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Exact solutions of stochastic Burgers–Korteweg de Vries type equation with variable coefficients

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A B S T R A C T

We will present exact solutions for three variations of the stochastic Korteweg de Vries–Burgers (KdV–Burgers) equation featuring variable coefficients. In each variant, white noise exhibits spatial uniformity, and the three categories include additive, multiplicative, and advection noise. Across all cases, the coefficients are time-dependent functions. Our discovery indicates that solving certain deterministic counterparts of KdV–Burgers equations and composing the solution with a solution of stochastic differential equations leads to the exact solution of the stochastic Korteweg de Vries–Burgers (KdV–Burgers) equations.

1. Introduction

The majority of physical and biological systems exhibit nonhomogeneity, often influenced by environmental fluctuations and the existence of nonuniform mediums. Consequently, the nonlinear equations relevant to practical applications typically involve coefficients that vary spatially and/or temporally along with stochastic terms. Reaction–diffusion equations are crucial in modeling heat diffusion and reaction processes in nonlinear acoustics, biology, chemistry, genetics, and various other research domains. However, like numerous mathematical models representing real-world phenomena, solving this problem explicitly poses a considerable challenge.

The Burgers–Korteweg de Vries equation (Burgers–KdV) arises from many physical contexts,[1] for example, the propagation of undular wells in shallow water,[2] the flow of liquids containing gas bubbles,[3] the propagation of waves in an elastic tube filled with a viscous fluid,[4] and weakly nonlinear plasma waves with certain dissipative effects.[5,6] It is also used as a non-linear model in crystal lattice theory, non-linear circuit theory, and turbulence.[7,8] Including stochastic white noise introduces real-life scenarios in which the system's parameters are influenced by environmental uncertainties or noise, leading to the need for a stochastic treatment.[9,10]

The goal of this paper is to introduce exact solutions for stochastic Burgers–KdV equation with variable coefficients with space-uniform white noise. We consider the following three different stochastic KdV–Burgers equations (additive, advection, and multiplicative noise) with a space-uniform white noise of the form

$$du = (\delta(t)\partial_{zzz}u + \beta(t)u\partial_z u + \mu(t)\partial_{zz}u + \alpha(t)\partial_z u + \gamma(t)u)dt + \sigma(t)\partial_z u dW_t \quad (1.1)$$

and

$$du = (\delta(t)\partial_{zzz}u + \beta(t)u\partial_z u + \mu(t)\partial_{zz}u + \alpha(t)\partial_z u + \gamma(t)u)dt + \sigma(t)dW_t \quad (1.2)$$

for $t \in [t_0, T]$ and $z \in \mathbb{R}$ with $u(0, z) = \phi(z)$ for $z \in \mathbb{R}$. We also consider a linear PDE in the KdV form of

$$du = (\delta(t)\partial_{zzz}u + \mu(t)\partial_{zz}u + \alpha(t)\partial_z u + \gamma(t)u)dt + \sigma(t)u dW_t \quad (1.3)$$

for $t \in [t_0, T]$ and $z \in \mathbb{R}$ with $u(0, z) = \phi(z)$ for $z \in \mathbb{R}$.

Our contribution in this paper is to extend the existing methodologies by deriving exact solutions for stochastic KdV–Burgers equations with variable coefficients and spatially uniform white noise. Using Itô calculus and transformation techniques, we decompose the stochastic equation into two equations, a deterministic PDE and a stochastic differential equation, and solve the deterministic counterparts, a method inspired by the works of Refs. 11, 12, see also Ref. 13. Our approach not only enhances theoretical understanding but also addresses more realistic scenarios that are highly relevant in the modeling of complex physical and biological systems.

In addition, we provide numerical simulations to demonstrate the practical implications of our solutions. These simulations serve as crucial tools for validating theoretical predictions and bridging the gap between mathematical models and their real-life applications.[14]

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This paper is organized as follows: In Section 2, we recall two lemmas that provide details and properties in solving SDEs and execute numerical simulations. In Section 3, We present exact solutions for stochastic Korteweg de Vries-Burgers (KdV–Burgers) Eqs. (1.1)–(1.3) featuring variable coefficients. In each variant, white noise exhibits spatial uniformity, and the three categories include additive, multiplicative, and advection noise. Across all cases, the coefficients are time-dependent functions. Our discovery indicates that solving certain deterministic counterparts of KdV–Burgers equations and composing the solution with a solution of stochastic differential equations leads to the exact solution of the stochastic Korteweg de Vries-Burgers (KdV–Burgers) equations. We provide several examples.

