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Contact Numbers for Packing of Spherical Particles

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CONTACT NUMBERS FOR PACKING OF SPHERICAL PARTICLES

A Thesis

by

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ABSTRACT

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This thesis covers packings of spherical particles. A packing of closed balls is a set of Euclidean balls with disjoint interiors. The main object of investigation is the contact number of such packings. The contact graph of a packing of closed balls is a graph with balls as vertices and pairs of tangent balls as edges. The case of packings of balls with 2 or 3 different radii in $\mathbb{R}^3$ is investigated and new upper bounds for the average degree of their contact graphs are obtained as the outcome of the research. The other direction covered here is the so-called soft packings of spherical particles. New upper bounds for contact numbers are obtained in this scenario as well.

DEDICATION

I dedicate this work to my family, who have helped me in every situation in life and to my closest friends, the Healers. Love you all!

ACKNOWLEDGMENTS

I thank God for His help and support on this work. Mom, Uncle, Grandmother, Brother, and Sister, thank you for the love and support you have given me throughout all these years. I also thank my professor and advisor, Dr. Alexey Glazyrin, for his constant motivation to get me involved with mathematical research and for encouraging me, with his example, to see how vast and beautiful geometry is. Dr. Jerzy Mogilski, thank you for guiding me on pursuing my mathematical career. Dr. Alexey Garber, thank you for always believing in me and for all the encouragement you given me.

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CHAPTER I

INTRODUCTION

This thesis surveys recent results on Ball Packings as defined by the mathematician Greg Kuperberg and Oded Schramm,in [1], with average Kissing numbers. Chapter 2 is dedicated to the definitions and notation that we will use in later chapters. Chapter 3 introduces Kuperberg and Shramm approach[6], with the emphasis on kissing numbers. Chapter 4 cover finitely sites that can be found with different radii. Chapter 5 classifies soft packings among the results in the former classification.

CHAPTER II

DEFINITION AND NOTATIONS

We will start with the definiton of of ball packings.

Definition 1. *A packing of balls P in $\mathbb{R}^d$ is a finite set of balls with non-overlapping interiors.*

Definition 2. *A contact graph of a packing of balls $G(P)$ is a graph where vertices represent balls from P and two vertices are connected by an edge if the corresponding balls are tangent to each other.*

Characterizing contact graphs is a problem that has been solved by the Koebe-Andreev-Thurson Theorem.[7,8]

Theorem 1. (Koebe-Andreev-Thurston). For all simple planar graph G, there exist a set of non-intersecting closed disks on the plane whose contact graph is G.

Theorem 2. (Kuperberg-Schramm [8])

$$12.566 \approx 666/53 \leq k_3 \leq 8 + 4\sqrt{3} \approx 14.928$$

Following the previews theorems, and definitions we will look into Archidemes spherical cap. A spherical cap is the region of a sphere which is located above or below from a given plane. The cap is called a hemisphere from a plane that goes through the center of the sphere. In addition, a spherical segment is form if the cap is cut by another plane. By Harris, J. W. and Stocker, H in [1], a spherical segment is used to describe the spherical cap and zone as well.

Theorem 3. Archimedes Spherical Cap

$$A = \pi(2rh))$$

Proof. Using Calculus we defined the following integral from a spherical cap.

$$\int_{r-h}^{r} 2\pi f(x)\sqrt{(1+f'(x)^2)}dx$$

where $f(x) = \sqrt{r^2 - x^2}$ and $f'(x) = -\frac{x}{\sqrt{r^2-x^2}}$. Thus, the result becomes

$$\int_{r-h}^{r} 2\pi \cdot \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}} dx =$$

$$= \int_{r-h}^{r} 2\pi r dx$$

$$= 2\pi r h$$

□

CHAPTER III

APPROACH OF GRAG KUPERBERG AND ODED SCHRAMM

In this chapter, we discuss the results from the theory of Kuperberg-Schramm.The results that they found, are the counter-stone that allowed them to prove that the upper bound is $8 + 4\sqrt{3}$ for k_3. Following from chapter 2, Archimedes' formula (3) is used for the areas of spherical caps, where R is the radius of the sphere and h is the height.

To start Kuperber-Schramm approach we need the following proposition.

Proposition 1.

$$k_3 \leq 8 + 4\sqrt{3}$$

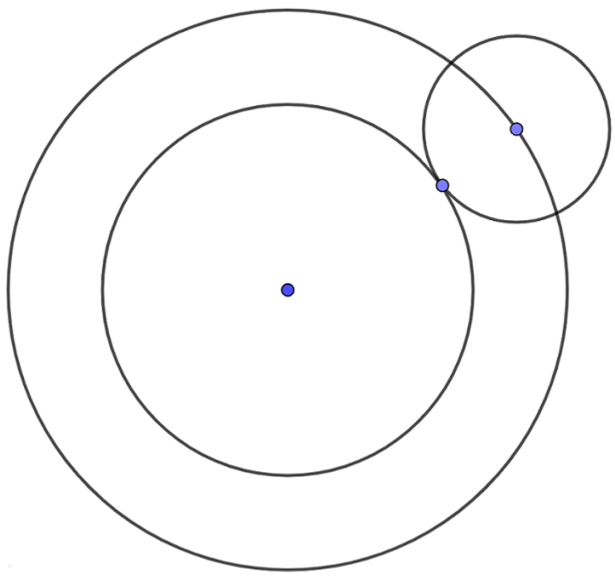
Proof. For a unit Ball A in $\mathbb{R}^3$, lets consider a concentric sphere with radius of $\sqrt{3}$. For any unit sphere tangent to A, there is an an intersection with the cocentric sphere by a spherical cap with angular spherical radius of $\pi/6$. The height of the spherical cap is equal to $\sqrt{3} - \frac{3}{2}$.[7] By Archimedes' formula, the area is $2\pi\sqrt{3}(\sqrt{3} - 3/2) = (6 - 3\sqrt{3})\pi$ for the spherical caps. The area of the concretic sphere equals 12π, thus no more than $\frac{12\pi}{(6-3\sqrt{3})\pi} = 8 + 4\sqrt{3}$ spherical caps [7] may fit in the surface of the concentric sphere. □

Remark 1. *Since kissing numbers are integers, then any upper bound can be substituted by the Proposition 1 [7]. Proposition 1 implies that k_3 must be $leq 14$. Proposition 1 emphasizes the idea of packings with different radii.*

When working with different radii, the bounding of numbers of tangent spheres is not applicable. For a unit ball, any number of small balls can be contructed tangent to it.However, for two tangent balls is compensated by a larger proportion of area taken by a smaller ball on a sphere concretic to a smaller ball is compensated by a larger proportion of the area taken by the larger ball on a sphere concretic to a smaller ball. This is the main approach of Kuperberg and Shramm.

