m = k + 1.$$

Notice that each n_i equals $k+1$. It happens because α_n contains m distinct elements from each side, but in each n_i both the end points are counted. Hence, by (11), we have $V_n = \frac{1}{3k^2}\sin^2\frac{\pi}{m}$. Thus, the proof of the proposition is complete. □

Theorem 5.7. *Let P be the uniform distribution on the boundary of a regular m-sided polygon inscribed in a unit circle. Then, the conditional quantization coefficient for P exists as a finite positive number and equals $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$, i.e., $\lim\limits_{n\to\infty} n^2(V_n - V_\infty) = \frac{1}{3}m^2\sin^2(\frac{\pi}{m})$.*

Proof. Let $n \in \mathbb{N}$ be such that $n \geq m$. Then, there exists a unique positive integer $\ell(n) \geq 2$ such that $m\ell(n) \leq n < m(\ell(n)+1)$. Then,

$$(m\ell(n))^2 V_{m(\ell(n)+1)} < n^2 V_n < (m(\ell(n)+1))^2 V_{m\ell(n)}. \tag{12}$$

Moreover, by squeeze theorem, we have $V_\infty = \lim_{n\to\infty} V_n = 0$. Recall Proposition 5.6. We have

$$\lim_{n\to\infty}(m\ell(n))^2(V_{m(\ell(n)+1)} - V_\infty) = \lim_{n\to\infty}(m\ell(n))^2\frac{1}{3(\ell(n)+1)^2}\sin^2\frac{\pi}{m} = \frac{1}{3}m^2\sin^2\frac{\pi}{m},$$

and

$$\lim_{n\to\infty}(m(\ell(n)+1))^2(V_{m\ell(n)} - V_\infty) = \lim_{n\to\infty}(m(\ell(n)+1))^2\frac{1}{3(\ell(n))^2}\sin^2\frac{\pi}{m} = \frac{1}{3}m^2\sin^2\frac{\pi}{m},$$

and hence, by (12), using squeeze theorem, we have $\lim\limits_{n\to\infty} n^2(V_n - V_\infty) = \frac{1}{3}m^2\sin^2\frac{\pi}{m}$, i.e., the conditional quantization coefficient exists as a finite positive number which equals $\frac{1}{3}m^2\sin^2\frac{\pi}{m}$. Thus, the proof of the theorem is complete. □

Remark 5.8. It is known that for an absolutely continuous probability measure, the quantization dimension equals the Euclidean dimension of the underlying space, and the quantization coefficient exists as a finite positive number (see [BW]). Since the conditional quantization dimension is same as the quantization dimension, and the conditional quantization coefficient is same as the quantization coefficient (see [PR4]), by Theorem 5.7, we can conclude that the quantization coefficient for the uniform distribution defined on the boundary of a regular m-sided polygon inscribed in a unit circle is $\frac{1}{3}m^2\sin^2\frac{\pi}{m}$, which depends on m and is an increasing function of m. Thus, we can conclude that for absolutely continuous probability measures given in an Euclidean space, the quantization dimensions remain constant and it is equal to the dimension of the underlying space, but the quantization coefficients can be different.

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