2. Preliminaries

Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ for which the Brownian motion $\{W_t, t \geq 0\}$ is defined and $E(W_s W_t) = \min(s,t)$ for all $s, t \geq 0$. Also consider the filtration $\mathcal{F}_t := \sigma(W_s : s \leq t)$ being the smallest $\sigma-$algebra to which W_s is measurable for $s \leq t$.

Then consider the stochastic differential equation (SDE) with variable coefficients[15,16]

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t, \tag{2.1}$$

with initial state X_{t_0} and for $t \in [t_0, T]$. The SDE in (2.1) has a general solution given by

$$X_t = X_{t_0} + \int_{t_0}^t \alpha(s, X_s)ds + \int_{t_0}^t \beta(s, X_s)dW_s,$$

for $t \leq T$. If $\alpha(t) := \alpha(t, X_t)$ and $\beta(t) := \beta(t, X_t)$, then Eq. (2.1) has a general solution given by

$$X_t = X_{t_0} + \int_{t_0}^t \alpha(s)ds + \int_{t_0}^t \beta(s)dW_s$$

for $t \leq T$. The process $\{W_t; t \geq 0\}$ is a Wiener process with respect to filtration $\{\mathcal{F}_t; t \geq 0\}$. The initial state X_{t_0} is $\mathcal{F}_{t_0}$ and the functions $\alpha(t)$ and $\beta(t)$ are Lebesgue measurable and bounded on $[t_0, T]$. The latter implies both the global Lipschitz and linearity growth conditions required to ensure the existence and (pathwise) uniqueness of a strong solution to (2.1), Ref. 15.

Let X_t and Y_t be any two diffusion processes such as those defined by the solution of Eq. (2.1). If $F(x,y)$ is a differentiable function that works as a transformation for two processes X_t and Y_t, then the general bi-variate Itô formula[15] gives

$$\begin{aligned} dF(X_t, Y_t) = {} & \partial_x F(X_t, Y_t)dX_t + \partial_y F(X_t, Y_t)dY_t + \frac{1}{2}\partial_{xx}F(X_t, Y_t)(dX_t)^2 \\ & + \frac{1}{2}\partial_{yy}F(X_t, Y_t)(dY_t)^2 + \partial_{xy}F(X_t, Y_t)dX_t dY_t. \end{aligned} \tag{2.2}$$

$F(x,y)$ is a differentiable function.

The following two lemmas are crucial to identify the solutions of the SDEs; they were introduced previously in Ref. 13. These two lemmas are fundamental for our simulations. We also use the lemmas to simulate the processes with $X_0 = x$ and then compose the exact solutions with the simulations based on Lemma 3.

Lemma 1.

(1) The stochastic process X_t solving

$$dX_t = C(t)dt + E(t)dW_t$$

with $X_{t_0} \sim N(x_{t_0}, \sigma_0^2)$ independent of W_t, is a non-stationary Gaussian process with mean $x_{t_0} + \int_{t_0}^t C(s)ds$ and variance $\sigma^2(X_t) = \sigma_0^2 + \int_{t_0}^t E^2(s)ds$.

(2) The covariance of the two processes X_t and W_t is

$$\sigma(X_t, W_t) = \int_{t_0}^t E(s)ds.$$

(3) Moreover,

$$[X_t | W_t = w] \sim N\left(x_{t_0} + \int_{t_0}^t C(s)ds + \frac{w\int_{t_0}^t E(s)ds}{t}, V_1^2(t)\right)$$

where $V_1^2(t) = \sigma_0^2 + \int_{t_0}^t E^2(s)ds - \frac{(\int_{t_0}^t E(s)ds)^2}{t}$.

Lemma 2.

(1) The position random process $Z_t := z + \int_{t_0}^t \bar{B}(s)K(s)dW_s$ solves the Langevin-type second-order SDE

$$\ddot{Z}_t = \frac{B'(t)}{B(t)}\dot{Z}_t + B(t)K(t)\dot{W}_t, \ t \in [t_0, T]$$

with initial state $Z_{t_0} = z$, where $\bar{B}(s) = \int_s^t B(r)dr$ for $t > s$.

(2) The process Z_t is a nonstationary Gaussian process with mean z and variance $\sigma^2(Z_t) = \int_{t_0}^t (\bar{B}(s)K(s))^2 ds$.

(3) Meanwhile,

$$\dot{Z}_t := B(t)\left(\int_{t_0}^t K(r)dW_r\right).$$

(4) The process $\dot{Z}_t$ is a non-stationary Gaussian process with mean zero and variance $\sigma^2(\dot{Z}_t) = (B(t))^2 \int_{t_0}^t (K(s))^2 ds$.

