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CONTACT NUMBERS FOR PACKING OF SPHERICAL PARTICLES

A Thesis

by

EDUARDO ALEJANDRO RAMIREZ MARTINEZ

Submitted to the Graduate College of
The University of Texas Rio Grande Valley
In partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2018

Major Subject: Mathematics

CONTACT NUMBERS FOR PACKING OF SPHERICAL PARTICLES

A Thesis

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December 2018

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ABSTRACT

Ramirez Martinez, Eduardo Alejandro, CONTACT NUMBERS FOR PACKING OF SPHERICAL PARTICLES. Master of Science (MS), December, 2018, 60 pp., 2 figures, 9 references.

This thesis covers packings of spherical particles. A packing of closed balls is a set of Euclidean balls with disjoint interiors. The main object of investigation is the contact number of such packings. The contact graph of a packing of closed balls is a graph with balls as vertices and pairs of tangent balls as edges. The case of packings of balls with 2 or 3 different radii in $\mathbb{R}^3$ is investigated and new upper bounds for the average degree of their contact graphs are obtained as the outcome of the research. The other direction covered here is the so-called soft packings of spherical particles. New upper bounds for contact numbers are obtained in this scenario as well.

DEDICATION

I dedicate this work to my family, who have helped me in every situation in life and to my closest friends, the Healers. Love you all!

ACKNOWLEDGMENTS

I thank God for His help and support on this work. Mom, Uncle, Grandmother, Brother, and Sister, thank you for the love and support you have given me throughout all these years. I also thank my professor and advisor, Dr. Alexey Glazyrin, for his constant motivation to get me involved with mathematical research and for encouraging me, with his example, to see how vast and beautiful geometry is. Dr. Jerzy Mogilski, thank you for guiding me on pursuing my mathematical career. Dr. Alexey Garber, thank you for always believing in me and for all the encouragement you given me.

TABLE OF CONTENTS

	Page
ABSTRACT	iii
DEDICATION	iv
ACKNOWLEDGMENTS	v
TABLE OF CONTENTS	vi
LIST OF FIGURES	vii
CHAPTER I. INTRODUCTION	1
CHAPTER II. DEFINITIONS AND NOTATIONS	2
CHAPTER III. APPROACH OF GREG KUPERBERG AND ODED SCHRAMM	4
CHAPTER IV. APPROACH TO FINITELY MANY SITES (2-3 BALLS)	10
CHAPTER V. SOFT PACKINGS	21
REFERENCES	23
MATLAB CODE	24
BIOGRAPHICAL SKETCH	60

LIST OF FIGURES

	Page
Figure 1: Sphere of Radius 1 and of radius R	19
Figure 2: Three radii 1,2, R	20

CHAPTER I

INTRODUCTION

This thesis surveys recent results on Ball Packings as defined by the mathematician Greg Kuperberg and Oded Schramm, in [1], with average Kissing numbers. Chapter 2 is dedicated to the definitions and notation that we will use in later chapters. Chapter 3 introduces Kuperberg and Schramm approach [6], with the emphasis on kissing numbers. Chapter 4 covers finitely sites that can be found with different radii. Chapter 5 classifies soft packings among the results in the former classification.

CHAPTER II

DEFINITION AND NOTATIONS

We will start with the definition of ball packings.

Definition 1. *A packing of balls P in $\mathbb{R}^d$ is a finite set of balls with non-overlapping interiors.*

Definition 2. *A contact graph of a packing of balls $G(P)$ is a graph where vertices represent balls from P and two vertices are connected by an edge if the corresponding balls are tangent to each other.*

Characterizing contact graphs is a problem that has been solved by the Koebe-Andreev-Thurston Theorem.[7,8]

Theorem 1. (Koebe-Andreev-Thurston). For all simple planar graph G , there exist a set of non-intersecting closed disks on the plane whose contact graph is G .

Theorem 2. (Kuperberg-Schramm [8])

$$12.566 \approx 666/53 \leq k_3 \leq 8 + 4\sqrt{3} \approx 14.928$$

Following the previous theorems, and definitions we will look into Archimedes spherical cap. A spherical cap is the region of a sphere which is located above or below from a given plane. The cap is called a hemisphere from a plane that goes through the center of the sphere. In addition, a spherical segment is formed if the cap is cut by another plane. By Harris, J. W. and Stocker, H in [1], a spherical segment is used to describe the spherical cap and zone as well.

Theorem 3. Archimedes Spherical Cap

$$A = \pi(2rh))$$

Proof. Using Calculus we defined the following integral from a spherical cap.

$$\int_{r-h}^r 2\pi f(x) \sqrt{1 + f'(x)^2} dx$$

where $f(x) = \sqrt{r^2 - x^2}$ and $f'(x) = -\frac{x}{\sqrt{r^2 - x^2}}$. Thus, the result becomes

$$\begin{aligned} \int_{r-h}^r 2\pi \cdot \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}} dx &= \\ &= \int_{r-h}^r 2\pi r dx \\ &= 2\pi r h \end{aligned}$$

□

CHAPTER III

APPROACH OF GRAG KUPERBERG AND ODED SCHRAMM

In this chapter, we discuss the results from the theory of Kuperberg-Schramm. The results that they found, are the counter-stone that allowed them to prove that the upper bound is $8 + 4\sqrt{3}$ for k_3 . Following from chapter 2, Archimedes' formula (3) is used for the areas of spherical caps, where R is the radius of the sphere and h is the height.

To start Kuperberg-Schramm approach we need the following proposition.

Proposition 1.

$$k_3 \leq 8 + 4\sqrt{3}$$

Proof. For a unit Ball A in $\mathbb{R}^3$, let's consider a concentric sphere with radius of $\sqrt{3}$. For any unit sphere tangent to A , there is an intersection with the concentric sphere by a spherical cap with angular spherical radius of $\pi/6$. The height of the spherical cap is equal to $\sqrt{3} - \frac{3}{2}$. [7] By Archimedes' formula, the area is $2\pi\sqrt{3}(\sqrt{3} - 3/2) = (6 - 3\sqrt{3})\pi$ for the spherical caps. The area of the concentric sphere equals 12π , thus no more than $\frac{12\pi}{(6-3\sqrt{3})\pi} = 8 + 4\sqrt{3}$ spherical caps [7] may fit in the surface of the concentric sphere. $\square$

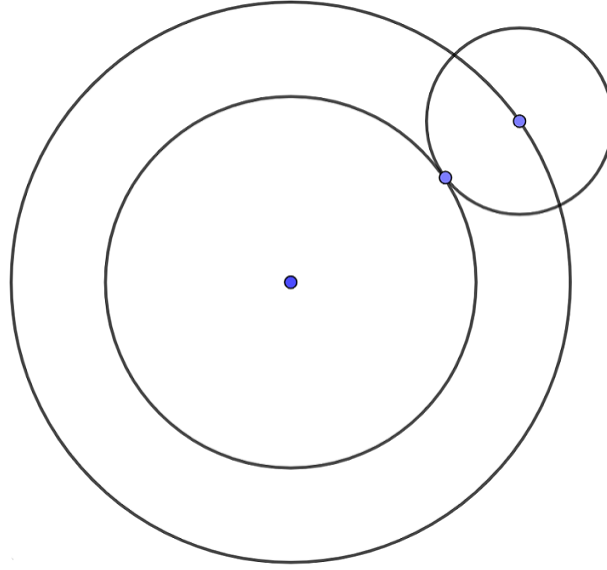
Remark 1. *Since kissing numbers are integers, then any upper bound can be substituted by the Proposition 1 [7]. Proposition 1 implies that k_3 must be less than 14. Proposition 1 emphasizes the idea of packings with different radii.*

When working with different radii, the bounding of numbers of tangent spheres is not applicable. For a unit ball, any number of small balls can be constructed tangent to it. However, for two tangent balls is compensated by a larger proportion of area taken by a smaller ball on a sphere concentric

to a smaller ball is compensated by a larger proportion of the area taken by the larger ball on a sphere concretion to a smaller ball. This is the main approach of Kuperberg and Shramm.