Let p be fixed, then $p > 1$. For each closed ball A, the concentric sphere can be denoted with radius ρ times larger by $S_p(A)$. For two tangents balls A_1 and A_2 with radius r_1 and r_2 by [7] it is defined as the following

$$\alpha(A_1, A_2) = \frac{area(S_p(A_1) \cap A_2)}{area(S_p(A_1))}$$



If both $S_p(A_1) \cap A_2$ and $S_p(A_2) \cap A_1$ are non-empty, then $\alpha(A_1, A_2)$+$\alpha(A_2, A_1)$ depends solely on ρ. To show this, the height of a spherical cap $S_p(A_1) \cap A_2 by h_1$ and the height of $S_p(A_2) \cap A_1$ by h_2. If the intersection is empty then the height must be 0.

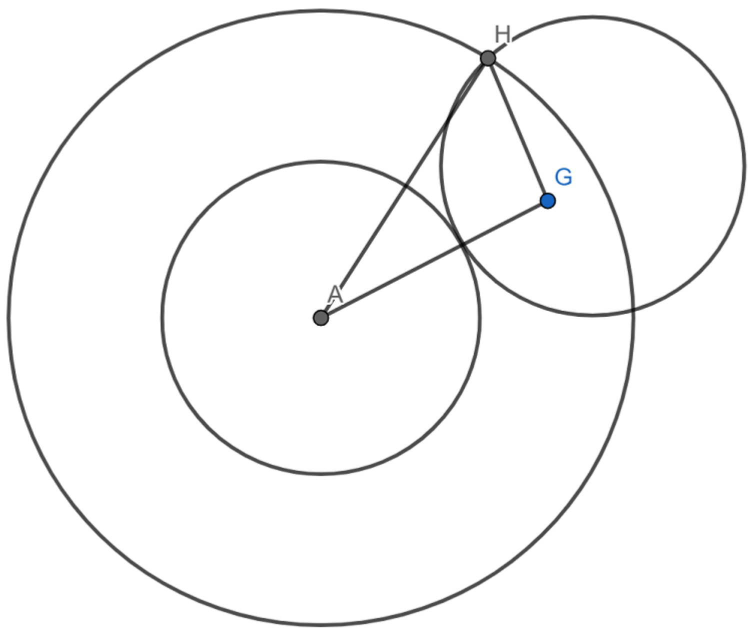
From this point forward, $p < 3$.This is beacause either or at least one of $\alpha(A_1, A_2)$ and $\alpha(A_2, A_1)$ has to be 0.

Lemma 1.

$$\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2} = \frac{-\rho^2 + 4\rho - 3}{2\rho}$$

if both $S_p(A_1) \cap A_2$ and $S_p(A_2) \cap A_1$ are not empty and the left hand side is greater than the right hand side otherwise.[7]

Proof. Let $\frac{h_1}{\rho r_1}$= 1-$\cos\alpha$, where α is the radius of the spherical of the cap. Such as the following image:



Since the centers A_1 and A_2 form a triangle, we can now use the the law of cosines:

$$(\rho r_1)^2 + (r_1 + r_2)^2 - 2\rho r_1(r_1 + r_2)\cos\alpha = r_2^2,$$

$$\cos(\alpha) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} = \frac{(\rho^2 + 1)r_1 + 2r_2}{2\rho(r_1 + r_2)}.$$

For the raidus of γ for the second cap, we obtained:

$$\cos(\gamma) = \frac{(\rho^2 + 1)r_2 + 2r_1}{2\rho(r_1 + r_2)}$$

Thus,

$$\cos(\alpha) + \cos(\beta) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} + \frac{(\rho^2 + 1)r_2 + 2r_1}{2\rho(r_1 + r_2)} = \frac{-p^2 + 4p - 3}{2p},$$

wehre,

$$\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2} = 2 - (\cos(\alpha) + \cos(\gamma)) = \frac{-\rho^2 + 4\rho - 3}{2\rho}$$

□

Lemma 2.

$$a(A_1, A_2) + a(A_2, A_1) = \frac{-\rho^2 + 4\rho - 3}{4\rho}$$

if both $S_\rho(A_1) \cap A_2$ and $S_\rho(A_2) \cap A_1$ are non-empty and the left side is greater than the right side otherwise.

Proof. Lemma 2 can be proven by lemma 1 and the Archimedes'found in chapter 2. □

Now having showed lemma 1 and lemma 2,a more precise and straight forward theorem can be introduced.

Theorem 4. Let

$$k_3 \leq inf\{\frac{8\rho}{-\rho^2 + 4\rho - 3} dens(\rho)\}$$

$dens(\rho)$ denotes the maximum proportion of the area of $S_\rho(A)$ covered by non-overlapping balls tangent to A

Proof. We denote by $dens(\rho)$ the maximum of $\sum_i a(A, A_i)$, where B_i are closed balls with disjoint interiors tangent to B.

If $G = (V, E)$ is the contact graph of a ball packing, then

$$\frac{-\rho^2 + 4\rho - 3}{4\rho} |E| \leq \sum_{\{X,Y\} \in E} (a(X, Y) + a(Y, X)) \leq dens(\rho) |V|;$$

$$\frac{2|E|}{|V|} \leq dens(\rho)\ \frac{8\rho}{-\rho^2+4\rho+3};$$

$$k_3 \leq \inf_{1<\rho<3} \left\{ dens(\rho)\ \frac{8\rho}{-\rho^2+4\rho-3} \right\} \leq$$

$$\leq 1 \cdot \frac{8\rho}{-\rho^2+4\rho-3}\ |_{\rho=\sqrt{3}} = 8+4\sqrt{3}.$$

Since kuperberg and Schramm used $dens(\rho) \leq 1$, [7], then ρ must be taken to be $= \sqrt{3}$ for this proves the upper bound they found. □

Following from Kurperberg-Schramm aprroach, Dr. Alexey Glazyrin in [7] and in [8] Kurperberg-Schramm discuss bounds on higher dimensions. The following theorems 5 and 6 are used in [7] for bounds in higher dimensions found by Glazyrin.

Theorem 5.

$$k_d \leq \inf_{1<p,<3} \{ \frac{2}{f_d(p)} dens(p) \},$$

$dens_d(p)$ is the maximum proportion of area of $s_p(A)$ covered by non-overlapping balls tangent to A.

Proof. Following from [7], for a contact graph $G = (V, E)$,

$$\sum_{\{X,Y\} \in E} (\alpha(X,Y) + \alpha(Y,X)) \geq f_d(\rho)|E|$$

$$\sum_{\{X,Y\} \in E} (\alpha(X,Y) + \alpha(Y,X)) \leq dens_(\rho)|V|$$

Therefore,

$$\frac{2|E|}{|V|} \leq \frac{2}{f_d(\rho)} dens_d(\rho)$$

□

$f_d(\rho)$, as a function of ρ, reaches its maximum when $1 - (\frac{p^2+3}{4p})^2$ $p = \sqrt{3}$ and $S_p(x) \cap Y$ is the spherical cap with radius $\pi/2$. Using $dens_d(\sqrt{3} \leq 1$ and theorem 5 for $p = \sqrt{3}$, we get the bound analogous to the Kuperberg-Schramm bound in higher dimension.