(5) The covariance of Z_t and W_t is

$$\sigma(Z_t, W_t) = \int_{t_0}^t \bar{B}(s)K(s)ds$$

and

$$\sigma(\dot{Z}_t, W_t) = B(t)\left(\int_{t_0}^t K(s)ds\right).$$

(6) The conditional distributions are given by

$$[Z_t | W_t = w] \sim N\left(z + w\frac{\int_{t_0}^t \bar{B}(s)K(s)ds}{t}, \int_{t_0}^t (\bar{B}(s)K(s))^2 ds - \frac{(\int_{t_0}^t \bar{B}(s)K(s)ds)^2}{t}\right),$$

and

$$[\dot{Z}_t | W_t = w] \sim N\left(\frac{wB(t)\left(\int_{t_0}^t K(s)ds\right)}{t}, (B(t))^2\left[\int_{t_0}^t (K(s))^2 ds - \frac{\left(\int_{t_0}^t K(s)ds\right)^2}{t}\right]\right).$$

3. Stochastic Burgers-KdV equation

In this Section, through the use of Ito calculus and interesting transformations. We present exact solutions for stochastic Korteweg de Vries-Burgers (KdV–Burgers) Eqs. (1.1)–(1.3) featuring variable coefficients[17]. As the following lemma shows, solving certain deterministic counterparts of KdV–Burgers equations and composing the solution with a solution of stochastic differential equations leads to the exact solution of the stochastic Korteweg de Vries-Burgers (KdV–Burgers) equations. We provide several examples.

Lemma 3. *Let $\alpha, \beta, \gamma, \delta, \mu, \sigma \in C^b([t_0, T])$ be bounded functions on $[t_0, T]$. Assume that $\beta(t) > 0$ for all $t \in [t_0, T]$. Then we have*

(1) The stochastic Burgers–KdV Eq. (1.1) has a solution $u(t,z) = U(t, X_t)$, where $U(t,x)$ is the solution of

$$\partial_t U = \delta(t)\partial_{xxx}U + (\mu(t) - \frac{1}{2}\sigma^2(t))\partial_{xx}U + \beta(t)U\partial_x U + \gamma(t)u, \quad U(0,x) = \phi(x) \tag{3.1}$$

and X_t is the solution of

$$dX_t = \alpha(t)dt + \sigma(t)dW_t, \tag{3.2}$$

with initial state $X_{t_0} = z$ and for $t \in [t_0, T]$.

(2) The stochastic Burgers–KdV equation with the initial value problem (1.2) has a solution

$$u(t,z) = R(t)\left(V(t,Z_t) + \frac{1}{\mathfrak{B}(t)}\dot{Z}_t\right), \tag{3.3}$$

where $V(t,x)$ is the solution of

$$\partial_t V = \delta(t)\partial_{xxx}V + \mu(t)\partial_{xx}V + \mathfrak{B}(t)V\partial_x V + \alpha(t)\partial_x V, \quad V(0,x) = \phi(x), \tag{3.4}$$

and Z_t is the solution of a second-order stochastic differential equation

$$\ddot{Z}_t = \frac{\mathfrak{B}'(t)}{\mathfrak{B}(t)}\dot{Z}_t + \frac{\mathfrak{B}(t)\sigma(t)}{R(t)}\dot{W}_t, \tag{3.5}$$

with initial state $Z_{t_0} = z$ and for $t \in [t_0, T]$. Also, $R(t) = \exp(\int_{t_0}^t \gamma(s)ds)$ and $\mathfrak{B}(t) = \beta(t)R(t)$.

Proof. For (1), apply Itô's formula to the X_t solution of (3.2) with the transformation $U(t,x)$ that solves the deterministic KdV–Burgers Eq. (3.1)

$$dU(t,X_t) = f(t,X_t)dt + g(t,X_t)dW_t, \tag{3.6}$$

where

$$f(t,x) = \partial_t U(t,x) + \alpha(t)\partial_x U(t,x) + \frac{1}{2}\sigma^2(t)\partial_{xx}U(t,x).$$

Note that,

$$\partial_t U(t,x) = \delta(t)\partial_{xxx}U + (\mu(t) - \frac{1}{2}\sigma^2(t))\partial_{xx}U(t,x) + \beta(t)U(t,x)\partial_x U(t,x) + \gamma(t)U(t,x).$$

Therefore,

$$f(t,x) = \delta(t)\partial_{xxx}U + \mu(t)\partial_{xx}U(t,x) + \beta(t)U(t,x)\partial_x U(t,x) + \alpha(t)\partial_x U(t,x) + \gamma(t)U(t,x).$$

Observe that

$$g(t,x) = \sigma(t)\partial_x U(t,x),$$

which proves (1).