Let p be fixed, then $p > 1$. For each closed ball A , the concentric sphere can be denoted with radius ρ times larger by $S_p(A)$. For two tangents balls A_1 and A_2 with radius r_1 and r_2 by [7] it is defined as the following

$$\alpha(A_1, A_2) = \frac{\text{area}(S_p(A_1) \cap A_2)}{\text{area}(S_p(A_1))}$$



If both $S_p(A_1) \cap A_2$ and $S_p(A_2) \cap A_1$ are non-empty, then $\alpha(A_1, A_2) + \alpha(A_2, A_1)$ depends solely on ρ . To show this, the height of a spherical cap $S_p(A_1) \cap A_2$ by h_1 and the height of $S_p(A_2) \cap A_1$ by h_2 . If the intersection is empty then the height must be 0.

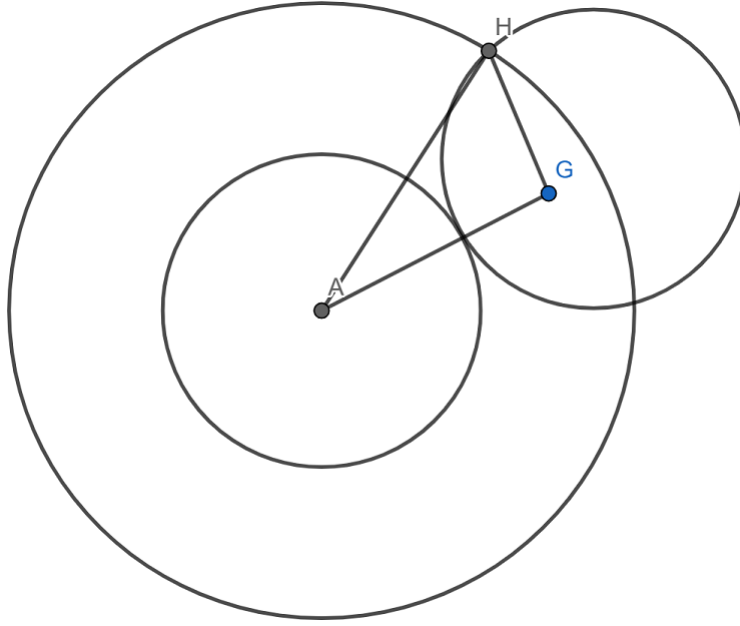
From this point forward, $p < 3$. This is because either or at least one of $\alpha(A_1, A_2)$ and $\alpha(A_2, A_1)$ has to be 0.

Lemma 1.

$$\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2} = \frac{-\rho^2 + 4\rho - 3}{2\rho}$$

if both $S_p(A_1) \cap A_2$ and $S_p(A_2) \cap A_1$ are not empty and the left hand side is greater than the right hand side otherwise.[7]

Proof. Let $\frac{h_1}{\rho r_1} = 1 - \cos \alpha$, where α is the radius of the spherical of the cap. Such as the following image:



Since the centers A_1 and A_2 form a triangle, we can now use the the law of cosines:

$$(\rho r_1)^2 + (r_1 + r_2)^2 - 2\rho r_1(r_1 + r_2) \cos \alpha = r_2^2,$$

$$\cos(\alpha) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} = \frac{(\rho^2 + 1)r_1 + 2r_2}{2\rho(r_1 + r_2)}.$$

For the raidus of γ for the second cap, we obtained:

$$\cos(\gamma) = \frac{(\rho^2 + 1)r_2 + 2r_1}{2\rho(r_1 + r_2)}$$

Thus,

$$\cos(\alpha) + \cos(\beta) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} + \frac{(\rho^2 + 1)r_2 + 2r_1}{2\rho(r_1 + r_2)} = \frac{-\rho^2 + 4\rho - 3}{2\rho},$$

wehre,

$$\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2} = 2 - (\cos(\alpha) + \cos(\gamma)) = \frac{-\rho^2 + 4\rho - 3}{2\rho}$$

□

Lemma 2.

$$a(A_1, A_2) + a(A_2, A_1) = \frac{-\rho^2 + 4\rho - 3}{4\rho}$$

if both $S_\rho(A_1) \cap A_2$ and $S_\rho(A_2) \cap A_1$ are non-empty and the left side is greater than the right side otherwise.

Proof. Lemma 2 can be proven by lemma 1 and the Archimedes' found in chapter 2. □

Now having showed lemma 1 and lemma 2, a more precise and straight forward theorem can be introduced.

Theorem 4. Let

$$k_3 \leq \inf\left\{\frac{8\rho}{-\rho^2 + 4\rho - 3} \text{dens}(\rho)\right\}$$

$\text{dens}(\rho)$ denotes the maximum proportion of the area of $S_\rho(A)$ covered by non-overlapping balls tangent to A

Proof. We denote by $\text{dens}(\rho)$ the maximum of $\sum_i a(A, B_i)$, where B_i are closed balls with disjoint interiors tangent to B .

If $G = (V, E)$ is the contact graph of a ball packing, then

$$\frac{-\rho^2 + 4\rho - 3}{4\rho} |E| \leq \sum_{\{X, Y\} \in E} (a(X, Y) + a(Y, X)) \leq \text{dens}(\rho) |V|;$$

$$\frac{2|E|}{|V|} \leq dens(\rho) \frac{8\rho}{-\rho^2 + 4\rho + 3};$$

$$k_3 \leq \inf_{1 < \rho < 3} \left\{ dens(\rho) \frac{8\rho}{-\rho^2 + 4\rho - 3} \right\} \leq$$

$$\leq 1 \cdot \frac{8\rho}{-\rho^2 + 4\rho - 3} \Big|_{\rho=\sqrt{3}} = 8 + 4\sqrt{3}.$$

Since Kuperberg and Schramm used $dens(\rho) \leq 1$, [7], then ρ must be taken to be $= \sqrt{3}$ for this proves the upper bound they found. $\square$

Following from Kurperberg-Schramm approach, Dr. Alexey Glazyrin in [7] and in [8] Kurperberg-Schramm discuss bounds on higher dimensions. The following theorems 5 and 6 are used in [7] for bounds in higher dimensions found by Glazyrin.

Theorem 5.

$$k_d \leq \inf_{1 < p < 3} \left\{ \frac{2}{f_d(p)} dens(p) \right\},$$

$dens_d(p)$ is the maximum proportion of area of $s_p(A)$ covered by non-overlapping balls tangent to A.

Proof. Following from [7], for a contact graph $G = (V, E)$,

$$\sum_{\{X,Y\} \in E} (\alpha(X, Y) + \alpha(Y, X)) \geq f_d(\rho) |E|$$

$$\sum_{\{X,Y\} \in E} (\alpha(X, Y) + \alpha(Y, X)) \leq dens(\rho) |V|$$

Therefore,

$$\frac{2|E|}{|V|} \leq \frac{2}{f_d(\rho)} dens_d(\rho)$$

$\square$

$f_d(\rho)$, as a function of ρ , reaches its maximum when $1 - (\frac{p^2+3}{4p})^2 p = \sqrt{3}$ and $S_p(x) \cap Y$ is the spherical cap with radius $\pi/2$. Using $dens_d(\sqrt{3}) \leq 1$ and theorem 5 for $p = \sqrt{3}$, we get the bound analogous to the Kuperberg-Schramm bound in higher dimension.

Theorem 6.

$$k_d \geq a(d) = \frac{2 \int_0^1 t^{\frac{d-3}{2}} (1-t)^{\frac{-1}{2}} dt}{\int_0^{1/4} t^{\frac{d-3}{2}} (1-t)^{\frac{-1}{2}} dt}$$

.