Theorem 6.

$$k_d \geq a(d) = \frac{2\int_0^1 t^{\frac{d-3}{2}}(1-t)^{\frac{-1}{2}}dt}{\int_0^{1/4} t^{\frac{d-3}{2}}(1-t)^{\frac{-1}{2}}dt}$$

.

The Kuperberg-Schramm upper bound is a generalization of $k_3 \leq 8 + 4\sqrt{3}$, theorem 5 and 6 are a generalization for the upper bound for kissing numbers on area estimates.

CHAPTER IV

APPROACH TO FINITELY MANY SITES (2-3) BALLS

In this chapter, we determine new bound in dimension 3.The area argument is the easiest way to find upper bounds on kissing numbers. Chapter 2 and 3, essentially explain how to extend this argument to packings with different radii, more specific 2 and 3.

The idea consists of constructing a certain tiling associated with a packing and bounding the density of the packing in each tile of a tiling [7]. This bound is used on the density as part of the general bound. This idea comes from, Fejes Toth [3], who showed that density of a packing of congruent circles of spherical radius α in the unit sphere can not be greater than the density of this packing in the regular spherical triangle of the side 2α with centers of the circle at the vertices of this triangle. The theorem that will be used for this results follows from Florian in [4,5], the we generalize it at [3] for circular caps with different sizes.

Theorem 7. Let $Q(\alpha)$ be a non-decreasing function defined on $I = [\alpha_{min}, \alpha_{max}]$, $0 < \alpha_{min} \leq \alpha_{max} \leq \pi/2$. For a packing C of a unit sphere with circles whose radii belong to I, the density is defined as:

$$d(C) = \frac{1}{4\pi} \sum_{C \in c} K(radius(C)).$$

For $x, y, z \in I$, we consider a spherical triangle $\triangle$ formed by centers of pairwise tangent circles of radii x, y, z. The density of this triangle is defined by:

$$D(x, y, z) = \frac{1}{2\pi \dot{a}rea(\triangle)}(K(x)\angle x + K(y)\angle y + K(z)\angle z)).$$

Then $d(C) \leq max_x, y, z \in ID(x, y, z)$.

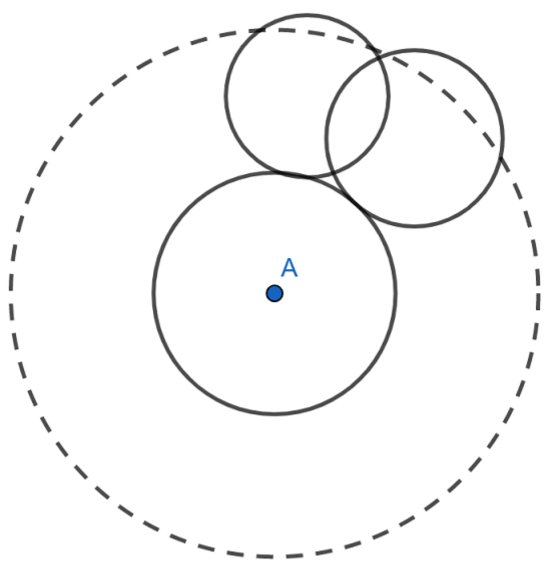
Proof. The proof of this theorem essentially consists of two parts. First, we can show that for any saturated packing with caps of radii between α_{min} and α_{max}, there exists a Delaunay-like triangulation [6]. The second part consists of proving that the maximal density among Delaunay-like triangles is attained on a triangle defined by three pairwise tangent caps. □

.

Remark 2. *Formally, Florian proved the theorem for the case when I is a finite set possible radii but theorem 7 follows immediately from his results.*

We will combine this bound on the density with theorem 4 to get the new bound in the dimension 3 [7]. Just to recall the notation used in the previous sections, by the $dens(\rho)$ we mean the maximum proportion of area of $S_p(A)$ covered by non-overlapping balls tangent to B.

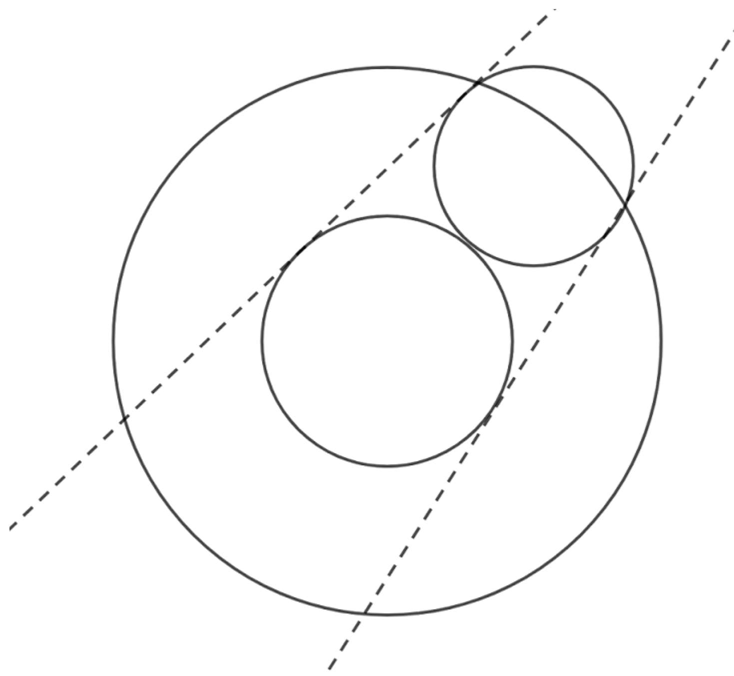
If we forget that circular caps on $S_p(A)$ were initially formed by non-overlapping balls and just try to find upper bounds for an arbitrary packing by circular caps, it is impossible to separate $dens(\rho)$ from 1. A spherical cap may have an arbitrarily small radius and thus the density of a packing may be arbitrarily to 1.



