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The creation and propagation of radiation: Fields inside and outside of sources

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We present an algorithm for computing the electromagnetic fields due to currents inside and outside of finite sources with a high degree of spatial symmetry for arbitrary time-dependent currents. The solutions for these fields do not involve the time derivatives of the currents but involve only the currents and their time integrals. We give solutions for moving planar sheets of charge, and a rotating spherical shell carrying a uniform charge density. We show that the general solutions reduce to the standard expressions for magnetic dipole radiation for slow time variations of the currents. If the currents are turned on very quickly, the general solutions show that the amount of energy radiated equals the magnetic energy stored in the static fields a long time after current creation. We give three problems which can be used in undergraduate courses and one problem suitable for graduate courses. These problems illustrate that because the generation of radiation depends on what has happened in the past, a system of currents can radiate even during time intervals when the currents are constant due to radiation associated with earlier acceleration.

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I. INTRODUCTION

There are almost no analytical solutions for electromagnetic fields everywhere in space which are generated by time-varying current systems with a non-vanishing extent in at least one spatial dimension. Such problems can be solved numerically. The standard approach is to first solve the time-harmonic problem and then finds the solution for an arbitrary time-dependence using inverse Fourier transforms.[1] In general, this approach involves numerical computations of much complexity. As a result, we avoid discussions of the fields associated with such problems in intermediate level undergraduate courses. At the graduate level, the physical meaning of the solutions is usually lost in a maze of mathematical complexity (for example, the solutions to the general problem involve vector spherical harmonics).

We would like students in intermediate and graduate level courses to be able to solve analytically problems in which an abrupt change in the current in one part of a system causes changes in the local fields. These field changes can be shown to propagate at the speed of light to other parts of the same system of currents, and interact with those currents, eventually producing radiation far from the source. Such problems illuminate the nature of reaction forces from first principles, instead of deriving the form for those forces from energy considerations. They also show that because radiation fields depend on what has happened in the past, finite current systems can radiate even during time intervals where the currents are not changing in time, due to radiation associated with earlier acceleration.

In this paper, we consider the properties of the complete analytic solutions to two systems of the type we have described: two oppositely moving infinite planes of charge separated by a distance $2a$; and a rotating, uniformly charged spherical shell with radius a. We consider only time-independent charge distributions with divergence-free current systems. For both systems, we determine the vector potential everywhere in space for arbitrary time dependence of the currents. In the slow-motion or dipole approximation limit, our complete solutions illuminate the properties of magnetic dipole radiation which are not accessible in the usual magnetic dipole radiation solutions. The complete solutions also yield new information about the energy radiated for an instantaneous turn-on. We give three undergraduate level problems related to the properties of these solutions. A fourth problem for a related spherical shell of current is appropriate for a graduate level electromagnetism course.

The kinds of systems we consider here, especially that of the spherical shell, were at the forefront of scientific research 100 years ago in the Abraham–Lorentz model of the electron. The Abraham–Lorentz model of the spinning electron has recently been rediscovered as a fascinating dynamical system.[2] Our purpose here is much more prosaic than current research addressing these dynamics. We will assume that the motions are given, and focus on causality, reaction forces, and transit time effects in the consequent generation of electromagnetic fields.

The remainder of this paper is organized as follows. The planar solution is derived in Sec. II, and the solution for the rotating spherical shell is sketched in Sec. III. These solutions are used in Sec. IV to examine issues of propagation, including what is hidden by the slow-motion or dipole approximation, energy considerations, and local reaction forces. Section V presents some numerical solutions for parameter domains of special interest. Section VI gives a summary of our results and insights, and Sec. VII presents four problems suitable for courses in electromagnetism.

II. THE PLANAR SOLENIOD

The problem of the fields of an infinite parallel current sheet changing arbitrarily in time has been treated by many authors.[3] We give a brief derivation of those fields here, using an approach that can also be applied to the problem of a rotating spherical shell. For time-independent charge distributions with divergence-free current systems with arbitrary time dependence, the differential equation for the vector potential $\mathbf{A}$ in the Lorentz gauge is

$$-\nabla\times\nabla\times\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}. \quad (1)$$

For a current sheet in the y-direction located at $x=0$, the current density is given by

$$\mathbf{J}(x,t)=\kappa(t)\delta(x)\hat{\mathbf{y}}, \quad (2)$$

where $\kappa(t)$ is the current per unit length and is an arbitrary function of time. For this current density, the vector potential $\mathbf{A}$ has only a y-component. We use Eq. (2) to write Eq. (1) as

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_y(x,t)=-\mu_0\kappa(t)\delta(x). \quad (3)$$

The general solution to the homogeneous equation in Eq. (3) has the form

$$G(t-x/c)+H(t+x/c), \quad (4)$$

where G and H are any functions of a single variable. To find the particular solution to Eq. (3) with the inhomogeneous source term, we construct it out of solutions to the homogeneous equation. We use causality and assume a function propagating to the left for $x<0$ and to the right for $x>0$. Continuity at the origin requires that

$$A_y(x,t)=G(t-|x|/c). \quad (5)$$

To determine $G(t)$, we note that Eq. (3) implies that there must be a jump in the z-component of the magnetic field (or in the x-derivative of A_y) across $x=0$ given by

$$\begin{aligned} B_z\Big|_{x=+\varepsilon}-B_z\Big|_{x=-\varepsilon}&=\frac{\partial A_y}{\partial x}\Big|_{x=+\varepsilon}-\frac{\partial A_y}{\partial x}\Big|_{x=-\varepsilon}\\ &=-\mu_0\kappa(t)=-\frac{2}{c}\frac{dG(t)}{dt}. \end{aligned} \quad (6)$$