The Kuperberg-Schramm upper bound is a generalization of $k_3 \leq 8 + 4\sqrt{3}$, theorem 5 and 6 are a generalization for the upper bound for kissing numbers on area estimates.

CHAPTER IV

APPROACH TO FINITELY MANY SITES (2-3) BALLS

In this chapter, we determine new bound in dimension 3. The area argument is the easiest way to find upper bounds on kissing numbers. Chapter 2 and 3, essentially explain how to extend this argument to packings with different radii, more specific 2 and 3.

The idea consists of constructing a certain tiling associated with a packing and bounding the density of the packing in each tile of a tiling [7]. This bound is used on the density as part of the general bound. This idea comes from, Fejes Toth [3], who showed that density of a packing of congruent circles of spherical radius α in the unit sphere can not be greater than the density of this packing in the regular spherical triangle of the side 2α with centers of the circle at the vertices of this triangle. The theorem that will be used for this results follows from Florian in [4,5], the we generalize it at [3] for circular caps with different sizes.

Theorem 7. Let $Q(\alpha)$ be a non-decreasing function defined on $I = [\alpha_{min}, \alpha_{max}]$, $0 < \alpha_{min} \leq \alpha_{max} \leq \pi/2$. For a packing C of a unit sphere with circles whose radii belong to I , the density is defined as:

$$d(C) = \frac{1}{4\pi} \sum_{C \in c} K(radius(C)).$$

For $x, y, z \in I$, we consider a spherical triangle $\triangle$ formed by centers of pairwise tangent circles of radii x, y, z . The density of this triangle is defined by:

$$D(x, y, z) = \frac{1}{2\pi \text{area}(\triangle)} (K(x)\angle x + K(y)\angle y + K(z)\angle z).$$

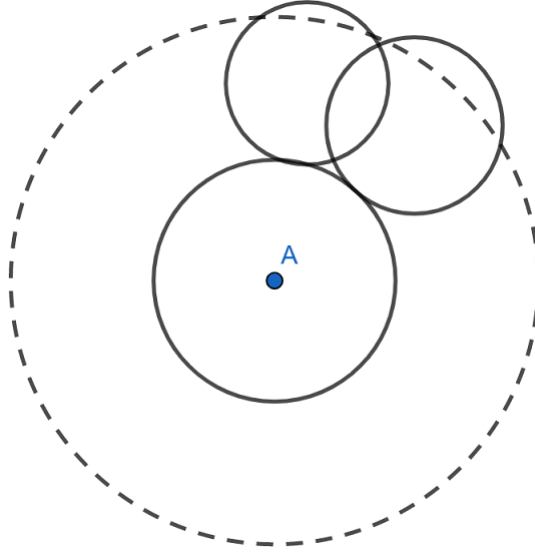
Then $d(C) \leq \max_{x, y, z \in I} D(x, y, z)$.

Proof. The proof of this theorem essentially consists of two parts. First, we can show that for any saturated packing with caps of radii between α_{min} and α_{max} , there exists a Delaunay-like triangulation [6]. The second part consists of proving that the maximal density among Delaunay-like triangles is attained on a triangle defined by three pairwise tangent caps. $\square$

Remark 2. *Formally, Florian proved the theorem for the case when I is a finite set possible radii but theorem 7 follows immediately from his results.*

We will combine this bound on the density with theorem 4 to get the new bound in the dimension 3 [7]. Just to recall the notation used in the previous sections, by the $dens(\rho)$ we mean the maximum proportion of area of $S_p(A)$ covered by non-overlapping balls tangent to B.

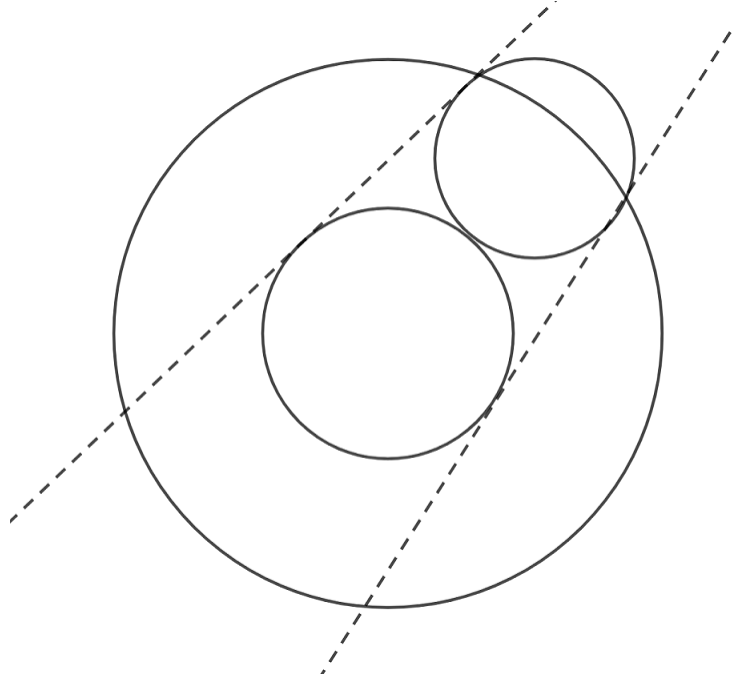
If we forget that circular caps on $S_p(A)$ were initially formed by non-overlapping balls and just try to find upper bounds for an arbitrary packing by circular caps, it is impossible to separate $dens(\rho)$ from 1. A spherical cap may have an arbitrarily small radius and thus the density of a packing may be arbitrarily to 1.



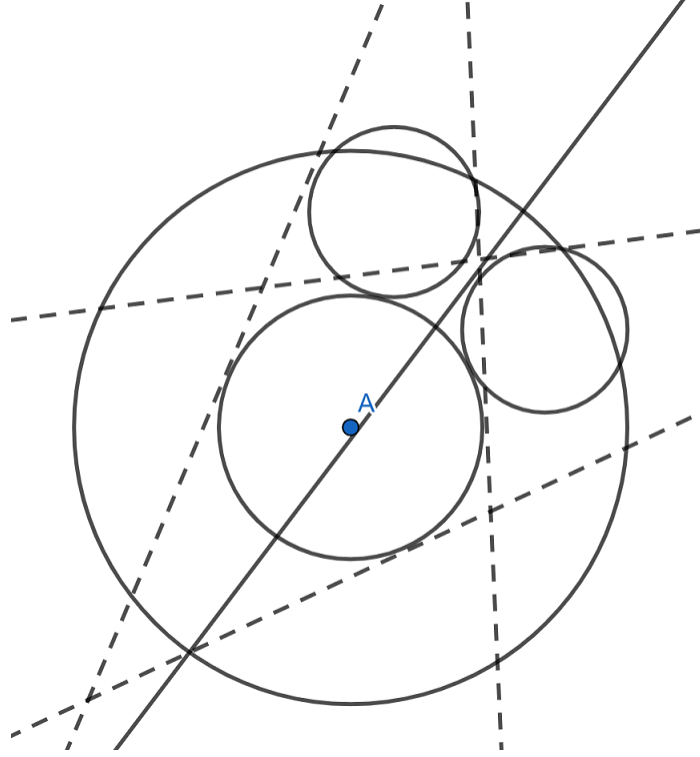
Relatively small caps can not be too close to each other.

The main idea finding an upper bound for $den(\rho)$ is to use an auxiliary circular packing which extends the original one and it can not contain circular caps of small sizes.

For each ball X tangent to a ball A , we define a circular cap $C_p(A, X)$ as a cap on $S_p(A)$ defined by tangent planes of A and X if a point of tangency when tangent line of such common tangent plane with X inside $S_p(A)$ [7]. Meaning, $C_p(A, X) = S_p(A) \cap X$.