To prove (2), let us take $u(t,z) = R(t)\,L(t,z)$. By the bi-variate general of Itô's formula we get

$$du(t,z) = R'(t)L(t,z)dt + R(t)dL(t,z). \tag{3.7}$$

In order to prove (1.2) we need to show

$$dL(t,z) = (\delta(t)\partial_{zzz}L(t,z) + \mu(t)\partial_{zz}L(t,z) + \mathfrak{B}(t)L(t,z)\partial_z L(t,z) + \alpha(t)\partial_z L(t,z))dt + \frac{\sigma(t)}{R(t)}dW_t. \tag{3.8}$$

Before proving (3.8), let us show how this statement will finish the proof of (1.2). Recalling that $R(t) = \exp(\int_{t_0}^t \gamma(s)ds)$ and using (3.8), from (3.7) we obtain

$$du = R(t)(\delta(t)\partial_{zzz}L(t,z) + \mu(t)\partial_{zz}L(t,z) + \mathfrak{B}(t)L(t,z)\partial_z L(t,z) + \alpha(t)\partial_z L(t,z))dt + \gamma(t)R(t)L(t,z)dt + \sigma(t)dW_t.$$

Hence, we obtain (3.7) as we wanted.

Let us proceed to prove (3.8). Eq. (3.5) can be written as a system of equations of the following form

$$dN_t = \frac{\mathfrak{B}'(t)}{\mathfrak{B}(t)}N_t dt + \frac{\mathfrak{B}(t)\sigma(t)}{R(t)}dW_t, \tag{3.9}$$

$$dZ_t = N_t dt. \tag{3.10}$$

Let $V(t,x)$ be the solution of

$$\partial_t V = \delta(t)\partial_{xxx}V + \mu(t)\partial_{xx}V + \mathfrak{B}(t)V\partial_x V + \alpha(t)\partial_x V, \quad V(0,x) = \phi(x).$$

Using the bivariate general Itô formula for $V(t,Z_t)$, we obtain

$$dV(t,Z_t) = \partial_t V(t,Z_t)dt + \partial_x V(t,Z_t)dZ_t$$

since $(dt)^2 = 0$, $dtdZ_t = N_t(dt)^2 = 0$ and $(dZ_t)^2 = (N_t)^2(dt)^2 = 0$. Thus,

$$dV(t,Z_t) = \big(\delta(t)\partial_{xxx}V(t,Z_t) + \mu(t)\partial_{xx}V(t,Z_t) + \mathfrak{B}(t)V(t,Z_t)\partial_x V(t,Z_t) + \alpha(t)\partial_x V(t,Z_t)\big)dt + \partial_x V(t,Z_t)N_t dt. \tag{3.11}$$

If we define $L(t,z) = V(t,Z_t) + \frac{1}{\mathfrak{B}(t)}\dot{Z}_t$, we can rewrite the right-hand side of Eq. (3.11) as

$$(\delta(t)\partial_{zzz}L(t,z) + \mu(t)\partial_{zz}L(t,z) + \mathfrak{B}(t)(L(t,z) - \frac{1}{\mathfrak{B}(t)}\dot{Z}_t)\partial_z L(t,z) + \alpha(t)\partial_z L(t,z) + \partial_z L(t,z)N_t)dt$$

$$= (\delta(t)\partial_{zzz}L(t,z) + \mu(t)\partial_{zz}L(t,z) + \mathfrak{B}(t)L(t,z)\partial_z L(t,z) + \alpha(t)\partial_z L(t,z))dt.$$

It follows that

$$dL(t,z) = (\delta(t)\partial_{zzz}L(t,z) + \mu(t)\partial_{zz}L(t,z) + \mathfrak{B}(t)L(t,z)\partial_z L(t,z) + \alpha(t)\partial_z L(t,z))dt + d\left(\frac{1}{\mathfrak{B}(t)}\dot{Z}_t\right).$$

Applying the bivariate general Itô formula Itô's formula once again, we obtain

$$d\left(\frac{1}{\mathfrak{B}(t)}N_t\right) = \frac{-\mathfrak{B}'(t)}{(\mathfrak{B}(t))^2}N_t dt + \frac{1}{\mathfrak{B}(t)}dN_t$$

since $(dt)^2 = 0$, $dtdN_t = 0$ and $\frac{d^2x/\mathfrak{B}(t)}{dx^2} = 0$. Therefore, we have

$$d\left(\frac{1}{\mathfrak{B}(t)}\dot{Z}_t\right) = \frac{-\mathfrak{B}'(t)}{(\mathfrak{B}(t))^2}N_t dt + \frac{1}{\mathfrak{B}(t)}\left(\frac{\mathfrak{B}'(t)}{\mathfrak{B}(t)}N_t dt + \frac{\mathfrak{B}(t)\sigma(t)}{R(t)}dW_t\right) = \frac{\sigma(t)}{R(t)}dW_t.$$

