

University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

Physics and Astronomy Faculty Publications
and Presentations

College of Sciences

12-6-2022

Non-Hermitian edge burst without skin localizations

Cem Yuce

H. Ramezani

The University of Texas Rio Grande Valley, hamidreza.ramezani@utrgv.edu

Follow this and additional works at: https://scholarworks.utrgv.edu/pa_fac



Part of the [Astrophysics and Astronomy Commons](#), and the [Physics Commons](#)

Recommended Citation

Yuce, C., and H. Ramezani. "Non-Hermitian edge burst without skin localizations." arXiv preprint arXiv:2212.02879 (2022).

This Article is brought to you for free and open access by the College of Sciences at ScholarWorks @ UTRGV. It has been accepted for inclusion in Physics and Astronomy Faculty Publications and Presentations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact justin.white@utrgv.edu, william.flores01@utrgv.edu.

Non-Hermitian edge burst without skin localizations

C. Yuce, H. Ramezani

*Department of Physics, Eskisehir Technical University, Eskisehir, Turkey and
Department of Physics and Astronomy, University of Texas Rio Grande Valley, Edinburg, Texas 78539, USA*
(Dated: December 7, 2022)

In a class of non-Hermitian quantum walk in lossy lattices with open boundary conditions, an unexpected peak in the distribution of the decay probabilities appears at the edge, dubbed edge burst. It is proposed that the edge burst is originated jointly from the non-Hermitian skin effect (NHSE) and the imaginary gaplessness of the spectrum [Wen-Tan Xue et al., Phys. Rev. Lett. 128, 120401 (2022)]. Using a particular one-dimensional lossy lattice with a nonuniform loss rate, we show that the edge burst can occur even in the absence of NHSE. Furthermore, we discuss that the edge burst may not appear if the spectrum satisfies the imaginary gaplessness condition. Aside from its fundamental importance, by removing the restrictions on observing the edge burst effect, our results open the door to broader design space for future applications of the edge burst effect.

I. INTRODUCTION

Non-Hermitian extension of the topological phase in one-dimensional systems has recently attracted a great deal of attention. In particular, NHSE was shown to break the standard bulk-boundary correspondence principle since eigenstates that are not at the Bloch points are localized at the edge of the open lattice [1, 2]. NHSE that can be predicted using the spectral winding number [3] implies that the spectra in the complex energy plane are highly sensitive to the boundary conditions [4–32]. The spectra form a loop/loops under the periodic boundary conditions, and becomes topologically nontrivial. On the other hand, a topological phase transition occurs and the system become topologically trivial in terms of a point gap under open boundary conditions [33–36].

Quantum dynamics in non-Hermitian systems are believed to be quite different from the standard Hermitian systems. Quantum walk originated from a generalization of the classical random walk has also been extended to non-Hermitian systems [37]. A quantum walker will completely leak out eventually from a bipartite lossy lattice with uniform loss rates [38]. The quantum walker in this system is expected to escape predominantly from nearby sites of a starting point that is far from the edges. However, numerical computations show that the decay probability distribution is left-right asymmetric, and a relatively large peak in the loss probability at the farthest edge from the starting point occurs. More unexpectedly, the relative height of this peak grows with the distance between the starting point and the edge. Originally, it was attributed to topological edge states [38], which is questioned in a recent paper [39]. The appearance of an edge peak (so-called the edge burst) was demonstrated to stem entirely from the interplay between two prominent non-Hermitian phenomena, NHSE and imaginary gap closing [39]. The left-right asymmetry is attributed to the NHSE since all eigenstates are localized at one edge of the system, and the large peak at the edge is due to the imaginary gap closing.

In this paper, we show that edge burst can occur even in the absence of NHSE. We consider the same lattice as

[38, 39] but with non-uniform loss rates. The NHSE disappears due to the nonuniform nature of the loss rates, but imaginary gap closing condition on the spectrum is satisfied. The left-right asymmetry of the decay probability occurs in the system due to the phase difference of the couplings in each unit cell. We also demonstrate that there exist systems with left-right asymmetric decay probability and without edge burst even if the imaginary gap closing condition on the spectrum is satisfied.

II. QUANTUM WALK

We consider a quantum walker in a tight-binding one-dimensional non-Hermitian lattice with N unit cells. The lattice as shown in Fig. 1 is composed of two sublattices A and B . The non-Hermiticity comes from lossy B sublattice with nonuniform loss rates. The dynamics of the quantum walker in this lattice obeys the following coupled equations

$$\begin{aligned}i \frac{d\psi_n^A}{dt} &= t_1 \psi_n^B + i \frac{t_2}{2} (\psi_{n-1}^A - \psi_{n+1}^A) + \frac{t_2}{2} (\psi_{n-1}^B + \psi_{n+1}^B) \\i \frac{d\psi_n^B}{dt} &= t_1 \psi_n^A + i \frac{t_2}{2} (\psi_{n-1}^B - \psi_{n+1}^B) + \frac{t_2}{2} (\psi_{n-1}^A - \psi_{n+1}^A) \\ &\quad - i \gamma_n \psi_n^B\end{aligned}\quad (1)$$

where $n = 1, 2, \dots, N$, $\psi_n^A(t)$ and $\psi_n^B(t)$ are time-dependent complex field amplitudes in the A and B sublattices, respectively, t_1 and t_2 are real positive parameters describing couplings and $\gamma_n > 0$ are site-dependent loss rates.