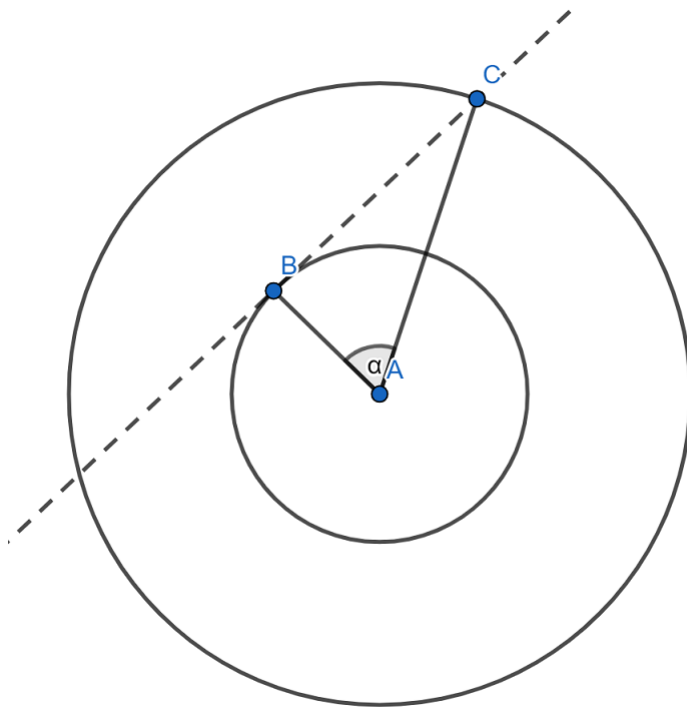
Lemma 3. *For any $p \geq 1$ and any non-overlapping balls X and Y tangent to A , the spherical caps $C_p(A, X)$ and $C_p(A, Y)$ can not overlap.*



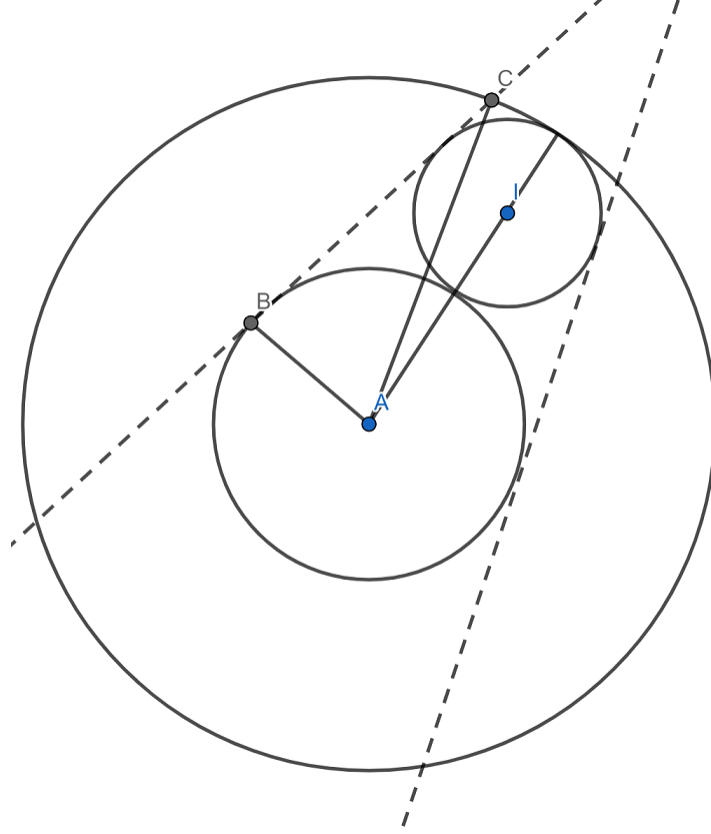
Proof. We start by considering a radical plane ρ of the boundaries of X and Y . We need show that ρ separates $C_p(A, X)$ and $C_p(A, Y)$. First, ρ must intersect A . If this was not the case then that means one of the two half-spaces formed by ρ does not have a common point with A but it can contain either X or Y , which both are tangent to A . As a radical plane of two non-overlapping spheres, ρ may not contain interior points of X or Y [7]. Thus, to show the complete proof of the lemma, it is sufficient to show that ρ does not have any interior points of $C_p(A, X)$ or $C_p(A, Y)$. We assume that ρ has an interior point of $C_p(A, X)$. We connect this point with an arbitrary point of $\rho \cap B$ by a line segment. This segment intersects X by an interior point so we get a contradiction to the fact that ρ contains no interior points of X [7].

□

Now we find all the functions needed to calculate the upper bound by theorem 7.

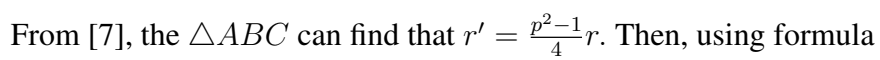


$$\alpha_{max} = \angle BAC = \arccos\left(\frac{1}{\rho}\right)$$



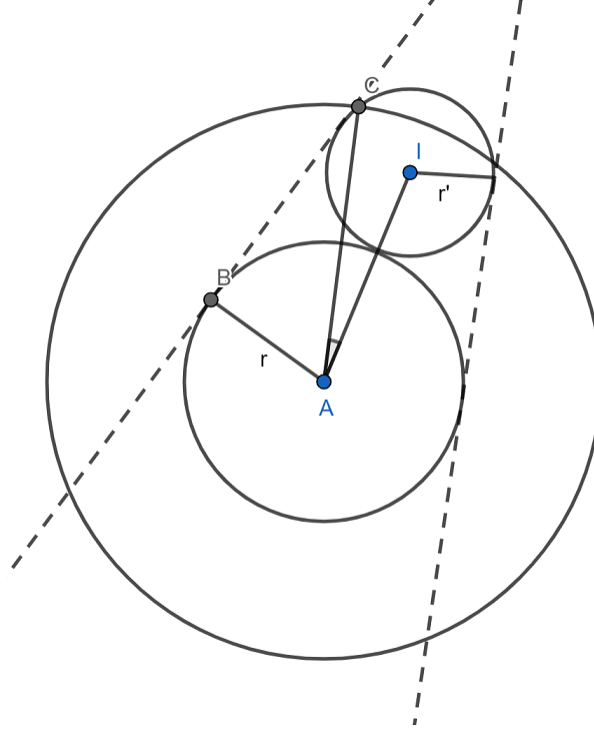
$$\alpha_{min} = \angle BAI - \angle BAC = \arccos\left(\frac{3-\rho}{1+\rho}\right) - \arccos\left(\frac{1}{\rho}\right)$$

We want is for the density of the initial circular packing to be bounded, the function $Q(\alpha)$ should be equal to the area of the circular cap given by the radius α of its auxiliary circular cap [7]. By definition, these two caps coincide if the common tangent plane of a ball A and a ball X tangent to A lies inside $S_p(A)$ [7]. This happens when α is greater or equal to a certain starting point α_i , when the tangent point is exactly on $S_p(A)$.



we get the formula for α_i :

Thus, $Q(\alpha)$ has an area of a circular cap with the spherical radius α where $\alpha \in [\alpha_i, \alpha_{max}]$:



Hence $r' = (\frac{\frac{1}{p} \cos \alpha - \sqrt{1 - \frac{1}{p^2}} \sin \alpha + 1}{\frac{1}{p} \cos \alpha - \sqrt{1 - \frac{1}{p^2}} \sin \alpha - 1})r$.