Hence, we obtain (3.8) as we wanted. □

Remark 1. If δ, μ and β are constants, then the KdV-Burgers equation

$$\partial_t U = \delta\partial_{xxx}U + \mu U_{xx} + \beta U\partial_x U \tag{3.12}$$

has the following explicit solution:

$$U(t,x) = \frac{3\mu^2}{25\beta\delta}\operatorname{sech}^2\left(\frac{\mu}{10\delta}x - \frac{6\mu^3}{250\delta^2}t\right) - \frac{6\mu^2}{25\beta\delta}\tanh\left(\frac{\mu}{10\delta}x - \frac{6\mu^3}{250\delta^2}t\right) + \frac{6\mu^2}{25\beta\delta}. \tag{3.13}$$

Part (3) of Lemma 1 and part (6) of Lemma 2 are used to simulate the stochastic processes for the solutions in Lemma 3 with its two parts. The following proposition will provide solutions for linear KdV-type equations.

Proposition 1. *Let us consider the stochastic process*

$$X_t(x) = x + \frac{1}{2}\int_0^t \sigma^2(s)ds - \int_0^t \sigma(s)W_s \tag{3.14}$$

for $x \in \mathbb{R}$ and the equation

$$du(t,x) = f(u(t,x))dt + \sigma(t)u(t,x)dW_t, \tag{3.15}$$

and f is linear function, Eq. (3.15) can be reduced to

$$dv(t,x) = f(v(t,x))dt$$

through the transformation

$$v(t,x) = u(t,x)e^{X_t(x)}.$$

Proof: By Itô's formula we see that

$$de^{X_t} = e^{X_t}(\sigma^2(t)dt - \sigma(t)dW_t). \tag{3.16}$$

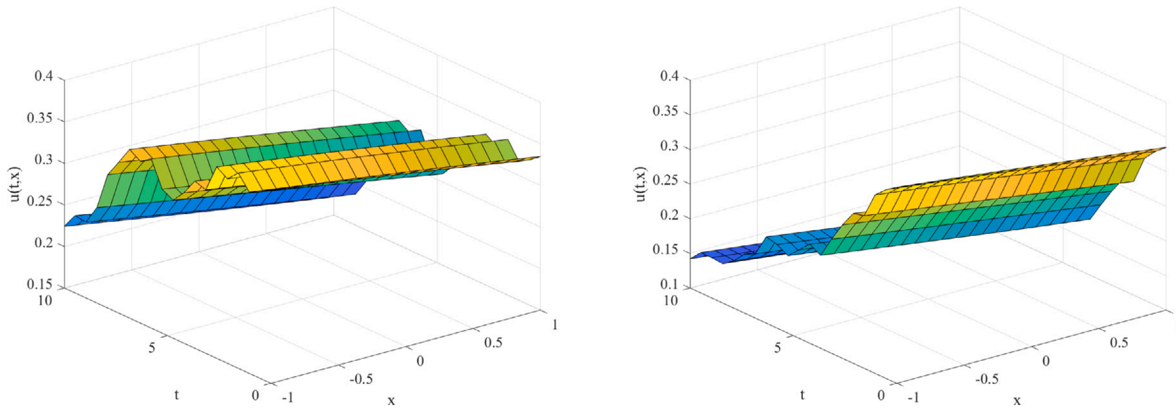

Fig. 1. Two realizations of the stochastic process in Eq. (3.22).

And by product rule $dv = ude^{X_t} + e^{X_t}du + de^{X_t}du$, *and after replacing* (3.15), (3.14) *and* (3.16), *we obtain*

$$dv = ue^{X_t}\left(\sigma^2 dt - \sigma dW_t\right) + e^{X_t}\left(f(u)dt + \sigma u dW_t\right) + \left(f(u)dt + \sigma u dW_t\right)e^{X_t}\left(\sigma^2 dt - \sigma dW_t\right).$$

Finally, using standard Itô calculus rules and simplifying, we obtain

$$dv = f(v)dt.$$

The following formula will be useful for the following examples, see Chapter 7 by Calin:

$$\int_a^b f(t)dW_t = f(t)W_t|_a^b - \int_a^b f'(t)W_t dt.$$

Next, we provide several examples.