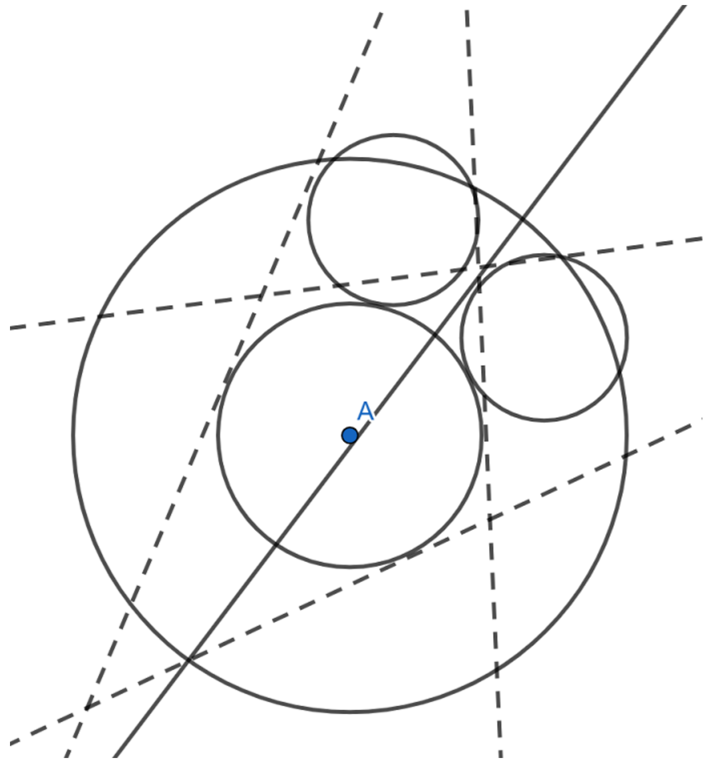
Relatively small caps can not be too close to each other.

The main idea finding a upper bound for $den(\rho)$ is to use an auxiliary circular packing which extends the original one and it can not contain circular caps of small sizes.

For each ball X tangent to a ball A, we define a circular cap $C_p(A, X)$ as a cap on $S_p(A)$ defined by tangent planes of A and X if a point of tanency when tangel lin of such common tangent plane with X inside $S_p(A)$ [7]. Meaning, $C_p(A, X) = S_p(A) \cap X$.



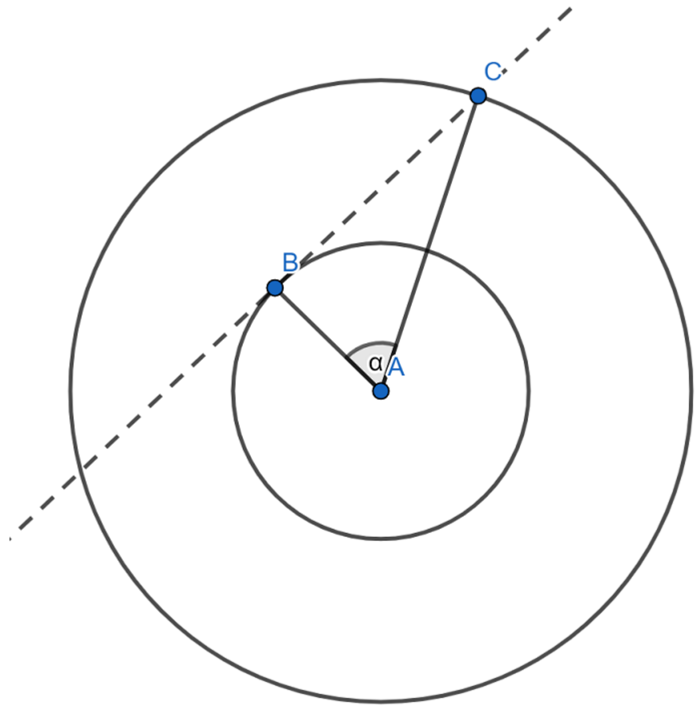
Lemma 3. *For any p¿1 and any non-overlapping balls X and Y tangent to A, the spherical caps $C_p(A, X)$ and $C_p(A, Y)$ can not overlap.*



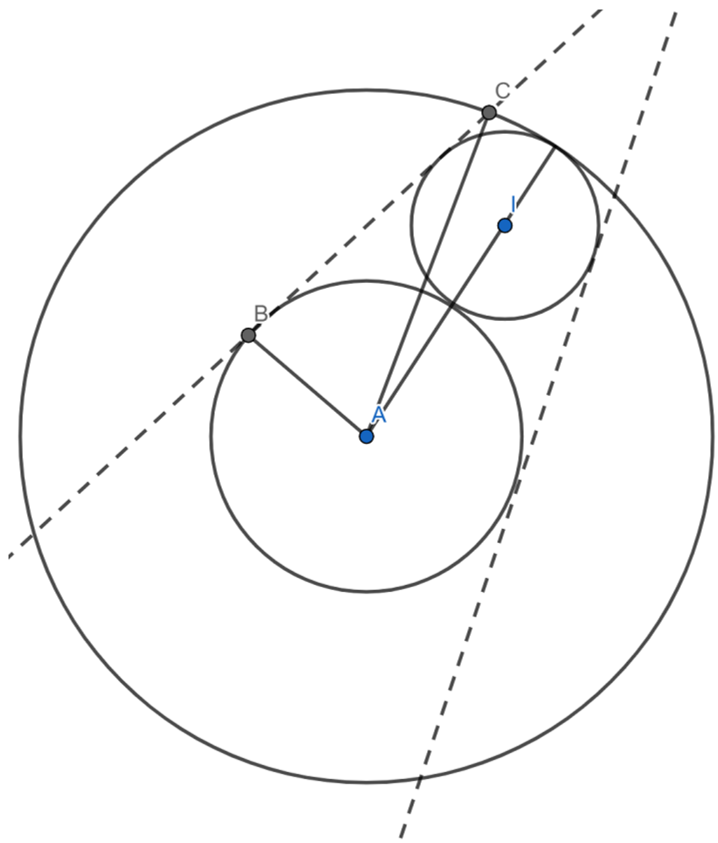
Proof. We start by considering a radical plane ρ of the boundaries of X and Y. We need show that ρ separates $C_p(A, X)$ and $C_p(A, Y)$. First, ρ must intersect A. If this was not the case then that means one of the two half-spaces formed by ρ does not have a common point with A but it can contain either X or Y, which both are tangent to A. As a radical plane of two non-overlapping spheres, ρ may not contain interior points of X or Y [7]. Thus, to show the complete proof of the lemma, it is sufficient to show that ρ does not have any interior points of $C_p(A, X)$ or $C_p(A, Y)$. We assume that ρ has an interior point if $C_p(A, X)$. We connect this point with an arbitrary point of $\rho \cap B$ by a line segment. This segment intersects X by an interior point so we get a contradiction to the fact that ρ contains no interior points of X [7].

□

Now we find all the functions needed to calculate the upper bound by theorem 7.