We derived the last term on the right-hand side of Eq. (6) using Eq. (5) to evaluate the left-most expression in Eq. (6). Thus, we have

$$A_y(x,t)=\frac{c\mu_0}{2}\kappa^{(1)}\left(t-\frac{|x|}{c}\right)+C, \quad (7)$$

where

$$\kappa^{(1)}(t)=\int_0^t\kappa(t')dt'. \quad (8)$$

Note that the solution for the vector potential depends on the first time integral of the current as a function of time and not on any of its time derivatives. We will see a similar behavior in the rotating spherical shell case, except we will have terms up to the third time integral of the current as a function of time in that case [cf. Eq. (19)].

The electric and magnetic fields are given by the spatial and temporal derivatives of the vector potential and will thus be proportional to $\kappa(t)$ and not to any of its time derivatives.[4]

Thus, the current sheet can radiate energy to infinity even when the current per unit length $\kappa(t)$ is constant in time. We might suspect that this behavior is a feature of the infinite planar geometry and does not appear in more realistic geometries. However, we will obtain the same behavior for a rotating spherical shell of charge, as we discuss in Sec. IV, where we will explain why this behavior (radiation without apparent acceleration) can be understood in terms of the usual relation between acceleration and radiation.

We now use superposition to determine the solution for two oppositely moving infinite planes of charge separated by a distance $2a$, which we call a planar solenoid. By a planar solenoid, we mean that we have two current sheets in the y-z plane, one located at $x=a$ and one located at $x=-a$, with the current sheet at $x=-a$ given by $-\kappa(t)\hat{\mathbf{y}}$ and the current sheet at $x=+a$ given by $+\kappa(t)\hat{\mathbf{y}}$. The solution for this case is a superposition of the two solutions of the type given in Eq. (7) and can be written in the form

$$\mathbf{A}(x,t)=\hat{\mathbf{y}}\frac{c\mu_0}{2}\left[\kappa^{(1)}\left(t-\frac{|x-a|}{c}\right)-\kappa^{(1)}\left(t-\frac{|x+a|}{c}\right)\right]. \quad (9)$$

III. A ROTATING SPHERICAL SHELL OF CHARGE

We turn to the problem of a uniformly charged rotating shell of radius a. We assume that the radius of the sphere and the distribution of charge remain constant even as the sphere rotates. A real sphere would deform under rapid rotation, so we require an external constraint to maintain the sphere's shape. This consideration is irrelevant here because our focus is on the generation of electromagnetic fields. Our rotating shell carries current in the azimuthal direction, with the current depending on the sine of the polar angle θ. Solutions to this problem in a very different form from ours have been given by Daboul and Jensen[5] and by Vlasov[6] (see also Ref. 2). The current density $\mathbf{J}$ has only a ϕ-component given by

$$J_\phi(r,\theta,t)=\kappa(t)\delta(r-a)\sin\theta. \quad (10)$$

If the sphere has charge per unit area σ and rotates with an angular speed $\Omega(t)$, then $\kappa(t)=\sigma a\Omega(t)$. For future use, we note that the magnetic dipole moment of the current distribution in Eq. (10) is

$$\mathbf{m}(t)=\frac{1}{2}\int\mathbf{r}'\times\mathbf{J}(\mathbf{r}',t)d^3x'=\hat{\mathbf{z}}\frac{4\pi a^3}{3}\kappa(t). \quad (11)$$

If the sphere has a total charge Q and is rotating at a fixed angular rotation rate Ω_0, the static magnetic dipole moment m_0 is given by

$$m_0=\frac{4\pi}{3}a^3\kappa_0=\frac{a^2\Omega_0 Q}{3}. \quad (12)$$

The vector potential $\mathbf{A}$ for the current density given in Eq. (10) has only a ϕ-component, and A_ϕ satisfies the differential equation

$$\begin{aligned}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_\phi)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}A_\phi\\ =-\mu_0 J_\phi.\end{aligned} \quad (13)$$

With the $\sin\theta$ dependence for $\mathbf{J}$ in Eq. (10), $A_\varphi(r,\theta,t)$ can be written in separable form as

$$A_\phi(r,\theta,t) = A(r,t)\sin\theta. \tag{14}$$

We find the differential equation for $A(r,t)$ by inserting Eq. (14) into Eq. (13) and using Eq. (10), yielding

$$\frac{\partial^2}{\partial r^2}(rA) - \frac{2}{r^2}(rA) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rA) = -a\delta(r-a)\mu_0\kappa(t). \tag{15}$$

Readers should compare Eq. (15) for the spherical case to Eq. (3) for the planar case.

The general solution to the homogeneous equation in Eq. (15) has the form

$$rA(r,t) = G'(t-r/c) + \frac{cG(t-r/c)}{r} + H'(t+r/c) - \frac{cH(t+r/c)}{r}. \tag{16}$$

As in the planar case, we find the particular solution to Eq. (15) by assuming the form in Eq. (16), but now we use causality to assume propagation both inward and outward for $r < a$ and only outward for $r > a$, and require regularity of the solution at $r = 0$. We then require continuity of $A(r,t)$ at $r = a$ as well as the jump in the r derivative of $A(r,t)$ at $r = a$ implied by the delta function in the source term in Eq. (15). As in the planar case, these requirements suffice to completely determine the particular solution.