Example 1. Consider the stochastic KdV–Burgers equation

$$du = (\delta\partial_{zzz}u + \beta u\partial_z u + \mu\partial_{zz}u + \alpha(t)\partial_z u)dt + \sigma\partial_z u\, dW_t, \tag{3.17}$$

where δ, β, σ and μ are real constants. By Lemma 1 part (1), Eq. (3.17) has a solution $u(t,z) = U(t, X_t)$, such that $U(t,x)$ is the solution of

$$\partial_t U = \delta\partial_{xxx}U + \left(\mu - \frac{\sigma^2}{2}\right)U_{xx} + \beta U\partial_x U. \tag{3.18}$$

In particular, the KdV–Burgers equation ($\delta = \beta = \mu = \alpha = \sigma = 1$)

$$du = (\partial_{zzz}u + u\partial_z u + \partial_{zz}u + \partial_z u)dt + \partial_z u\, dW_t \tag{3.19}$$

has a solution $u(t,z) = U(t, X_t)$, such that $U(t,x)$ is given by

$$U(t,x) = \frac{3}{25}\operatorname{sech}^2\left(\frac{1}{10}x - \frac{6}{250}t\right) - \frac{6}{25}\tanh\left(\frac{1}{10}x - \frac{6}{250}t\right) + \frac{6}{25} \tag{3.20}$$

and $X_t = z + t + W_t$ is the solution of ($\alpha = 1$ and $\sigma = 1$)

$$dX_t = dt + dW_t, \tag{3.21}$$

with initial state $X_0 = z$ and for $t \in [0,1]$.

Finally, the explicit solution of (3.19) is given by

$$u(t,z) = U(t,X_t) = \frac{3}{25}\operatorname{sech}^2\left(\frac{z+t+W_t}{10} - \frac{6t}{250}\right) - \frac{6}{25}\tanh\left(\frac{z+t+W_t}{10} - \frac{6t}{250}\right) + \frac{6}{25} \tag{3.22}$$

for $t \in [0,1]$ and $z \in \mathbb{R}$.

Fig. 1 shows two realizations of the general solution in (3.22).

Example 2. Consider another stochastic KdV–Burgers equation

$$du = (\partial_{zzz}u + \partial_{zz}u + u\partial_z u)dt + \sigma(t)dW_t, \tag{3.23}$$

$$u(0,z) = \frac{3}{25}\operatorname{sech}^2\left(\frac{z}{10}\right) - \frac{6}{25}\tanh\left(\frac{z}{10}\right) + \frac{6}{25} \tag{3.24}$$

for $t \in [0,1]$. By Lemma 3 part (2), Eq. (3.32) has a solution

$$u(t,z) = V(t, Z_t) + \dot{Z}_t,$$

where $R(t) = 1$, such that $V(t,x)$ is the solution of

$$\partial_t V = \partial_{xxx}V + \partial_{xx}V + V\partial_x V, \tag{3.25}$$

$$V(0,z) = \frac{3}{25}\operatorname{sech}^2\left(\frac{z}{10}\right) - \frac{6}{25}\tanh\left(\frac{z}{10}\right) + \frac{6}{25}, \tag{3.26}$$

and Z_t is the solution of

$$\ddot{Z}_t = \sigma(t)\dot{W}_t, \tag{3.27}$$

with initial state $Z_0 = z$ and for $t \in [0,1]$.

Again, Eq. (3.33) has the general solution

$$V(t,z) = \frac{3}{25}\operatorname{sech}^2\left(\frac{z}{10} - \frac{6t}{250}\right) - \frac{6}{25}\tanh\left(\frac{z}{10} - \frac{6t}{250}\right) + \frac{6}{25} \tag{3.28}$$

for $z \in \mathbb{R}$. Also, $\dot{Z}_t = \int_0^t \sigma(r)dW_r$ due to Lemma 2. Thus, Eq. (3.32) has a solution

$$u(t,z) = \frac{3}{25}\operatorname{sech}^2\left(\frac{z + \int_0^t \sigma(s)dW_s}{10} - \frac{6t}{250}\right) - \frac{6}{25}\tanh\left(\frac{z + \int_0^t \sigma(s)dW_s}{10} - \frac{6t}{250}\right) + \frac{6}{25} + \int_0^t \sigma(r)dW_r. \tag{3.29}$$

If $\sigma(t) = 1$, by Lemma 2 the stochastic differential Eq. (3.34) has a solution given by

$$Z_t = z - \int_0^t W_s ds$$

for $t \in [0,1]$.

The general solution of the stochastic KdV–Burgers equation (3.32) is given by

$$u(t,z) = \frac{3}{25}\operatorname{sech}^2\left(\frac{z - \int_0^t W_s ds}{10} - \frac{6t}{250}\right) - \frac{6}{25}\tanh\left(\frac{z - \int_0^t W_s ds}{10} - \frac{6t}{250}\right) + \frac{6}{25} + \dot{Z}_t \tag{3.30}$$

for $t \in [0,1]$ and $z \in \mathbb{R}$.