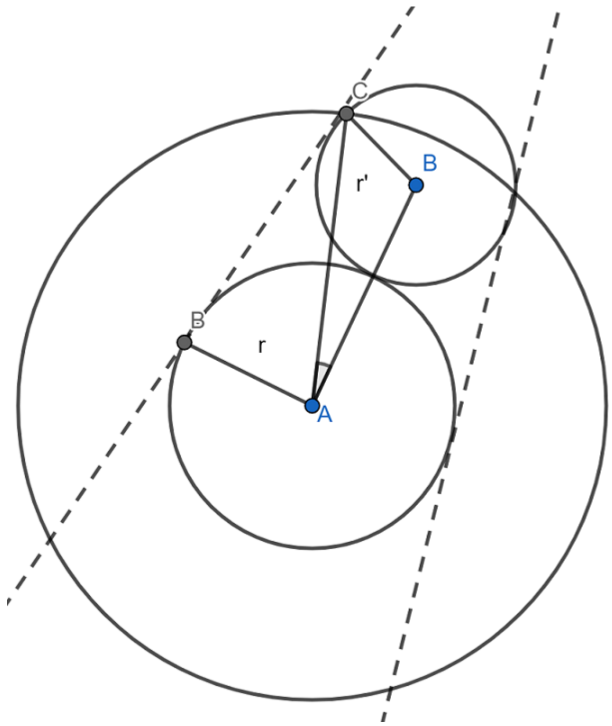


$$\alpha_{max} = \angle BAC = \arccos(\frac{1}{\rho})$$



$$\alpha_{min} = \angle BAI - \angle BAC = \arccos(\frac{3-\rho}{1+\rho}) - \arccos(\frac{1}{\rho})$$

We want is for the density of the initial circular packing to be bounded, the function $Q(\alpha)$ should be equal to the area of the circular cap given by the radius α of its auxiliary circular cap [7]. By definition, these two caps coincide if the common tangent plane of a ball A and a ball X tangent to A lies inside $S_p(A)$ [7]. This happens when α is greater or equal to a certain starting point α_i, when the tangent point is exactly on $S_p(A)$.



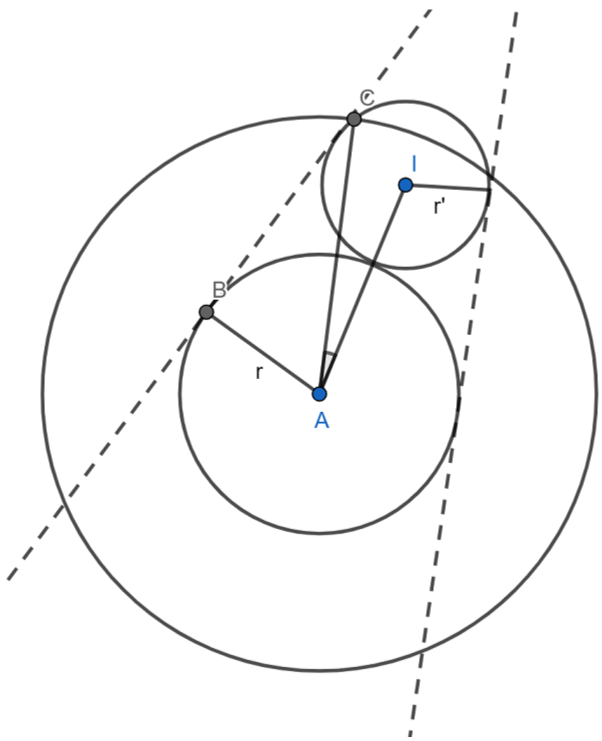
From [7], the $\triangle ABC$ can find that $r' = \frac{p^2-1}{4}r$. Then, using formula

$$\cos(\alpha) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} = \frac{(\rho^2 + 1)r_1 + 2r_2}{2\rho(r_1 + r_2)},$$

we get the formula for α_i:

$$\alpha_i = arccos\frac{3\rho^2 + 1}{\rho(\rho^2 + 3)}$$

Thus, $Q(\alpha)$ has an area of a circular cap with the spherical radius α where $\alpha \in [\alpha_i, \alpha_{max}]$:



For the case when the initial cap and its auxiliary cap do not coincide, on the other hand, $\cos\angle BAI = \cos(\angle BAC + \alpha) = \frac{1}{p}\cos\alpha - \sqrt{1 - \frac{1}{p^2}}\sin\alpha$. On the other hand, $\cos\angle BAI = \frac{r-r'}{r+r'}$.

Hence $r' = (\frac{2}{\frac{1}{p}\cos\alpha - \sqrt{1-\frac{1}{p^2}}\sin\alpha + 1} - 1)r$.

Combining this with the formula

$$\cos(\alpha) = \frac{(\rho r_1)^2 + (r_1 + r_2)^2 - r_2^2}{2\rho r_1(r_1 + r_2)} = \frac{(\rho^2 + 1)r_1 + 2r_2}{2\rho(r_1 + r_2)},$$

we find $Q(\alpha)$ for $\alpha \in [\alpha_{min}, \alpha_i]$;

$$Q(\alpha) = 2\pi(1 - \frac{(\rho^2 - 1)(\frac{1}{\rho}\cos\alpha - \sqrt{1 - \frac{1}{\rho^2}}\sin\alpha + 1) + 4}{4\rho} if \alpha \in [\alpha_{min}, \alpha_i].$$

The angles $\angle x$,$\angle y$, $\angle z$ are found by the Spherical Law of Cosines [7]:

$$\angle x = \arccos\frac{\cos(y + z) - \cos(x + z)\cos(x + y)}{\sin(x + z)\sin(x + y)}; \quad (1)$$

$$\angle y = \arccos\frac{\cos(x + z) - \cos(x + z)\cos(y + z)}{\sin(x + y)\sin(y + z)}; \quad (2)$$

$$\angle z = \arccos \frac{\cos(x+y) - \cos(x+z)\cos(y+z)}{\sin(x+z)\sin(y+z)}. \tag{3}$$

With this, the formula for the general bound in dimension 3 can be found as seen in Glazyrin's work [7].

Theorem 8. For any $p \in (1,3)$, we define $D_p(x,y,z)$ for all triples $x, y, z \in I_p[\alpha_{min}, \alpha_{max}]$,

$$D_p(x,y,z) = \frac{1}{2\pi(\angle x + \angle y + \angle z - \pi)}(K(x)\angle x + K(y)\angle y + K(z)\angle z,$$

where $\alpha_m in, \alpha_m ax, K(\alpha), \angle x, \angle y, \angle z)$ are defined by Spherical Law of Cosines. Then,

$$k_3 \leq \inf_{1<p<3}\{\max_{x,y,z \in I_p} D_p(x,y,z)\frac{8p}{-p^2+4p-3}\}$$

Proof. Using the both theorems 4 and 7 [7], we obtaine the following result:

$$k_3 \leq \inf_{1<p<3}\{dens(\rho)\frac{8p}{-p^2+4p-3}\} \leq$$

$$\leq \inf_{1<p<3}\{\max_{x,y,z \in I_p} D_p(x,y,z)\frac{8p}{-p^2+4p-3}\}$$

□

Now using Theorems 4, 7, and 8 it can shown that $k_3 \leq 13.955$. The approximation for infimum in Theorem 8, MATLAB sofware was used. The value obtained using $\rho = 1.755$ is of 13.908.For theorems 4, 7, and 8 we have I as any possible value for our boundaries. Now, we take the I to be numbers between 2 and 3. Using this method we are able to find a more precise solutions even if we look into 2 and 3 radii.