We can also find the solution to Eq. (15) by Laplace transforming and inverting. Because the details of either derivation are involved, and we are mainly interested in the properties of the solutions, we give only the results here and give the details of the time-domain and the Laplace transform approach in a technical note.[7]

We use Eq. (11) to write the solution in terms of $m(t)$ instead of $\kappa(t)$ to facilitate comparison with the classic magnetic dipole solutions, which involve $m(t)$ and its time derivatives. If we define

$$r_< = \min(r,a) \text{ and } r_> = \max(r,a), \tag{17}$$

and

$$t_\pm = t - \frac{r_>}{c} \pm \frac{r_<}{c}, \tag{18}$$

We can write the solution for the vector potential of our spherical shell, which has only a ϕ-component, as

$$A_\phi^{\text{shell}}(r,\theta,t) = \frac{3c\mu_0\sin\theta}{8\pi a^2 r}\left[\begin{array}{l}\left[m^{(1)}(t_+) + m^{(1)}(t_-)\right] - \dfrac{c}{r_<}\left[m^{(2)}(t_+) - m^{(2)}(t_-)\right] \\ + \dfrac{c}{r_>}\left\{\left[m^{(2)}(t_+) + m^{(2)}(t_-)\right] - \dfrac{c}{r_<}\left[m^{(3)}(t_+) - m^{(3)}(t_-)\right]\right\}\end{array}\right], \tag{19}$$

where we have defined the successive integrals of the dipole moment $m(t)$ by

$$m^{(0)}(t) = m(t) = \frac{4\pi a^3}{3}\kappa(t) \tag{20}$$

$$m^{(n+1)}(t) = \int_0^t m^{(n)}(t')dt' \qquad (n = 0,1,2\ldots). \tag{21}$$

Even for a constant magnetic dipole moment, the expression for the vector potential in Eq. (19) appears to be a function of time, because the successive time integrals of a constant $m(t)$ in Eq. (21) are functions of time, even if the magnetic dipole moment is not. Closer examination of Eq. (19) reveals the following. Suppose that the magnetic dipole moment $m(t)$ is zero for $t < 0$, varies in the time interval $0 \le t < T$ in an arbitrary manner, and reaches a constant value m_0 for $t > T$. It can be shown that[7]

$$A_\phi^{\text{shell}}(r,\theta,t) = \begin{cases} 0 & t < \frac{|r-a|}{c} \\ \dfrac{\mu_0 m_0 \sin\theta}{4\pi a^2}\dfrac{r_<^2}{r r_>} & t > \frac{(r+a)}{c} + T. \end{cases} \tag{22}$$

At very long times, Eq. (22) is the correct solution for the vector potential of a uniformly charged spherical shell spinning at a constant rate. As we expect, we see time changes in the vector potential at a given radius r only in the time interval

$$\frac{|r-a|}{c} \le t \le \frac{r+a}{c} + T. \tag{23}$$

For an instantaneous spin-up ($T = 0$) an observer will see fields changing in time only over the time $2a/c$ if $r > a$; if $r < a$, the observer will see fields changing in time only over the time $2r/c$. For future use we also define

$$m^{(-n)}(t) = \frac{d^n m}{dt^n} \qquad (n = 1,2\ldots). \tag{24}$$

We will switch back and forth between the notation in Eq. (24) and the notation in which we use dots over the function to denote time differentiation, for example,

$$\ddot{m}(t) = \frac{d^2 m(t)}{dt^2} = m^{(-2)}(t). \tag{25}$$

The associated electric and magnetic fields can be found by taking the negative time derivative of $\mathbf{A}$ and the curl of $\mathbf{A}$, respectively. We give the explicit forms for these fields in Ref. 7. We also show there that the solution in Eq. (19) is identical to that given in Ref. 5, although they look very different.

IV. THE COMPLETE FIELD SOLUTIONS

It is remarkable that the solution for the spherical shell potential in Eq. (19) does not involve any time derivatives of the magnetic dipole moment $m(t)$. In the context of the classic theory of magnetic dipole radiation, this result is unexpected, which raises three obvious questions.

Can we recover the classic magnetic dipole radiation formulae from the complete spherical shell solution when we go to the slow motion or dipole approximation limit? Can the complete solution of the spherical shell case illuminate features of magnetic dipole radiation which we cannot answer with the classic slow motion solution? Is the planar solenoid solution we have given contained in the spherical shell solution in some limit?

We consider each of these three questions in turn. In the standard development leading to classic magnetic dipole radiation, we assume that if T is the time scale for changes in $\mathbf{J}$, then T is much greater than a/c. We call this approximation the slow motion or dipole approximation. We also assume that $\mathbf{J}$ is well behaved; that is, we may use a Taylor series expansion for the time dependence of $\mathbf{J}$ about the time $t-r/c$. If we also assume that the dipole moment $\mathbf{m}(t)$ is always along the z-axis, and that a/r and a/cT are small compared to unity, then the vector potential has only a ϕ-component, which can be shown to be (Ref. 8)

$$A_\phi^{\text{classic}}(r,\theta,t)=\frac{\mu_0\sin\theta}{4\pi}\left[\frac{m(t-r/c)}{r^2}+\frac{\dot{m}(t-r/c)}{rc}\right]. \quad (26)$$

Because we have shrunk the radius of the sphere to zero, the only information we have about the spatial structure of the current distribution inside a is its spatial moments, through $m(t)$. In contrast to the complete solutions, we have no information about the structure of the fields for r on the order of a. The total rate at which energy is radiated to infinity in classic magnetic dipole radiation is $\mu_0\ddot{m}^2/6\pi c^3$.