Combining this with the formula

$$\cos(\alpha) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} = \frac{(\rho^2 + 1)r_1 + 2r_2}{2\rho(r_1 + r_2)},$$

we find $Q(\alpha)$ for $\alpha \in [\alpha_{min}, \alpha_i]$;

$$Q(\alpha) = 2\pi(1 - \frac{(\rho^2 - 1)(\frac{1}{\rho} \cos \alpha - \sqrt{1 - \frac{1}{\rho^2}} \sin \alpha + 1) + 4}{4\rho}) \text{ if } \alpha \in [\alpha_{min}, \alpha_i].$$

The angles $\angle x, \angle y, \angle z$ are found by the Spherical Law of Cosines [7]:

$$\angle x = \arccos \frac{\cos(y + z) - \cos(x + z) \cos(x + y)}{\sin(x + z) \sin(x + y)}; \quad (1)$$

$$\angle y = \arccos \frac{\cos(x + z) - \cos(x + y) \cos(y + z)}{\sin(x + y) \sin(y + z)}; \quad (2)$$

$$\angle z = \arccos \frac{\cos(x+y) - \cos(x+z) \cos(y+z)}{\sin(x+z) \sin(y+z)}. \quad (3)$$

With this, the formula for the general bound in dimension 3 can be found as seen in Glazyrin's work [7].

Theorem 8. For any $p \in (1, 3)$, we define $D_p(x, y, z)$ for all triples $x, y, z \in I_p[\alpha_{min}, \alpha_{max}]$,

$$D_p(x, y, z) = \frac{1}{2\pi(\angle x + \angle y + \angle z - \pi)} (K(x)\angle x + K(y)\angle y + K(z)\angle z,$$

where $\alpha_{min}, \alpha_{max}, K(\alpha), \angle x, \angle y, \angle z$ are defined by Spherical Law of Cosines. Then,

$$k_3 \leq \inf_{1 < p < 3} \left\{ \max_{x, y, z \in I_p} D_p(x, y, z) \frac{8p}{-p^2 + 4p - 3} \right\}$$

Proof. Using the both theorems 4 and 7 [7], we obtaine the following result:

$$\begin{aligned} k_3 &\leq \inf_{1 < p < 3} \left\{ dens(\rho) \frac{8p}{-p^2 + 4p - 3} \right\} \leq \\ &\leq \inf_{1 < p < 3} \left\{ \max_{x, y, z \in I_p} D_p(x, y, z) \frac{8p}{-p^2 + 4p - 3} \right\} \end{aligned}$$

□

Now using Theorems 4, 7, and 8 it can shown that $k_3 \leq 13.955$. The approximation for infimum in Theorem 8, MATLAB software was used. The value obtained using $\rho = 1.755$ is of 13.908. For theorems 4, 7, and 8 we have I as any possible value for our boundaries. Now, we take the I to be numbers between 2 and 3. Using this method we are able to find a more precise solutions even if we look into 2 and 3 radii.

First, for this results the program MATLAB was used. Using MATLAB, we were able to use the formulas from angles $\angle x, \angle y, \angle z$ 1, 2, and 3 respectevly. Using them we were able to find an optimal 4ρ . Using the MATLAB CODE, in the appendix section, the following results was obtained:

Example 0.1. $R = 1.2$ for packings in $\mathbb{R}^3$.

(Why 1.2? For all $R \leq 1.141$, all degrees cannot be greater than 13 because of the solution of the Tammes problem for 14 [7].)

Using $\rho = 1.731$ we get that the average degree is less than 13.43.

Example 0.2. $R = 2$ for packings in $\mathbb{R}^3$.

Using $\rho = 1.714$ we get that the average degree is less than 13.61.

Question 1. For a given finite set $\mathcal{R} = \{R_i\}$, what is the maximal average degree of a graph realized as a contact graphs of packings of balls in $\mathbb{R}^d$ with radii from $\mathcal{R}$?

Example 0.3. $\mathcal{R} = \{1, 1.2\}$ for packings in $\mathbb{R}^3$.

Using $\rho = 1.731$ we get that the average degree is less than 13.43.

Example 0.4. $\mathcal{R} = \{1, 2\}$ for packings in $\mathbb{R}^3$.

Using $\rho = 1.841$ we get that the average degree is less than 13.57.

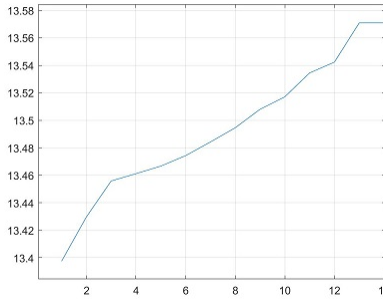


Figure 1: Sphere of radius 1 and of radius R

As shown in figure 1 and figure 2. The average degree of contact graph of packing balls (along with different radii) such as 2 and 3 in $\mathbb{R}^3$ is not greater than 13.955. MATLAB was used as the computation will take a long time without software assistance.

The way the code works is that ρ has to be narrowed down to $[1.562, 1.928]$ as shown in [7]. Using $\rho = 1.721$ and $R = \{1, 2\}$ and the code will in turn give that the average degree is less than 13.57. This will also work for radii 3 and the calculations will take more time to get results.

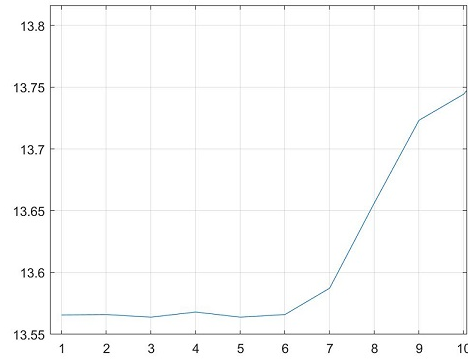


Figure 2: Three radii 1,2,R

from. As shown in the MATLAB code, ρ and R can be substituted for the wanted values. The density will have to be tweaked to not overflow the density. With all the calculations, 2 and 3 radii, their average degree is still less 13.955, thus as shown in [7] by Glazyrin, $k_3 < 13.955$ and with different radii the average number is still not above 13.955

CHAPTER V

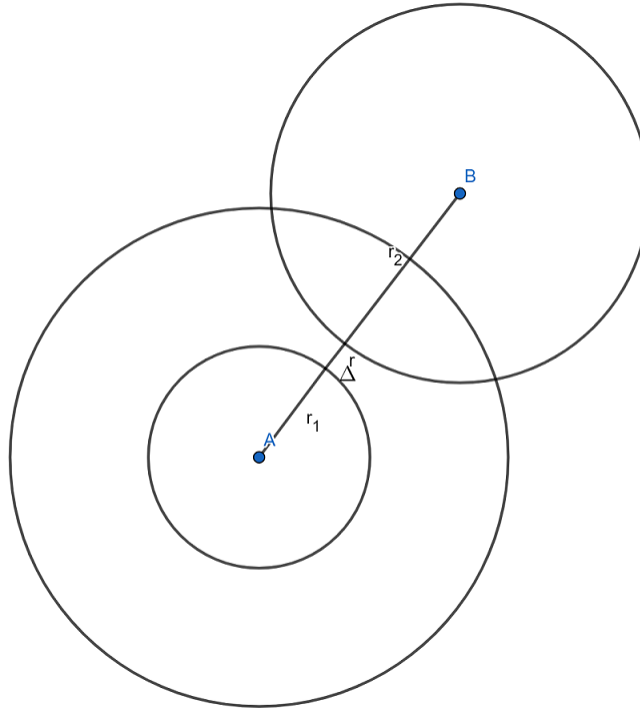
SOFT PACKINGS

In this chapter we classify soft packings. Following the Greg Kuperberg and Oded Schramm approach, and the packings with restricted radii we can cover soft packing.

Fix $\delta > 0$. Each soft ball consists of a hard ball of some radius r and a concentric soft ball of radius $(1 + \delta)r$.

Two soft balls are called tangent if hard interiors are disjoint and both soft shells intersect hard interiors of their counterparts. For weaker conditions on tangency, a complete graph of any size is realized.

$$\alpha(A_1, A_2) + \alpha(A_2, A_1) \geq \frac{-p^2+4p-3}{4p} - \frac{p^2-1}{4p}\delta + O(\delta^2)$$



If we use matlab software we obtained the following for the average degree. It shows that it still

has to be < 15.26 once δ is given a certain value in range and ρ as 1.721

Example 0.5. For $\delta = 0.01$, using $\rho = 1.721$ we get that the average degree is less than 15.26.