First, for this results the program MATLAB was used. Using MATLAB, we were able to use the formulas from angles $\angle x, \angle y, \angle z$ 1,2, and 3 respectevely. Using them we were able to find an optimal 4ρ. Using the MATLAB CODE, in the appendix section, the following results was obtained:

Example 0.1. *$R = 1.2$ for packings in $\mathbb{R}^3$.*

(Why 1.2? For all $R \leq 1.141$, all degrees cannot be greater than 13 because of the solution of the Tammes problem for 14 [7].)

Using $\rho = 1.731$ we get that the average degree is less than 13.43.

Example 0.2. *$R = 2$ for packings in $\mathbb{R}^3$.*

Using $\rho = 1.714$ we get that the average degree is less than 13.61.

Question 1. *For a given finite set $\mathcal{R} = \{R_i\}$, what is the maximal average degree of a graph realized as a contact graphs of packings of balls in $\mathbb{R}^d$ with radii from $\mathcal{R}$?*

Example 0.3. *$\mathcal{R} = \{1, 1.2\}$ for packings in $\mathbb{R}^3$.*

Using $\rho = 1.731$ we get that the average degree is less than 13.43.

Example 0.4. *$\mathcal{R} = \{1, 2\}$ for packings in $\mathbb{R}^3$.*

Using $\rho = 1.841$ we get that the average degree is less than 13.57.

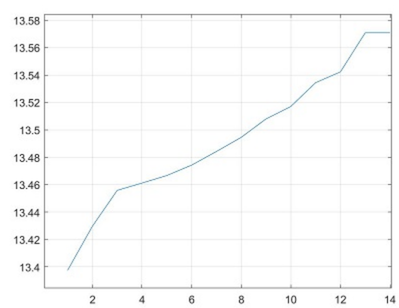


Figure 1: Sphere of radius 1 and of radius R

A shown in figure 1 and figure 2. The average degree of contact graph pf packing balls (along with different radii) such as 2 and 3 in $\mathbb{R}^3$ is not greater that 13.955. MATLAB was used as the computation will take a long time without software assiatant.

The way the code works is that ρ has to be narrowed down to [1.562,1.928] as shown in [7]. Using $\rho = 1.721$ and $R = \{1, 2\}$ and the code will in turn give that the average degree is less than 13.57. This will also work for radii 3 and the calculations will be take more time to get results

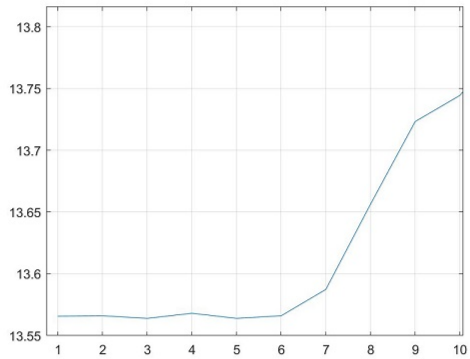


Figure 2: Three radii 1,2,R

from. As shown in the MATLAB code, ρ and R can be substituted for the wanted values.The density will have to be tweaked to not overflow the density. With all the calculations, 2 and 3 radii, their average degree is still less 13.955, thus as shown in [7] by Glazyrin, $k_3 < 13.955$ and with different radii the average number is still not above 13.955

CHAPTER V

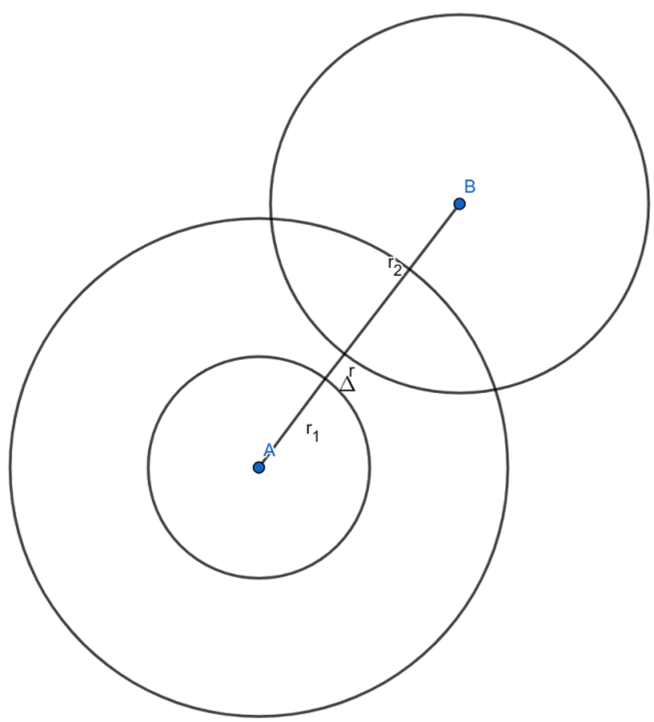
SOFT PACKINGS

In this chapter we classify soft packings. Following the Greg Kuperberg and Oded Schramm approach, and the packings with restricted radii we can cover soft packing.

Fix $\delta > 0$. Each soft ball consists of a hard ball of some radius r and a concentric soft ball of radius $(1+\delta)r$.

Two soft balls are called tangent if hard interiors are disjoint and both soft shells intersect hard interiors of their counterparts. For weaker conditions on tangency, a complete graph of any size is realized.

$\alpha(A_1, A_2) + \alpha(A_2, A_1) \geq \frac{-p^2+4p-3}{4p} - \frac{p^2-1}{4p}\delta + O(\delta^2)$



If we use matlab sofware we obtained the following for the average degree. It shows that it still has to be < 15.26 once $delta$ is given a certain value in range and ρ as 1.721

Example 0.5. *For* $\delta = 0.01$, *using* $\rho = 1.721$ *we get that the average degree is less than* 15.26.