Now let us turn to the three questions. The complete spherical shell solution involves $m(t)$ and its time integrals, whereas the classic magnetic dipole radiation involves $m(t)$ and its time derivatives. If we assume that we can expand $m(t)$ in a Taylor series, we can show that Eq. (19) can be written for $r\geq a$ as

$$A_\phi^{\text{shell}}(r\geq a,\theta,t)=\frac{\mu_0\sin\theta}{4\pi}$$
$$\left\{\frac{m(t')}{r^2}+\frac{\dot{m}(t')}{rc}+\frac{1}{10}\left(\frac{a}{c}\right)^2\frac{d}{dt'}\left[\frac{\dot{m}(t')}{r^2}+\frac{\ddot{m}(t')}{rc}\right]+\ldots\right\}, \quad (27)$$

where $t'=t-r/c$. In the limit $a\to 0$, we see from Eq. (27) that the vector potential for the complete spherical shell solutions reduces to the familiar expression for the potential associated with classic magnetic dipole radiation, as given in Eq. (26). Thus, we recover the classic magnetic dipole radiation formulae from the complete spherical shell solution in this limit.

We next turn to the question of whether the complete solution illuminates features of magnetic dipole radiation which were not accessible before. The answer to this question is a resounding yes. If we assume that $a/cT\ll 1$, we can find an explicit expression for the electric field at $r=a$ as follows:

$$E_\phi^{\text{shell}}(a,\theta,t)=-\frac{\mu_0\sin\theta}{4\pi a^2}\left[\dot{m}(t)-\frac{2}{5}\left(\frac{a}{c}\right)^2\dddot{m}(t)+\frac{1}{3}\left(\frac{a}{c}\right)^3\ddddot{m}(t)+\ldots\right]. \quad (28)$$

Equation (28) gives the electric field at the surface of the sphere. We can use Eq. (28) to calculate the rate at which work is being done by the external agents spinning up the sphere.[7] This work is equal to the energy stored in the static field after spin-up plus the energy carried away to infinity by the radiation, as we would expect, but in the classic calculation this equality has been a supposition, whereas now we can do an explicit calculation.

Finally, the planar solenoid solution is contained in our complete solution. To see this, we let the radius of the sphere a and the distance from the origin r go to infinity in such a way that the difference is finite. In this limit, if we are at the equatorial plane where $\sin\theta=1$, at a distance $|x|=|r-a|\ll a$ from the spherical surface, the spherical surface looks planar. In this limit, the only term to survive in the equation for E_ϕ^{shell} [which is the negative time derivative of the expression in Eq. (19)] is

$$\begin{aligned}E_\phi^{\text{shell}}(r,\pi/2,t)&=-\frac{3c\mu_0}{8\pi a^2 r}m(t-|x|/c)\\&=-\frac{c\mu_0}{2}\kappa(t-|x|/c).\end{aligned} \quad (29)$$

We used Eq. (11) to replace $m(t)$ by $4\pi a^3\kappa(t)/3$, and assumed that $m(t)$ is zero in the far distant past. In this limit, we recover the behavior of the planar solenoid case, that is, radiation fields which are not zero even when the sphere is rotating at a constant rate.

How do we account for this situation? Radiation from a planar sheet moving at constant speed has been discussed,[3] but it is well known that problems involving infinite distributions of current can lead to unphysical paradoxes. At first glance, radiation to infinity by an infinite current sheet with a constant current appears to be a candidate for this kind of unphysical behavior. However, the result we have just obtained for spherical geometry dispels any suspicion that the phenomenon is an artifact of ambiguities associated with infinite extent. Given that a facile escape from paradoxes associated with infinite extent is not possible, our result is novel and initially surprising. After all, when the surface is moving at a constant speed, the fields in the vicinity of the surface are not changing in time, yet at and near the surface there is a radiation field E_ϕ, which is a radiation field and produces a radiation reaction.

We first emphasize what is interesting about these solutions, in particular the dynamical aspects related to the creation and flow of momentum. Then we will explain why we are discussing the radiation generated by acceleration, and how this fact is hidden in the mathematics of the final solutions.

One interesting feature of the solutions concerns Lorentz transformations and emerges most transparently for the planar geometry. If the sheet of charge has never been moving, then in the frame in which the sheet is stationary the electric field has only an x-component E_x. A Lorentz transformation in the y-direction does not change this fact. In a frame in which the sheet is moving in the y-direction with some velocity (and has always been moving in the y-direction with that same velocity) there is still no E_y component. The solution in Eq. (9) has a y-component, one that is not the result of observing in one frame or another.

This insight can be applied to a sheet that starts from rest in the lab frame, and then settles down to some constant nonzero speed in this frame. The E_y component cannot be removed by going to a frame co-moving with the sheet at late times. That E_y is unambiguously present. The E_y field, a true radiation field, "knows" not only about the motion of the sheet but also about its history. These counter-intuitive features of the radiation underscore the subtlety of the relation of radiation and its sources.