Suppose that the quantum walker is initially placed in the A sublattice at the starting unit cell S that is supposed to be close to the right edge. Therefore, the initial conditions are given by $\psi_n^A(t=0) = \delta_{n,S}$ and $\psi_n^B(t=0) = 0$. To study the dynamics, we numerically solve Eq. (1) subject to the open boundary conditions and the above initial conditions. During the quantum walk, the walker moves in discrete steps in both sublattices and escape only from lossy B sites. As $t \rightarrow \infty$, the walker completely leaks out from the system. The decay probability that

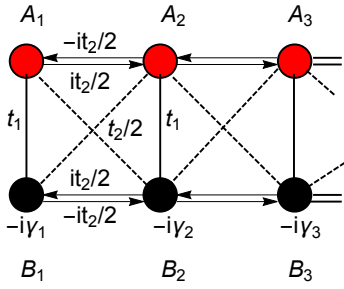


FIG. 1: The lossy finite tight-binding lattice with two sublattices. Losses occur only in B sublattice with nonuniform loss rates $\gamma_n = \{\gamma_1, \gamma_2, \dots, \gamma_N\}$ with N being the total number of unit cells. A quantum walker starts in the A sublattice A_S that is supposed to be close to the right edge. As $t \rightarrow \infty$, the walker completely leaks out from the system.

the quantum walker escapes from the leaky B sublattice with the site number n is given by [38]

$$P_n = 2 \gamma_n \int_0^\infty |\psi_n^B(t)|^2 dt \quad (2)$$

with total decay probability conservation $\sum_{n=1}^N P_n = 1$.

As a special case, the system exhibits left-right asymmetry in decay probabilities when the loss rate is uniform ($\gamma_n = \gamma_1$) [38, 39]. In this case, P_n is maximum at $n = S$ and decreases algebraically in the bulk as n decreases from S , and then makes a sharp peak at the left edge (edge burst). On the other hand, P_n is very small for $n > S$. We are interested in exploring the edge burst for the system with nonuniform loss rates. We think that we can enhance or suppress the edge burst by a proper choice of γ_n . For example, one intuitively expects that it can be suppressed if γ_n are large around the starting point S but small around the left edge. Let us specifically suppose that the loss rate increases linearly from the left edge with a constant rate of change γ

$$\gamma_n = \gamma n \quad (3)$$

from which one may naively say that the edge burst can be suppressed by thinking that the quantum walker escapes from the system before reaching the left edge of the lattice ($N \gg 1$ and $S \gg 1$). But we numerically see that this is not the case. In fact, the system has radically different behavior than its uniformly lossy analogue and edge burst can be enhanced instead.

To quantify the edge burst, we use the relative height, defined as P_1/P_{min} , where $P_{min} = \min\{P_1, P_2, \dots, P_S\}$ is the minimum of P_n between the left edge and the starting point S . We note that $P_1/P_{min} \gg 1$ and $P_1/P_{min} \sim 1$ are the evidence of the existence and absence of the edge burst, respectively [39]. We can also define another ratio P_1/P_S to compare the decay probabilities at the left edge and starting point. For the uniform loss rate, this ratio is always smaller than 1, indicating that the quantum

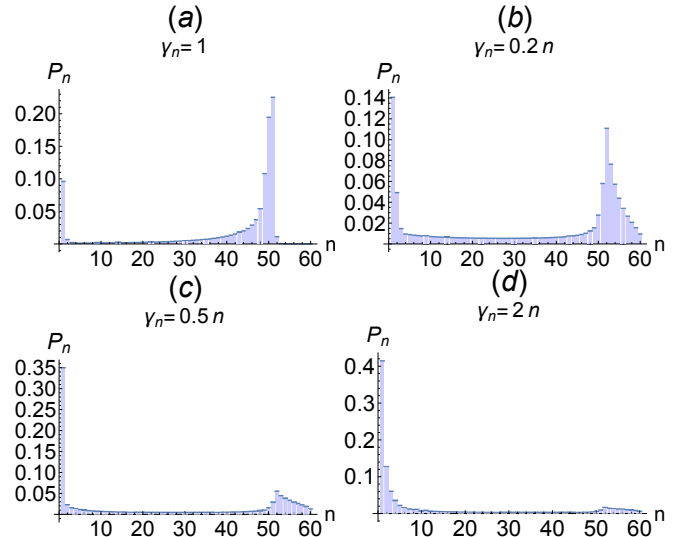


FIG. 2: The distribution of the decay probabilities when the loss rates are uniform (a) and nonuniform (b,c,d). The ratio P_1/P_{min} are 18, 25, 76 and 124 for (a,b,c) and (d), respectively. The other ratio P_1/P_S is smaller than 1 for (a), whereas bigger than 1 and increases with γ for (b-d). The edge burst is enhanced with increasing γ . 14, 35 and 40 percentages of the total decay occur at the left edge for (b), (c) and (e), respectively. All the systems have left-right asymmetry, but they cannot attributed to the NHSE for the systems with nonuniform loss rates. The decay probabilities don't go to zero sharply in (b-d) as opposed to the case in (a). The parameters are $t_1 = 0.3$, $t_2 = 0.5$, $S = 50$ and $N = 60$.