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Matlab Code

```
rho=1.2;
R=1;
Delta=0.01;
angle=@(a,b,c)  acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+c)));
%angley=@(a,b,c)  acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c)  acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin=14;
rhomin=rho;
while  (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r)  1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
```

```
t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
%if (1/x0>(rho-1)/2)
%     crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
```

```
        y(3)=acos(t(1/R));
        z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
            density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
            density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
            density([y(3),y(3),y(3)])];
        maxd=-min(z);
        if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin)
            rhomin=rho;
            absmin=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end
%plot(,)
%hold on
%rhomin;
%%%%%%%%%
%%%%%%%%%
rho=1.2;
R=1.2;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
```

```
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin1=14;
rhomin=rho;
while (rho<=2)
      maxd=0;
      cr0=(3*rho^2+1)/(rho*(rho^2+3));
      crmax=1/rho;
      crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
      g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
      t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
      %if (1/x0>(rho-1)/2)
      %     crmin=t(1/x0);
      %end
      %crmax=t(R/x0);
      density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
```

```
(x(3),x(2),x(1))/(2*pi)*g(x(3)))))*(cos(x(1))<=crmin)*(cos(
x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3)
)/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+
angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
)/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
density([y(3),y(3),y(3)])];
maxd=-min(z);
if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin1)
  rhomin=rho;
  absmin1=2*maxd*4*rho/(-rho^2+4*rho-3);
```

```
        end
        rho=rho+Delta;
end
%%%%%%%%
%%%%%%%%
%hold on

%%%%%%%%%
%%%%%%%%%%
rho=1.2;
R=1.4;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
absmin2=14;
rhomin=rho;
while (rho<=2)
        maxd=0;
```

```
cr0=(3*rho^2+1)/(rho*(rho^2+3));
crmax=1/rho;
crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
%if (1/x0>(rho-1)/2)
%     crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
```

```
        %density3=@(y)  (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
        y(1)=acos(t(1));
        y(2)=acos(t(R));
        y(3)=acos(t(1/R));
        z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),density([y(3),y(3),y(3)])];
        maxd=-min(z);
        if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin2)
            rhomin=rho;
            absmin2=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end
%%%%%%%%%%%%%
%%%%%%%%%%%%%
%hold on
rho=1.2;
R=1.43;
```

```
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin3=14;
rhomin=rho;
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
    %if (1/x0>(rho-1)/2)
```

```
%       crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
```

```
            density([y(3),y(3),y(3)])];
        maxd=-min(z);
        if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin3)
            rhomin=rho;
            absmin3=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end
%%%%%%%%
%%%%%%%%
%hold on
rho=1.2;
R=1.46;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
absmin4=14;
rhomin=rho;
```

```
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
    %if (1/x0>(rho-1)/2)
    %     crmin=t(1/x0);
    %end
    %crmax=t(R/x0);
    density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
    %density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
```

```
    *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
    *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
    %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
    )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
    anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
    *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
    *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
    y(1)=acos(t(1));
    y(2)=acos(t(R));
    y(3)=acos(t(1/R));
    z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
    density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
    density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
    density([y(3),y(3),y(3)])];
    maxd=-min(z);
    if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin4)
        rhomin=rho;
        absmin4=2*maxd*4*rho/(-rho^2+4*rho-3);
    end
    rho=rho+Delta;
end
%%%%%%%%
%%%%%%%%
%hold on
```

```
rho=1.2;
R=1.5;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin5=14;
rhomin=rho;
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4)
```

```
;
%if (1/x0>(rho-1)/2)
%     crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
    /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
    (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
    x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
    (2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3)
    )/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+
    angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
    *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
    *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
    )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
    anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
    *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
    *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
```

```
            density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
            density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
            density([y(3),y(3),y(3)])];
        maxd=-min(z);
        if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin5)
            rhomin=rho;
            absmin5=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end

rho=1.2;
R=1.55;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin6=14;
rhomin=rho;
```

```
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
    %if (1/x0>(rho-1)/2)
    %    crmin=t(1/x0);
    %end
    %crmax=t(R/x0);
    density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
    %density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
```

```
        *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
        *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
    %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
        )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
        anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
        *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
        *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
    y(1)=acos(t(1));
    y(2)=acos(t(R));
    y(3)=acos(t(1/R));
    z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
    maxd=-min(z);
    if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin6)
        rhomin=rho;
        absmin6=2*maxd*4*rho/(-rho^2+4*rho-3);
    end
    rho=rho+Delta;
end

rho=1.2;
R=1.6;
```

```
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin7=14;
rhomin=rho;
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
    %if (1/x0>(rho-1)/2)
```

```
%       crmin=t(1/x0);
%end
%crmax=t(R/x0);
 density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
```

```
            density([y(3),y(3),y(3)])];
        maxd=-min(z);
        if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin7)
            rhomin=rho;
            absmin7=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end


rho=1.2;
R=1.66;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
absmin8=14;
rhomin=rho;
while (rho<=2)
```

```
maxd=0;
cr0=(3*rho^2+1)/(rho*(rho^2+3));
crmax=1/rho;
crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
%if (1/x0>(rho-1)/2)
%     crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
```

```
            *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
        %density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
            )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
            anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
            *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
            *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
        y(1)=acos(t(1));
        y(2)=acos(t(R));
        y(3)=acos(t(1/R));
        z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
            density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