Before discussing this subtlety, we examine the solutions in terms of momentum creation and flow for a charged sheet. In the following, we keep only terms to first order in v_0/c, but neglect terms of higher order. In the planar case, a single current sheet can be thought of as a moving uniformly charged sheet with positive charge per unit area σ_0, moving at velocity $\mathbf{v}(t) = v(t)\hat{\mathbf{y}}$, with the current per unit length given by $\kappa(t) = \sigma_0 v(t)$. In the discussion of the planar sheet in Sec. II, we assumed that the moving positive charge σ_0 is balanced by a stationary negative charge $-\sigma_0$, so that there is no net electrostatic field (see Ref. 4 for other possibilities). But consider the case for which there is no stationary negative charge, so that there is an electrostatic field. The unbalanced positive charge per unit area σ_0 produces an electrostatic field of magnitude $\sigma_0/2\varepsilon_0$ perpendicular to the sheet in the $\pm x$-direction. Suppose that at $t = 0$, we bring this sheet of charge instantaneously from zero speed to constant speed v_0. At time $t = T > 0$, for $|x| > cT$, we will observe the electrostatic field in the x-direction, and for $|x| < cT$, we will observe both this field and the radiation electric and magnetic fields. The ratio of the magnitude of the radiation electric field in the y-direction to the electrostatic field in the x-direction for $|x| < cT$ is given by [see Eq. (29)]

$$\frac{|E_y|}{|E_x|} = \left[\frac{c\mu_0\kappa_0}{2}\right] \Big/ \left[\frac{\sigma_0}{2\varepsilon_0}\right] = \frac{v_0}{c}. \tag{30}$$

This ratio is what we expect from simple geometry. The foot of the electrostatic field line is rooted in the charges making up the sheet, and the parts of the field line for $|x| < cT$ are moving along with the foot in the y-direction at speed v_0 after $t = 0$. Thus we expect from geometry alone to see the ratio given in Eq. (30). This behavior is like waves on a string, and we would have an analogous situation if a post supporting a string is suddenly set in motion with speed v_0 perpendicular to the string at $t = 0$, with the speed c replaced by the speed of waves on the string.

From this point of view, the reason that the unbalanced positive sheet of charge continues to radiate even after it has been brought up to constant speed is that even though the external agents for a time just greater than zero have already put in the energy required to get the sheet itself up to constant speed, they must continue to do work for $t > 0$ to bring the electrostatic fields associated with the unbalanced charges in the sheet up to speed. The rate at which they continue to add momentum in the y-direction per unit area in the y-z plane is the force that they must exert to counterbalance the reaction electric field. That is,

$$\sigma_0 E_y\Big|_{x=0} = \frac{2\varepsilon_0 E_0^2}{c^2} v_0 c. \tag{31}$$

Where does the momentum provided by the external agents go? The total electromagnetic momentum in the y-direction per unit area in the y-z plane at time $t = T > 0$ is

$$\begin{aligned} 2cT[\varepsilon_0 \mathbf{E} \times \mathbf{B}]_y = 2cT[\varepsilon_0 E_0 B_z] = 2cT\left[\frac{\varepsilon_0 E_0 E_y}{c}\right] \\ = T\left[\frac{2\varepsilon_0 E_0^2}{c^2} v_0 c\right]. \end{aligned} \tag{32}$$

We see from Eq. (32) that the electromagnetic momentum in the y-direction per unit area in the y-z-plane is continuously increasing for $t = T > 0$, as the radiation fields move outward at the speed of light, and that the rate at which the electromagnetic momentum is increasing equals the rate at which the external agents are providing momentum at $x = 0$ [see Eq. (31)].

Even long after the unbalanced charged sheet is up to speed, the sheet continues to radiate. This radiation carries momentum outward at a rate sufficient to get the more and more distant electrostatic fields "up to speed." For a single sheet this radiation never stops, because there is an infinite amount of electrostatic energy to get up to speed. For a two-sheet planar solenoid, there is a finite amount of electrostatic energy per unit area in the y-z plane, and those fields are up to speed after a time $2a/c$, so that the sheets cease to radiate after this time. We expect similar behavior in spinning up a spherical shell of charge.

The dynamical aspects of the fields are reasonable, but what about the fundamental origins of these fields? It appears that the equations tell us that radiation fields can have source currents which are constant in time, but this conclusion is incorrect as can be seen in Eq. (7). At $x = 0$, Eq. (7) tells us that A_y is proportional to $\kappa^{(1)}(t)$ and that, therefore, the radiation electric field at $x = 0$, which is proportional to the time derivative of A_y, depends only on $\kappa(t)$, the present value of the current. It thus appears that the electric field at $x = 0$ does not depend on what happened in the past.

Physically we know that the electric field does depend on the past. Let us focus on a particular observation point, at a particular value of $x = 0$, y, and z and a particular time t. At this place and time, the sources of the field include any point $(0, y', z')$ on the current sheet at a distance L from our point, at a time $t' = t - L/c$, where L is the distance from source to observation point and the time t' is the retarded time for that source point relative to our observation point. For an infinite sheet, no matter how long ago the acceleration stopped, there will always be sufficiently distant source locations which were accelerating when they made their contribution to the field at $(0, y, z)$ at time t.

This result is in the mathematics, but it is not readily apparent in the final solution. Suppose that the current is constant in time and that there never was any acceleration. In Eq. (6), we have that $\dot{G}(t)$ is a constant for all time, and hence A_y is meaningless; it would be infinite in the infinite past. Equivalently, the time integral in Eq. (8) would have to extend back to infinity, and hence would have no meaning; our method would break down. Our method thus requires that there has to be a change in the past, and therefore an acceleration, and that the integral back to the finite past is finite. What is particularly interesting about this insight is that the details of that history do not matter. It only matters that, due to its properties, the integral of $\kappa(t)$ is finite. If this requirement is satisfied, the solution in Eq. (7) is valid.