If $\sigma(t) = t^n$, by Lemma 2 the stochastic differential Eq. (3.34) has a solution given by

$$Z_t = z - \int_0^t (t-s)s^n dW_s = z - t\int_0^t s^n dW_s + \int_0^t s^{n+1}dW_s = z - t\left(t^n W_t - \int_0^t ns^{n-1}W_s ds\right) + t^{n+1}W_t - \int_0^t (n+1)s^n W_s ds$$

for $t \in [0,1]$.

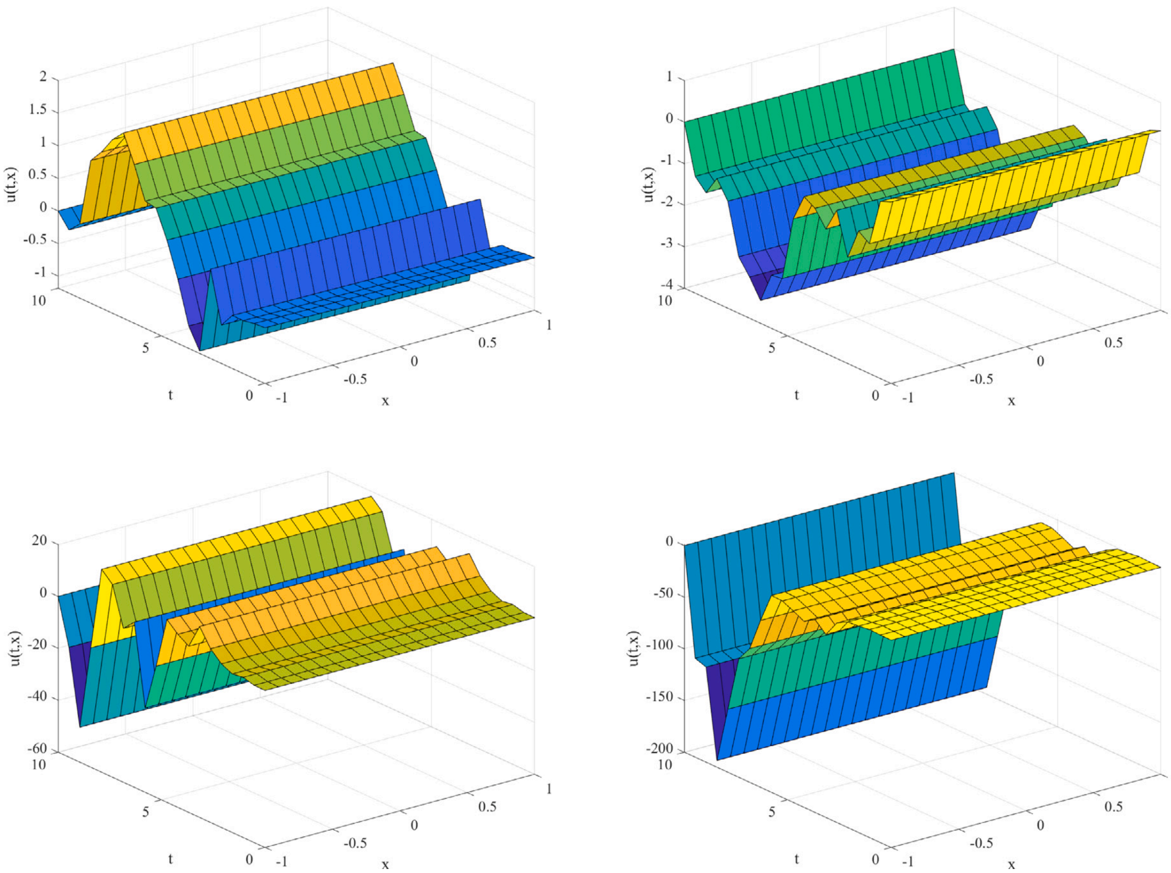

Fig. 2. Four realizations of the stochastic process in Eq. (3.29) when $\sigma(t) = 1$ (top) and $\sigma(t) = t^2$ (bottom).

In that case, the general solution of the stochastic KdV–Burgers equation (3.32) is given by

$$u(t,z) = \frac{3}{25}\operatorname{sech}^2\left(\frac{Z_t}{10} - \frac{6t}{250}\right) - \frac{6}{25}\tanh\left(\frac{Z_t}{10} - \frac{6t}{250}\right) + \frac{6}{25} + \dot{Z}_t, \quad (3.31)$$

where $Z_t = z - t\left(t^n W_t - \int_0^t n s^{n-1} W_s ds\right) + t^{n+1} W_t - \int_0^t (n+1) s^n W_s ds$ for $t \in [0,1]$ and $z \in \mathbb{R}$. Fig. 2 shows two realizations of the general solution in (3.29).

Another example for the stochastic forced term is given here. See[18–23].