            density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
            density([y(3),y(3),y(3)])];
        maxd=-min(z);
        if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin8)
            rhomin=rho;
            absmin8=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end

rho=1.2;
R=1.7;
Delta=0.01;
```

```
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin9=14;
rhomin=rho;
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
    %if (1/x0>(rho-1)/2)
    %    crmin=t(1/x0);
```

```
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
    density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
    density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
    density([y(3),y(3),y(3)])];
```

```
        maxd=-min(z);
        if  (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin9)
            rhomin=rho;
            absmin9=2*maxd*4*rho/(-rho^2+4*rho-3);
        end
        rho=rho+Delta;
end

rho=1.2;
R=1.77;
Delta=0.01;
angle=@(a,b,c)  acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c)  acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
%anglez=@(a,b,c)  acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
absmin10=14;
rhomin=rho;
while  (rho<=2)
        maxd=0;
        cr0=(3*rho^2+1)/(rho*(rho^2+3));
```

```
crmax=1/rho;
crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
%if (1/x0>(rho-1)/2)
%     crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
```

```
    )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
    anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
    *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
    *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
  y(1)=acos(t(1));
  y(2)=acos(t(R));
  y(3)=acos(t(1/R));
  z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
    density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
    density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
    density([y(3),y(3),y(3)])];
  maxd=-min(z);
  if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin10)
    rhomin=rho;
    absmin10=2*maxd*4*rho/(-rho^2+4*rho-3);
  end
  rho=rho+Delta;
end

rho=1.2;
R=1.8;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
  c)));
```

```
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b+c))));
%anglez=@(a,b,c) acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c) 1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin11=14;
rhomin=rho;
while (rho<=2)
    maxd=0;
    cr0=(3*rho^2+1)/(rho*(rho^2+3));
    crmax=1/rho;
    crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
    g=@(r) 1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
    t=@(y) cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
    %if (1/x0>(rho-1)/2)
    %     crmin=t(1/x0);
    %end
    %crmax=t(R/x0);
```

```
density=@(x) (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))
    /(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
    (x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(
    x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x
    (2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3)
    )/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+
    angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)
    *(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)
    *(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3)
    )/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+
    anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
    *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
    *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
    density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
    density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
    density([y(3),y(3),y(3)])];
maxd=-min(z);
if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin11)
```

```
            rhomin=rho;
            absmin11=2*maxd*4*rho/(-rho^2+4*rho-3);
      end
      rho=rho+Delta;
end

rho=1.2;
R=1.9;
Delta=0.01;
angle=@(a,b,c)  acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c)  acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
%anglez=@(a,b,c)  acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b
    +c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(
    c/2)));
absmin12=14;
rhomin=rho;
while  (rho<=2)
      maxd=0;
      cr0=(3*rho^2+1)/(rho*(rho^2+3));
      crmax=1/rho;
      crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)
```

```
^2/(rho+1)^2);
g=@(r)  1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
t=@(y)  cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
%if  (1/x0>(rho-1)/2)
%      crmin=t(1/x0);
%end
%crmax=t(R/x0);
density=@(x)  (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y)  (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y)  (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)
```

```
        *(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)
        *(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
    y(1)=acos(t(1));
    y(2)=acos(t(R));
    y(3)=acos(t(1/R));
    z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
        density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
        density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
        density([y(3),y(3),y(3)])];
    maxd=-min(z);
    if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin12)
        rhomin=rho;
        absmin12=2*maxd*4*rho/(-rho^2+4*rho-3);
    end
    rho=rho+Delta;
end

rho=1.2;
R=2;
Delta=0.01;
angle=@(a,b,c) acos((cos(b+c)-cos(a+b)*cos(a+c))/(sin(a+b)*sin(a+
    c)));
%angley=@(a,b,c) acos((cos(a+c)-cos(a+b)*cos(b+c)/(sin(a+b)*sin(b
    +c))));
```

```
%anglez=@(a,b,c)  acos((cos(a+b)-cos(a+c)*cos(b+c)/(sin(a+c)*sin(b+c))));
area=@(a,b,c)1/pi*atan(sqrt(tan((a+b+c)/2)*tan(a/2)*tan(b/2)*tan(c/2)));
absmin13=14;
rhomin=rho;
while  (rho<=2)
      maxd=0;
      cr0=(3*rho^2+1)/(rho*(rho^2+3));
      crmax=1/rho;
      crmin=1/rho*(3-rho)/(rho+1)+sqrt(1-1/rho^2)*sqrt(1-(3-rho)^2/(rho+1)^2);
      g=@(r)  1/2*(1-cos(r))*(cos(r)<=cr0)+1/2*(1-1/rho+(1-rho^2)/(4*rho)*(1-sin(r)*sqrt(1-1/rho^2)+cos(r)*1/rho))*(cos(r)>cr0);
      t=@(y)  cos(acos((1-y)/(1+y))-acos(1/rho))*(y<(rho^2-1)/4)*(y>(rho-1)/2)+(rho^2+1+2*y)/(2*rho*y+2*rho)*(y>=(rho^2-1)/4);
      %if  (1/x0>(rho-1)/2)
      %      crmin=t(1/x0);
      %end
      %crmax=t(R/x0);
      density=@(x)  (-1/area(x(1),x(2),x(3))*(angle(x(1),x(2),x(3))/(2*pi)*g(x(1))+angle(x(2),x(1),x(3))/(2*pi)*g(x(2))+angle
```

```
(x(3),x(2),x(1))/(2*pi)*g(x(3))))*(cos(x(1))<=crmin)*(cos(x(2))<=crmin)*(cos(x(3))<=crmin)*(cos(x(1))>=crmax)*(cos(x(2))>=crmax)*(cos(x(3))>=crmax);
%density2=@(y) (-1/are(y(1),y(2),y(3))*(angley(y(1),y(2),y(3))/(2*pi)*g(y(1))+angley(y(2),y(1),y(3))/(2*pi)*g(y(2))+angley(y(3),y(2),y(1))/(2*pi)*g(y(3))))*(cos(y(1))<=crmin)*(cos(y(2))<=crmin)*(cos(y(3))<=crmin)*(cos(y(1))>=crmax)*(cos(y(2))>=crmax)*(cos(y(3))>=crmax);
%density3=@(y) (-1/are(y(1),y(2),y(3))*(anglez(z(1),z(2),z(3))/(2*pi)*g(z(1))+anglez(z(2),z(1),z(3))/(2*pi)*g(z(2))+anglez(z(3),z(2),z(1))/(2*pi)*g(z(3))))*(cos(z(1))<=crmin)*(cos(z(2))<=crmin)*(cos(z(3))<=crmin)*(cos(z(1))>=crmax)*(cos(z(2))>=crmax)*(cos(z(3))>=crmax);
y(1)=acos(t(1));
y(2)=acos(t(R));
y(3)=acos(t(1/R));
z=[density([y(1),y(1),y(1)]),density([y(1),y(1),y(2)]),
    density([y(1),y(2),y(2)]),density([y(2),y(2),y(2)]),
    density([y(1),y(1),y(3)]),density([y(1),y(3),y(3)]),
    density([y(3),y(3),y(3)])];
maxd=-min(z);
if (2*maxd*4*rho/(-rho^2+4*rho-3)<absmin13)
    rhomin=rho;
    absmin13=2*maxd*4*rho/(-rho^2+4*rho-3);
```

```
        end
        rho=rho+Delta;
end

a1=[absmin absmin1 absmin2 absmin3 absmin4 absmin5 absmin6
    absmin7 absmin8 absmin9 absmin10 absmin11 absmin12 absmin13];
plot(a1)
grid on
```

BIOGRAPHICAL SKETCH

Eduardo Alejandro Ramirez Martinez was born in H. Matamoros, Tamaulipas, Mexico, on November 4,, 1993, and he was the third child of Dr. Jesus Angel Ramirez and Dr. Telma Ofelia Martinez, both of who have medical profession. He studied in the bilingual school Colegio Mexico and in Colegio Jevenal Rendon in the same cit, then in Valley Christian High School in Brownsville, Texas. Ramirez studied mathematics in the 4 Plus 1 Program in Mathematics and received both the Bachelor and Master of Science degree in Mathematics from The University of Texas Rio Grande Valley on December 14, 2018.

While at UTB and UTRGV, Ramirez worked as a tutor in various programs. He worked for the Math Department as a Teacher Assistant and helped in areas such as calculus, discrete math, modern algebra and statistics. He was also a tutor for the College of Science, Mathematics and Technology, the Learning Enrichment Program and Title V. In addition, Ramirez worked as a graduate lecturer in Fall 2018 as a college algebra instructor.

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