The appearance of radiation without apparent acceleration is not as awkward for the sphere as for the plane, and the way in which the mathematics is compatible with the physics is more transparent in the spherical case. For the latter case, when the time T in which we spin up the sphere is short compared with the light transit time across the radius of the sphere, this radiation without apparent acceleration should cease after a time $2a/c$. This conclusion should be clear. Suppose we take any finite distribution of current and turn the currents on instantaneously. Far away from this event we expect to see a burst of radiation which lasts a time the order of the finite size of the system divided by the speed of light. Thus, the current distribution must be radiating after the currents are no longer changing in time, because the time interval when the currents are changing is zero (or at least arbitrarily small). Even though this conclusion is clear, we have in Eq. (19) for the first time a complete analytic solution which allows us to examine the process in detail.

V. NUMERICAL SOLUTIONS

We now turn to numerical solutions to further explore the details of the compete solutions. We consider two different time behaviors for the turn-on of $m(t)$. The first has well-behaved time derivatives to all orders.

$$m_{\text{smooth}}(t) = \frac{m_0}{2}\left[\frac{2}{\pi}\arctan\left(\frac{t}{T/5}\right) + 1\right]. \quad (33)$$

The second is a ramp turn-on of $m(t)$ which has derivatives which are not well behaved.

$$m_{\text{ramp}}(t) = m_0 \begin{cases} 0 & t < 0 \\ t/T & 0 < t < T \\ 1 & t > T. \end{cases} \quad (34)$$

For both of these time behaviors, we compute the total energy radiated to infinity in the complete spherical shell solutions and compare it to the amount radiated in the classic magnetic dipole radiation solutions. We normalize the radiated energy in all cases to the energy contained in the static magnetic field for constant rotation rate, which is given by $\mu_0 m_0^2/4\pi a^3$.

The normalized radiated energy for both time behaviors is plotted as a function of the ratio a/cT in Fig. 1, with the two curves labeled as smooth and ramp. For comparison, the dashed straight line labeled dipole in Fig. 1 is the normalized radiated energy using the classic expression for the magnetic dipole radiation rate for the smooth time behavior. It is easily seen that the classic dipole total energy radiated scales as $(a/cT)^3$. For small values of a/cT, the smooth curve in Fig. 1 shows this behavior and is identical with the classical result. As a/cT becomes comparable with and much greater than unity, our numerical result for the energy radiated in the complete spherical shell solutions does not increase without limit, but instead approaches the energy stored in the static magnetic field after spin-up. For the ramp spin-up, the classic expression for the radiated power in the dipole limit cannot be evaluated because the second time derivative of $m(t)$ is ill-defined. However, there is no problem in evaluating our complete spherical shell expressions for the ramp turn-on because we can easily integrate the ramp successive times. The behavior at small values of a/cT for the ramp turn-on is proportional to $(a/cT)^2$, which can be obtained analytically with some effort. The behavior at large values of a/cT is the same as for the smooth function, approaching the same constant, as we expect, because in this limit both $m_{\text{ramp}}(t)$ and $m_{\text{smooth}}(t)$ approach a step function at $t = 0$.

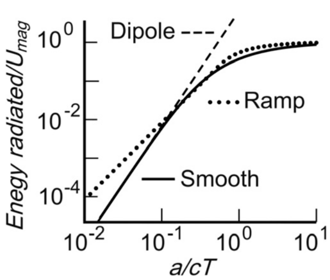

Fig. 1. The normalized total energy radiated by a spherical shell as a function of a/cT for the smooth and ramp spin-up with characteristic time T. The curve labeled dipole is the total normalized energy radiated by a point magnetic dipole computed for the smooth spin-up time function.

Because our solutions are novel for $a/cT \gg 1$, the opposite of the slow motion approximation, we also show the radial profiles of the fields for very rapid turn-on. In Fig. 2, we have taken the ramp turn-on with $a/cT = 10$, or $T = 0.1a/c$. We plot $\tilde{E}_\varphi^{\text{shell}}$, the ϕ component of the electric field of the shell divided by $c\mu_0\kappa_0$, and $\tilde{B}_\theta^{\text{shell}}$, the θ component of the magnetic field of the shell divided by $\mu_0\kappa_0$, at the time $t = 0.5a/c$ after the ramp turn-on. Both of the field components are evaluated in the equatorial plane at $\theta = \pi/2$. Figure 2 is one frame of a video that can be found online.[7]

What we see in Fig. 2 is approximately the fields we would see around a single infinite plane of current located at $x = a$ with a turn-on of the current per unit length given by the ramp turn-on of Eq. (34). In the planar case, at $t = 0.5a/c$, the normalized electric field is constant at a value of $-1/2$ in the interval $0.6a < x < 1.4a$, linearly decreasing from that value to zero at $x = 0.5a$ and $x = 1.5a$. Similarly, in the planar case at $t = 0.5a/c$, the magnitude of the normalized magnetic field is constant at a value of $1/2$ in the interval $0.6a < x < 1.4a$, reversing sign across the current sheet at $x = a$, and then linearly decreasing to zero at $x = 0.5a$ and $x = 1.5a$, respectively. The departures from this planar behavior in Fig. 2 are caused by the spherical geometry, but the similarities are clear.