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Matlab Code

```
1 rho=1.2;
2 R=1;
3 Delta=0.01;
4 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
5 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
6 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
7 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
8 absmin=14;
9 rhomin=rho;
10 while (rho <=2)
11     maxd=0;
12     cr0=(3*rho^2+1)/(rho*(rho^2+3));
13     crmax=1/rho;
14     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
15     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
```

```

16      t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
      >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
      ;
17  %if (1/x0>(rho-1)/2)
18  %      crmin=t(1/x0);
19  %end
20  %crmax=t(R/x0);
21  density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
      /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
      (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
      x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
      (2))>=crmax)*(cos(x(3))>=crmax);
22  %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3)
      )/(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
      angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
      *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
      *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
23  %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
      )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
      anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
      *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
      *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
24  y(1)=acos(t(1));
25  y(2)=acos(t(R));

```

```

26     y(3)=acos(t(1/R));
27     z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
28     maxd=-min(z);
29     if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin)
30         rhomin=rho;
31         absmin=2*maxd*4*rho/(-rho^2+4*rho-3);
32     end
33     rho=rho+Delta;
34 end
35 %plot(,)
36 %hold on
37 %rhomin;
38 %%%%%%%%%%
39 %%%%%%%%%%
40 rho=1.2;
41 R=1.2;
42 Delta=0.01;
43 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
44 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));

```

```

45 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
+c)))));
46 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
c/2)));
47 absmin1=14;
48 rhomin=rho;
49 while (rho <=2)
50     maxd=0;
51     cr0=(3*rho^2+1)/(rho*(rho^2+3));
52     crmax=1/rho;
53     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
^2/(rho+1)^2);
54     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
cr0);
55     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
;
56 %if (1/x0>(rho-1)/2)
57 %     crmin=t(1/x0);
58 %end
59 %crmax=t(R/x0);
60 density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle

```

```

(x(3),x(2),x(1))/(2*pi)*g(x(3)))*(cos(x(1))<=crmin)*(cos(
x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
(2))>=crmax)*(cos(x(3))>=crmax);
61 %density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3)
)/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+
angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
62 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
)/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
63 y(1)=acos(t(1));
64 y(2)=acos(t(R));
65 y(3)=acos(t(1/R));
66 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
density([y(3),y(3),y(3)])];
67 maxd=-min(z);
68 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin1)
69     rhomin=rho;
70     absmin1=2*maxd*4*rho/(-rho^2+4*rho-3);

```

```

71     end
72     rho=rho+Delta;
73 end
74 %%%%%%%%%%
75 %%%%%%%%%%
76 %hold on
77
78 %%%%%%%%%%
79 %%%%%%%%%%
80 rho=1.2;
81 R=1.4;
82 Delta=0.01;
83 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
84 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
85 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
86 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
87 absmin2=14;
88 rhomin=rho;
89 while (rho <=2)
90     maxd=0;

```

```

91     cr0=(3*rho^2+1)/(rho*(rho^2+3));
92     crmax=1/rho;
93     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
94     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
95     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
96     %if (1/x0>(rho-1)/2)
97     %     crmin=t(1/x0);
98     %end
99     %crmax=t(R/x0);
100    density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
        /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
        (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
        x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
        (2))>=crmax)*(cos(x(3))>=crmax);
101    %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3))
        /(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
        angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
        *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
        *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);

```

```

102    %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))
        )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
        anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
        *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
        *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
103    y(1)=acos(t(1));
104    y(2)=acos(t(R));
105    y(3)=acos(t(1/R));
106    z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
107    maxd=-min(z);
108    if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin2)
109        rhomin=rho;
110        absmin2=2*maxd*4*rho/(-rho^2+4*rho-3);
111    end
112    rho=rho+Delta;
113 end
114 %%%%%%%%%%
115 %%%%%%%%%%
116 %hold on
117 rho=1.2;
118 R=1.43;

```

```

119 Delta=0.01;
120 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
121 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
122 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
123 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
124 absmin3=14;
125 rhomin=rho;
126 while (rho <=2)
127     maxd=0;
128     cr0=(3*rho^2+1)/(rho*(rho^2+3));
129     crmax=1/rho;
130     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
131     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
132     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
133     %if (1/x0>(rho-1)/2)

```

```

134 %      crmin=t (1/x0);
135 %end
136 %crmax=t (R/x0);
137 density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
      /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
      (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
      x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
      (2))>=crmax)*(cos(x(3))>=crmax);
138 %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3)
      )/(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
      angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
      *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
      *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
139 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
      )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
      anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
      *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
      *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
140 y(1)=acos(t(1));
141 y(2)=acos(t(R));
142 y(3)=acos(t(1/R));
143 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
      density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
      density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),

```

```

        density ([ y(3) , y(3) , y(3) ] ) ];
144     maxd=-min(z) ;
145     if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin3)
146         rhomin=rho ;
147         absmin3=2*maxd*4*rho/(-rho^2+4*rho-3);
148     end
149     rho=rho+Delta ;
150 end
151 %%%%%%%%%%
152 %%%%%%%%%%
153 %hold on
154 rho=1.2;
155 R=1.46;
156 Delta=0.01;
157 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
158 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
159 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
160 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
161 absmin4=14;
162 rhomin=rho ;

```

```

163 while (rho <=2)
164     maxd=0;
165     cr0=(3*rho^2+1)/(rho*(rho^2+3));
166     crmax=1/rho;
167     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
168     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
169     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
170     %if (1/x0>(rho-1)/2)
171     %     crmin=t(1/x0);
172     %end
173     %crmax=t(R/x0);
174     density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
        /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
        (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
        x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
        (2))>=crmax)*(cos(x(3))>=crmax);
175     %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3))
        /(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
        angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)

```

```

        *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
        *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
176 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
        )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
        anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
        *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
        *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
177 y(1)=acos(t(1));
178 y(2)=acos(t(R));
179 y(3)=acos(t(1/R));
180 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
181 maxd=-min(z);
182 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin4)
183     rhomin=rho;
184     absmin4=2*maxd*4*rho/(-rho^2+4*rho-3);
185 end
186 rho=rho+Delta;
187 end
188 %%%%%%%%%%
189 %%%%%%%%%%
190 %hold on

```

```

191 rho=1.2;
192 R=1.5;
193 Delta=0.01;
194 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
195 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
196 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
197 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
198 absmin5=14;
199 rhomin=rho;
200 while (rho <=2)
201     maxd=0;
202     cr0=(3*rho^2+1)/(rho*(rho^2+3));
203     crmax=1/rho;
204     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
205     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
206     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)

```

```

;
207 %if (1/x0>(rho-1)/2)
208 %    crmin=t(1/x0);
209 %end
210 %crmax=t(R/x0);
211 density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
    /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
    (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
    x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
    (2))>=crmax)*(cos(x(3))>=crmax);
212 %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3))
    /(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
    angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
    *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
    *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
213 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))
    /(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
    anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
    *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
    *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
214 y(1)=acos(t(1));
215 y(2)=acos(t(R));
216 y(3)=acos(t(1/R));
217 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),

```

```

        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
218     maxd=-min(z);
219     if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin5)
220         rhomin=rho;
221         absmin5=2*maxd*4*rho/(-rho^2+4*rho-3);
222     end
223     rho=rho+Delta;
224 end
225
226 rho=1.2;
227 R=1.55;
228 Delta=0.01;
229 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
230 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
231 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
232 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
233 absmin6=14;
234 rhomin=rho;

```

```

235 while (rho <=2)
236     maxd=0;
237     cr0=(3*rho^2+1)/(rho*(rho^2+3));
238     crmax=1/rho;
239     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
240     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
241     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
242     %if (1/x0>(rho-1)/2)
243     %     crmin=t(1/x0);
244     %end
245     %crmax=t(R/x0);
246     density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
        /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
        (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
        x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
        (2))>=crmax)*(cos(x(3))>=crmax);
247     %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3))
        /(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
        angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)