Example 3. Consider another stochastic Burgers equation

$$du = (\exp(t)\partial_{zz}u + \exp(t)u\partial_z u)dt + dW_t, \quad u(0,z) = \frac{2}{1+\exp(-2-z)} \quad (3.32)$$

for $t \in [0,1]$. By Lemma 3 part (2), Eq. (3.32) has a solution

$$u(t,z) = V(t, Z_t) + \frac{1}{\exp(t)}\dot{Z}_t,$$

where $R(t) = 1$, such that $V(t,x)$ is the solution of

$$\partial_t V = \exp(t)\partial_{xx}V + \exp(t)V\partial_x V, \quad V(0,x) = \frac{2}{1+\exp(-2-x)}, \quad (3.33)$$

and Z_t is the solution of

$$\ddot{Z}_t = \dot{Z}_t + \exp(t)\dot{W}_t \quad (3.34)$$

with initial state $Z_0 = z$ and for $t \in [0,1]$.

Again, Eq. (3.33) has the general solution

$$V(t,x) = \frac{2}{1+\exp(-1-x-\exp(t))}$$

for $x \in \mathbb{R}$.

By Lemma 2, the stochastic differential Eq. (3.34) has a solution given by

$$Z_t = z + \exp(t)W_t - \int_0^t \exp(s)dW_s,$$

and

$$\dot{Z}_t = \exp(t)W_t$$

for $t \in [0,1]$.

Therefore, the general solution of the stochastic Burgers Eq. (3.32) is given by

$$u(t,z) = W_t + \frac{2}{1+\exp(-1-z-\exp(t)-\exp(t)W_t + \int_0^t \exp(s)dW_s)} \quad (3.35)$$

for $t \in [0,1]$ and $z \in \mathbb{R}$.

Fig. 3 shows two realizations of the general solution in (3.35).

4. Conclusion

In this paper, we carried out a study on exact solutions to a class of stochastic Burgers–Korteweg de Vries (KdV–Burgers) equations. The analysis we have carried out clearly demonstrates the effectiveness of Itô calculus and different transformation techniques in developing explicit solutions by splitting the random element and solving the deterministic kinetic part.

Including exact solutions to the stochastic Burgers–KdV equation not only contributes a theoretical aspect but also provides essential insights for various physical and biological applications of these equations. The introduction of spatially uniform noise and variable coefficients signifies a more plausible environment, which often characterizes complex systems in real life due to inherent stochasticity or uncertainty.

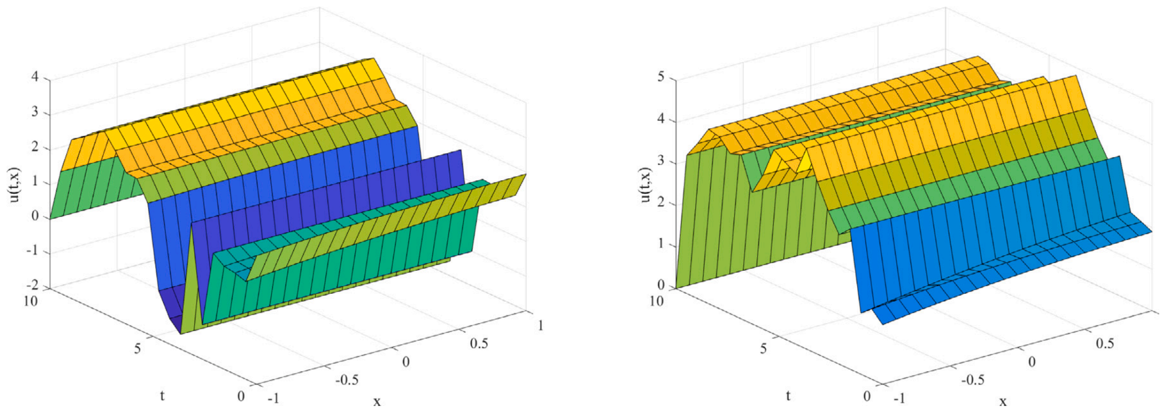

Fig. 3. Two realizations of the stochastic process in Eq. (3.35).

Moreover, the figures made using stochastic simulations based on exact solutions of the Burgers–KdV equation are very interesting. These figures form a logical link between analytical solutions, which are not always feasible to investigate in the physical domain, and actual applications of the derived method, attesting to the corroboration of exact solutions in diverse environments.

In conclusion, this research significantly contributes to the study of stochastic partial differential equations and paves the way for establishing a basis for the solution of equations with common noise structures and variable coefficients. Future research may involve generalization of the methodology developed here for higher-order space dimensions. This may also consist of the dynamics in the stochastic Kudryashov–Sinelshchikov equation or in using nonuniform spatial noise. Future perspectives will enrich the fields of stochastic analysis and mathematical physics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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