We also give some feel for the overall topology of the field at the time shown in Fig. 2. To illustrate the structure of the magnetic fields, Fig. 3 uses the line integral convolution method,[9] a method that shows the field as streaks of pixels and provides a visual representation of the field close to the limits of the display. At this time, the magnetic field is only

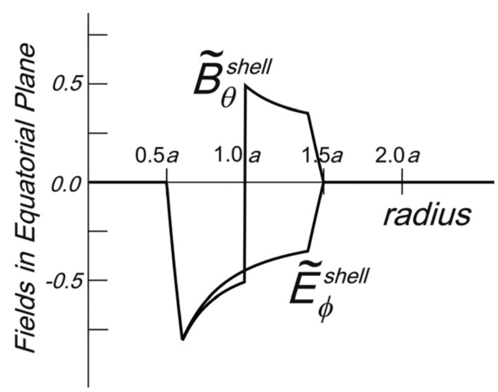

Fig. 2. The normalized θ and ϕ components of the electric and magnetic field of the shell in the equatorial plane for the ramp turn-on with $T = 0.1a/c$. We show the fields at a time $t = 0.5a/c$ after the turn-on starting at $t = 0$ (enhanced online) [URL: http://dx.doi.org/10.1119/1.3682326.1].

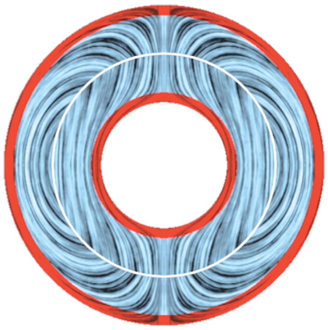

Fig. 3. A line integral convolution representation of the magnetic field topology for a ramp turn-on in a time $T = 0.1a/c$. The field is shown at a time $t = 0.5a/c$ after the turn-on starting at $t = 0$. The sphere is indicated by the circle [URL: http://dx.doi.org/10.1119/1.3682326.2].

non-zero in the interval $0.5a < r < 1.5a$. This spatial behavior of the magnetic field is consistent with the fact that the information that the spherical surface at $r = a$ has started spinning propagates away from the surface of the sphere at the speed of light, beginning at time $t = 0$, and we are looking at a later time $t = 0.5a/c$. We have coded Fig. 3 so that the darker zones correspond to fields associated with the times at which the currents in the sphere are in the process of turning on, and the lighter zones are the fields associated with the times at which the currents in the spherical shell are constant in time. Note that most of the fields created by the spin-up of the sphere are associated with times after the sphere is rotating at constant angular speed and is no longer accelerating. Figure 3 is one frame of a video that is available online.[7]

VI. SUMMARY

We have given the complete solutions everywhere in space for a planar solenoid and a spherical shell of current. Unexpectedly, we find that the solutions for these electromagnetic fields are free of the concepts of differential calculus, and our solutions involve only the currents and their time integrals, and do not involve the time derivatives of the currents. These problems are of interest for several reasons: they present examples which exhibit radiation near a current source at a time when the current is constant. The details of the solution show that this radiation, while ultimately due to changes (accelerations) in the past, is insensitive to the details of those changes. They explicitly display transit time delays across a source associated with its finite dimensions; allow students to see directly the origin of the reaction forces for time-varying systems; and allow for the direct calculation of the ratio of the energy radiated to the energy stored for arbitrary time dependence.

VII. SUGGESTED PROBLEMS

A. Problem 1: The planar solenoid and Faraday's law

This problem is appropriate for the typical junior/senior level course taken by physics majors. The problem probes the quasi-static approximation for the magnetic field of the planar solenoid. In this approximation, we ignore propagation effects and consider the magnetic field inside the solenoid to be spatially uniform and the magnetic field outside the solenoid to be zero. Because we have the exact solutions for the planar solenoid, we can compare the exact solutions to the fields in the quasi-static approximation and draw conclusions about the validity of the approximation. In particular, we will compare the exact solutions to the calculation of the electric field at the surface of the solenoid using Faraday's Law applied assuming the quasi-static approximation for the magnetic field.

There are two current sheets in the y-z plane as we have described. We assume that student has been led through the derivation of Eq. (9).

(a) Assume that to a good approximation the magnetic field as a function of time and space is given by $\mathbf{B}(x,t) = \mu_0\kappa(t)\hat{\mathbf{z}}$ for $|x| \le a$ and is zero otherwise. Use Faraday's Law to find the electric field at $x = \pm a$.

(b) Find an expression for $\mathbf{E}(x, t)$ from the exact solution for the vector potential in Eq. (9). Assume that the time scale T for significant variation in $\kappa(t)$ is much greater than a/c and expand your exact expression for $\mathbf{E}$ using a Taylor series, keeping only leading order terms in a/cT. Show that if you keep terms to this order, then the exact solution for the electric field assuming $T \gg a/c$ reduces to the electric field obtained in (a) at $x = \pm a$.

B. Problem 2: Comparison of the radiated and stored energy for a planar solenoid

Consider two current sheets in the y-z plane. We assume that students have been led through the derivation of Eq. (9) and the associated fields.