```

```

        *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
        *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
248 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
        )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
        anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
        *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
        *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
249 y(1)=acos(t(1));
250 y(2)=acos(t(R));
251 y(3)=acos(t(1/R));
252 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
253 maxd=-min(z);
254 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin6)
255     rhomin=rho;
256     absmin6=2*maxd*4*rho/(-rho^2+4*rho-3);
257 end
258 rho=rho+Delta;
259 end
260
261 rho=1.2;
262 R=1.6;

```

```

263 Delta=0.01;
264 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
265 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
266 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
267 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
268 absmin7=14;
269 rhomin=rho;
270 while (rho <=2)
271     maxd=0;
272     cr0=(3*rho^2+1)/(rho*(rho^2+3));
273     crmax=1/rho;
274     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
275     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
276     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
277     %if (1/x0>(rho-1)/2)

```

```

278 %      crmin=t ( 1 / x0 ) ;
279 %end
280 %crmax=t ( R / x0 ) ;
281 density=@(x) ( -1/area ( x ( 1 ) , x ( 2 ) , x ( 3 ) ) *( angle ( x ( 1 ) , x ( 2 ) , x ( 3 ) )
    /( 2 * pi ) * g ( x ( 1 ) ) + angle ( x ( 2 ) , x ( 1 ) , x ( 3 ) ) / ( 2 * pi ) * g ( x ( 2 ) ) + angle
    ( x ( 3 ) , x ( 2 ) , x ( 1 ) ) / ( 2 * pi ) * g ( x ( 3 ) ) ) ) *( cos ( x ( 1 ) ) <= crmin ) *( cos (
    x ( 2 ) ) <= crmin ) *( cos ( x ( 3 ) ) <= crmin ) *( cos ( x ( 1 ) ) >= crmax ) *( cos ( x
    ( 2 ) ) >= crmax ) *( cos ( x ( 3 ) ) >= crmax ) ;
282 %density2=@(y) ( -1/are ( y ( 1 ) , y ( 2 ) , y ( 3 ) ) *( angley ( y ( 1 ) , y ( 2 ) , y ( 3 )
    ) / ( 2 * pi ) * g ( y ( 1 ) ) + angley ( y ( 2 ) , y ( 1 ) , y ( 3 ) ) / ( 2 * pi ) * g ( y ( 2 ) ) +
    angley ( y ( 3 ) , y ( 2 ) , y ( 1 ) ) / ( 2 * pi ) * g ( y ( 3 ) ) ) ) *( cos ( y ( 1 ) ) <= crmin )
    *( cos ( y ( 2 ) ) <= crmin ) *( cos ( y ( 3 ) ) <= crmin ) *( cos ( y ( 1 ) ) >= crmax )
    *( cos ( y ( 2 ) ) >= crmax ) *( cos ( y ( 3 ) ) >= crmax ) ;
283 %density3=@(y) ( -1/are ( y ( 1 ) , y ( 2 ) , y ( 3 ) ) *( anglez ( z ( 1 ) , z ( 2 ) , z ( 3 )
    ) / ( 2 * pi ) * g ( z ( 1 ) ) + anglez ( z ( 2 ) , z ( 1 ) , z ( 3 ) ) / ( 2 * pi ) * g ( z ( 2 ) ) +
    anglez ( z ( 3 ) , z ( 2 ) , z ( 1 ) ) / ( 2 * pi ) * g ( z ( 3 ) ) ) ) *( cos ( z ( 1 ) ) <= crmin )
    *( cos ( z ( 2 ) ) <= crmin ) *( cos ( z ( 3 ) ) <= crmin ) *( cos ( z ( 1 ) ) >= crmax )
    *( cos ( z ( 2 ) ) >= crmax ) *( cos ( z ( 3 ) ) >= crmax ) ;
284 y ( 1 ) = acos ( t ( 1 ) ) ;
285 y ( 2 ) = acos ( t ( R ) ) ;
286 y ( 3 ) = acos ( t ( 1 / R ) ) ;
287 z = [ density ( [ y ( 1 ) , y ( 1 ) , y ( 1 ) ] ) , density ( [ y ( 1 ) , y ( 1 ) , y ( 2 ) ] ) ,
    density ( [ y ( 1 ) , y ( 2 ) , y ( 2 ) ] ) , density ( [ y ( 2 ) , y ( 2 ) , y ( 2 ) ] ) ,
    density ( [ y ( 1 ) , y ( 1 ) , y ( 3 ) ] ) , density ( [ y ( 1 ) , y ( 3 ) , y ( 3 ) ] ) ,

```

```

        density ([ y(3) , y(3) , y(3) ] ) ];
288     maxd=-min(z) ;
289     if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin7)
290         rhomin=rho ;
291         absmin7=2*maxd*4*rho/(-rho^2+4*rho-3);
292     end
293     rho=rho+Delta ;
294 end
295
296
297 rho = 1.2;
298 R = 1.66;
299 Delta = 0.01;
300 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
301 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
302 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
303 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
304 absmin8=14;
305 rhomin=rho ;
306 while (rho <=2)

```

```

307     maxd=0;
308     cr0=(3*rho^2+1)/(rho*(rho^2+3));
309     crmax=1/rho;
310     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
311     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
312     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
313     %if (1/x0>(rho-1)/2)
314     %     crmin=t(1/x0);
315     %end
316     %crmax=t(R/x0);
317     density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
        /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
        (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
        x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
        (2))>=crmax)*(cos(x(3))>=crmax);
318     %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3))
        /(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
        angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
        *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)

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        *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
319 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
        )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
        anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
        *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
        *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
320 y(1)=acos(t(1));
321 y(2)=acos(t(R));
322 y(3)=acos(t(1/R));
323 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
324 maxd=-min(z);
325 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin8)
326     rhomin=rho;
327     absmin8=2*maxd*4*rho/(-rho^2+4*rho-3);
328 end
329 rho=rho+Delta;
330 end
331
332 rho=1.2;
333 R=1.7;
334 Delta=0.01;

```

```

335 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
336 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
337 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
338 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
339 absmin9=14;
340 rhomin=rho;
341 while (rho <=2)
342     maxd=0;
343     cr0=(3*rho^2+1)/(rho*(rho^2+3));
344     crmax=1/rho;
345     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
346     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
347     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
348     %if (1/x0>(rho-1)/2)
349     %     crmin=t(1/x0);

```

```

350 %end
351 %crmax=t (R/ x0) ;
352 density=@(x) (-1/area (x(1) ,x(2) ,x(3) )*(angle (x(1) ,x(2) ,x(3) )
    /(2* pi)*g(x(1) )+angle (x(2) ,x(1) ,x(3) )/(2* pi)*g(x(2) )+angle
    (x(3) ,x(2) ,x(1) )/(2* pi)*g(x(3) ) ))*(cos (x(1) )<=crmin)*(cos (
    x(2) )<=crmin)*(cos (x(3) )<=crmin)*(cos (x(1) )>=crmax)*(cos (x
    (2) )>=crmax)*(cos (x(3) )>=crmax) ;
353 %density2=@(y) (-1/are (y(1) ,y(2) ,y(3) )*(angley (y(1) ,y(2) ,y(3)
    )/(2* pi)*g(y(1) )+angley (y(2) ,y(1) ,y(3) )/(2* pi)*g(y(2) )+
    angley (y(3) ,y(2) ,y(1) )/(2* pi)*g(y(3) ) ))*(cos (y(1) )<=crmin)
    *(cos (y(2) )<=crmin)*(cos (y(3) )<=crmin)*(cos (y(1) )>=crmax)
    *(cos (y(2) )>=crmax)*(cos (y(3) )>=crmax) ;
354 %density3=@(y) (-1/are (y(1) ,y(2) ,y(3) )*(anglez (z(1) ,z(2) ,z(3)
    )/(2* pi)*g(z(1) )+anglez (z(2) ,z(1) ,z(3) )/(2* pi)*g(z(2) )+
    anglez (z(3) ,z(2) ,z(1) )/(2* pi)*g(z(3) ) ))*(cos (z(1) )<=crmin)
    *(cos (z(2) )<=crmin)*(cos (z(3) )<=crmin)*(cos (z(1) )>=crmax)
    *(cos (z(2) )>=crmax)*(cos (z(3) )>=crmax) ;
355 y(1)=acos (t(1) ) ;
356 y(2)=acos (t(R) ) ;
357 y(3)=acos (t(1/R) ) ;
358 z=[ density ([y(1) ,y(1) ,y(1) ]) , density ([y(1) ,y(1) ,y(2) ]) ,
    density ([y(1) ,y(2) ,y(2) ]) , density ([y(2) ,y(2) ,y(2) ]) ,
    density ([y(1) ,y(1) ,y(3) ]) , density ([y(1) ,y(3) ,y(3) ]) ,
    density ([y(3) ,y(3) ,y(3) ]) ] ;

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```

359     maxd=-min(z);
360     if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin9)
361         rhomin=rho;
362         absmin9=2*maxd*4*rho/(-rho^2+4*rho-3);
363     end
364     rho=rho+Delta;
365 end
366
367 rho=1.2;
368 R=1.77;
369 Delta=0.01;
370 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
371 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c))/(sin(a+b)*sin(b
    +c)));
372 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c))/(sin(a+c)*sin(b
    +c)));
373 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
374 absmin10=14;
375 rhomin=rho;
376 while (rho <=2)
377     maxd=0;
378     cr0=(3*rho^2+1)/(rho*(rho^2+3));

```

```

379     crmax=1/rho ;
380     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
381     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
382     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
383     %if (1/x0>(rho-1)/2)
384     %     crmin=t(1/x0);
385     %end
386     %crmax=t(R/x0);
387     density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
        /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
        (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
        x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
        (2))>=crmax)*(cos(x(3))>=crmax);
388     %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3)
        )/(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
        angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
        *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
        *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
389     %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)

```

```

)/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
anglez(z(3),z(2),z(1))/(2*pi)*g(z(3)))*(cos(z(1))<=crmin)
*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
390 y(1)=acos(t(1));
391 y(2)=acos(t(R));
392 y(3)=acos(t(1/R));
393 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
density([y(3),y(3),y(3)])];
394 maxd=-min(z);
395 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin10)
396     rhomin=rho;
397     absmin10=2*maxd*4*rho/(-rho^2+4*rho-3);
398 end
399 rho=rho+Delta;
400 end
401
402 rho=1.2;
403 R=1.8;
404 Delta=0.01;
405 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
c)));

```

```

406 %angle y=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
      +c))));
407 %angle z=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
      +c))));
408 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
      c/2)));
409 absmin11=14;
410 rhomin=rho;
411 while (rho <=2)
412     maxd=0;
413     cr0=(3*rho^2+1)/(rho*(rho^2+3));
414     crmax=1/rho;
415     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
      ^2/(rho+1)^2);
416     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
      /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
      cr0);
417     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
      >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
      ;
418     %if (1/x0>(rho-1)/2)
419     %     crmin=t(1/x0);
420     %end
421     %crmax=t(R/x0);

```

```

422 density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
    /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
    (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
    x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
    (2))>=crmax)*(cos(x(3))>=crmax);
423 %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3))
    /(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
    angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
    *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
    *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
424 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))
    /(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
    anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
    *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
    *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
425 y(1)=acos(t(1));
426 y(2)=acos(t(R));
427 y(3)=acos(t(1/R));
428 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
    density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
    density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
    density([y(3),y(3),y(3)])];
429 maxd=-min(z);
430 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin11)

```

```

431         rhomin=rho ;
432         absmin11=2*maxd*4*rho/(-rho^2+4*rho-3);
433     end
434     rho=rho+Delta ;
435 end
436
437 rho=1.2;
438 R=1.9;
439 Delta=0.01;
440 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c))));
441 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
442 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
443 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
444 absmin12=14;
445 rhomin=rho ;
446 while (rho <=2)
447     maxd=0;
448     cr0=(3*rho^2+1)/(rho*(rho^2+3));
449     crmax=1/rho ;
450     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)

```

```

^2/(rho+1)^2);
451 g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
cr0);
452 t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
;
453 %if (1/x0>(rho-1)/2)
454 % crmin=t(1/x0);
455 %end
456 %crmax=t(R/x0);
457 density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
(2))>=crmax)*(cos(x(3))>=crmax);
458 %density2=@(y) (-1/are(y(1),y(2),y(3))*(angle(y(1),y(2),y(3)
)/(2*pi)*g(y(1))+angle(y(2),y(1),y(3))/(2*pi)*g(y(2))+
angle(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
459 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
)/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)

```

```

*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);

460 y(1)=acos(t(1));
461 y(2)=acos(t(R));
462 y(3)=acos(t(1/R));
463 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
      density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
      density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
      density([y(3),y(3),y(3)])];
464 maxd=-min(z);
465 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin12)
466     rhomin=rho;
467     absmin12=2*maxd*4*rho/(-rho^2+4*rho-3);
468 end
469 rho=rho+Delta;
470 end
471
472 rho=1.2;
473 R=2;
474 Delta=0.01;
475 angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
      c)));
476 %angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
      +c))));

```

```

477 %anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c)))));
478 area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
479 absmin13=14;
480 rhomin=rho;
481 while (rho <=2)
482     maxd=0;
483     cr0=(3*rho^2+1)/(rho*(rho^2+3));
484     crmax=1/rho;
485     crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
        ^2/(rho+1)^2);
486     g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)
        /(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>
        cr0);
487     t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y
        >(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
        ;
488     %if (1/x0>(rho-1)/2)
489     %     crmin=t(1/x0);
490     %end
491     %crmax=t(R/x0);
492     density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
        /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle

```

```

(x(3),x(2),x(1))/(2*pi)*g(x(3)))*(cos(x(1))<=crmin)*(cos(
x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
(2))>=crmax)*(cos(x(3))>=crmax);
493 %density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3)
)/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+
angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
494 %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
)/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
495 y(1)=acos(t(1));
496 y(2)=acos(t(R));
497 y(3)=acos(t(1/R));
498 z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
density([y(3),y(3),y(3)])];
499 maxd=-min(z);
500 if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin13)
501     rhomin=rho;
502     absmin13=2*maxd*4*rho/(-rho^2+4*rho-3);

```

```

503     end
504     rho=rho+Delta;
505 end
506
507 a1=[absmin absmin1 absmin2 absmin3 absmin4 absmin5 absmin6
      absmin7 absmin8 absmin9 absmin10 absmin11 absmin12 absmin13];
508 plot(a1)
509 grid on

```

BIOGRAPHICAL SKETCH

Eduardo Alejandro Ramirez Martinez was born in H. Matamoros, Tamaulipas, Mexico, on November 4,, 1993, and he was the third child of Dr. Jesus Angel Ramirez and Dr. Telma Ofelia Martinez, both of who have medical profession. He studied in the bilingual school Colegio Mexico and in Colegio Jevenal Rendon in the same cit, then in Valley Christian High School in Brownsville, Texas. Ramirez studied mathematics in the 4 Plus 1 Program in Mathematics and received both the Bachelor and Master of Science degree in Mathematics from The University of Texas Rio Grande Valley on December 14, 2018.

While at UTB and UTRGV, Ramirez worked as a tutor in various programs. He worked for the Math Department as a Teacher Assistant and helped in areas such as calculus, discrete math, modern algebra and statistics. He was also a tutor for the College of Science, Mathematics and Technology, the Learning Enrichment Program and Title V. In addition, Ramirez worked as a graduate lecturer in Fall 2018 as a college algebra instructor.

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