(a) Assume that $T \gg a/c$ and show that for $x > a$, Eq. (9) can be written to leading order as

$$\mathbf{A}(x,t) = c\mu_0 a\kappa\left(t - \frac{x}{c}\right)\hat{\mathbf{y}}. \tag{35}$$

(b) Assume the functional form for $\kappa(t)$ is similar to the smooth turn-on form given in Eq. (33), and that $T \gg a/c$. Use the solution for the vector potential in Eq. (35), and its associated fields, and calculate the ratio of the energy radiated away to infinity per unit area perpendicular to the x direction to the energy in the final magnetic field per unit area perpendicular to the x direction for $t > T$ after the current has stopped increasing in time. (Hint: $\int_{-\infty}^{\infty} d\eta/(1+\eta^2)^2 = \pi/2$). [Answer: $(5/\pi)a/cT$].

(c) Now assume that $\kappa(t)$ goes instantaneously from 0 to κ_0 at $t = 0$. Use the fields derived from your exact solution for the potential in Eq. (9) to calculate the same ratio as in (b). [Answer: 1.]

C. Problem 3: Reaction forces for the uniformly-charged rotating spherical shell

Consider a spherical shell carrying a current as described in Eq. (10). We assume that the student has been led through the derivation of Eq. (19) using the time domain method.[7]

(a) For the case that the time scale T for significant variation in $m(t)$ is much greater than a/c, use Taylor series expansions of the exact solutions in Eq. (19) to show that the electric field at $r = a$ is given by Eq. (28).

(b) Assume that current is turned on over a time T with $\kappa(t) = 0$ for $t < 0$ and $\kappa(t) = \kappa_0$ for $t > T$. Assume also that the turn on and the leveling off to constant current are smooth in the sense that the first through third time derivatives of $\kappa(t)$ vanish both at $t = 0$ and at $t = T$. Use Eqs. (10) and (28) and calculate the work done (that is, the time and space integral of $-\mathbf{J} \cdot \mathbf{E}$) assuming that the time-scale for change T is much larger than a/c. Relate your answer to the energy stored in the magnetic field after $t = T$ and to the energy radiated away in magnetic dipole radiation between 0 and T. [Hints: the rate at which energy is radiated into all solid angles in magnetic dipole radiation is given by $\mu_0 \ddot{m}^2 / 6\pi c^3$, and the magnetic dipole moment for this problem is given in Eq. (11)].

(c) Now assume that $\kappa(t)$ goes instantaneously from 0 to κ_0 at $t = 0$. Consider only the radiation terms in the electric field associated with the potential in Eq. (19), that is, the terms going as $1/r$. For these terms assume that the corresponding field radiation terms are in magnitude the E field radiation terms divided by c. Use your exact solutions to calculate the energy radiated away to infinity for this process of instantaneously spinning up the sphere, and show that it is equal to the energy stored in the magnetic field long after the current has stopped increasing in time.

D. Problem 4: The spherical shell (graduate level)

Consider a current carrying shell in which the current in the ϕ-direction depends on the polar angle as

$$J_\phi(r, \theta, t) = \kappa(t)\delta(r - a)P_l^1(\cos\theta), \tag{36}$$

where $P_l^m(\cos\theta)$ is an associated Legendre polynomial.

(a) Show that the solution for the vector potential $\mathbf{A}$ is given by

$$A_\phi(r,t) = \frac{3c\mu_0}{8\pi a^3 r} P_l^1(\cos\theta) \sum_{k=0}^{l} \sum_{m=0}^{l} \frac{\gamma_{l,k}\gamma_{l,m} c^{k+m}}{(2r_>)^k (2r_<)^m} \begin{bmatrix} (-1)^m m^{(k+m+1)}(t_+) \\ +(-1)^{l+1} m^{(k+m+1)}(t_-) \end{bmatrix}, \tag{37}$$

where $m^{(n)}(t)$ is defined in Eqs. (20) and (21), and

$$\gamma_{l,m} = \frac{(l+m)!}{m!(l-m)!}. \tag{38}$$

(b) How would you solve the general problem for any axially symmetric current distribution on a spherical shell by superposition of the solutions given in Eq. (37)?

(c) Now assume that $\kappa(t)$ goes instantaneously from 0 to κ_0 at $t = 0$. Consider only the radiation terms in Eq. (37). Use your exact solutions and show that the energy radiated away to infinity for this process of instantaneously spinning up the sphere is equal to the energy stored in the static magnetic field a long time after the current has stopped increasing in time. That energy in the static magnetic field for general l is given by

$$U_l^{\text{mag}} = \frac{l(l+1)}{(2l+1)^2} \frac{9\mu_0 m_0^2}{8\pi a^3}. \tag{39}$$

(d) Argue from energy considerations that for any axially symmetric current distribution on a spherical shell, the energy required to establish the currents in a time very short compared with a/c must be approximately equal to twice the energy stored in the magnetic field after they the currents are established.

(e) Think of a way to show part (d) directly from your solution in Eq. (37), rather than relying on an energy argument.

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[4]An x-component of the electric field would arise from the scalar potential ϕ and from the relation $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$. We ignore ϕ and hence omit the x-component of E. Because we treat the charge density as constant in time, the scalar potential and E_x are constant, hence irrelevant to our considerations. Note that we could have chosen to have the surface current consist of opposite charge densities driven into motion in opposite directions, producing the same total current density as in Eq. (2), but without any charge density, and hence without any E_x. These same considerations, with appropriate modifications, justify our omission of the radial electric field for the spinning spherical shell considered in Sec. III.

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[7]See supplementary material at http://dx.doi.org/10.1119/1.3682326 for the method for solving the spherical case, for the solutions to the problems posed in the text, and for animations of Figs. 2 and 3.

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