walker leaks out more from the starting unit cell (Fig.2 (a)). However, the ratio P_1/P_S can be bigger than 1 for the nonuniform loss rate (3) as can be seen from Fig.2 (b-d). This is counterintuitive as it is natural to expect the quantum walker to decay mostly from the starting unit cell, and not from the farthest unit cell, which has also the least loss rate. It seems that the quantum walker reaches the left edge with less losses if we increase γ and waits there until it decays completely from there. We find that P_1 grows with γ , whereas P_S decreases with it. At quite large values of the loss rate ($\gamma > 3$), the peak at the starting point disappears. We can also compare the behavior of P_n in the bulk. P_n decreases algebraically in the bulk as n decreases from S when $\gamma_n = \gamma_1$ (Fig.2 (a)). However, it stays almost constant at many sites in the bulk for the system with the nonuniform loss rate (3) (Fig.2 (b-d)).

In each plot in Fig.2, the distribution of P_n is left-right asymmetric. Besides, P_n sharply falls to be almost zero value in the right of the starting point ($n > S$) in Fig.2 (a) because of the strong NHSE. However, we see a tail for the last three plots in Fig.2. In fact, NHSE is absent when the loss rates are nonuniform (3) and hence we can not attribute the left-right asymmetry in Fig. 2 (b-d) directly to NHSE. This is in stark contrast to the case considered in [39], in which the authors conclude that NHSE is necessary for the edge burst. Below, we explore

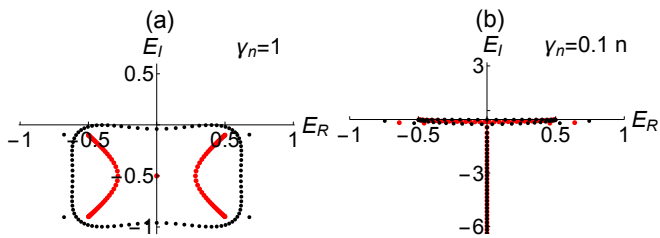


FIG. 3: The energy spectra for the uniform (a) and nonuniform loss rates (b), where the black and red points are for the ring and open geometries, respectively. The spectra for open and ring lattices are drastically different only for the uniform lattice, indicating that NHSE occurs only for the uniform one. The spectra are T -shaped for the nonuniform lattice, and close the imaginary gap. The parameters are $t_1 = 0.3$, $t_2 = 0.5$ and $N = 96$.

this issue in more detail.

Let us find the energy spectra for the ring and open configurations. The ring configuration is the one for which the left and right edges of the lattice are connected ($\psi_{N+1}^A = \psi_1^A$, $\psi_{N+1}^B = \psi_1^B$) and the open configuration is the one with two open edges ($\psi_0^A = \psi_{N+1}^A = \psi_0^B = \psi_{N+1}^B = 0$). Therefore, the Hamiltonians for the ring and open lattices are perturbatively different from each other (there are extra couplings between the edges in the ring lattices). However, the corresponding spectra can be quite different from each other. In the case of the uniform loss rate $\gamma_n = \gamma_1$, the spectrum for the ring lattice can make a loop in complex energy plane, whereas the spectrum for the open lattice is placed inside this loop (Fig.3 (a)). This drastic spectral difference leads to the NHSE, which means that an extensive number of eigenstates are localized at one edge of the open lattice, whereas eigenstates are extended for the ring lattice. On the other hand, in the case of the nonuniform loss rate (3), the spectra for the ring and open lattices almost coincide, indicating that NHSE doesn't appear. The spectra are T -shaped in the complex plane, and the band gap is closed in the imaginary axis (Fig.3 (b)). Let us now study the eigenstates for the open lattice. The states with imaginary (real) eigenvalues occupies mostly in B (A) sublattice. In order to quantify the localization, we use the mean displacement $\langle n \rangle$ for an eigenstate with energy E and the averaged mean displacement over all energy eigenvalues $\overline{\langle n \rangle}$ for both sublattices

$$\langle n \rangle_j = \sum_n n |\phi_n^j|^2, \quad \overline{\langle n \rangle_j} = \frac{1}{N} \sum_E \langle n \rangle_j \quad (4)$$

where $j = A, B$ and the stationary solutions are given by $\psi_n^A = e^{-iEt} \phi_n^A$ and $\psi_n^B = e^{-iEt} \phi_n^B$. Let us discuss them qualitatively. A skin state localized at the left edge has definitely a small mean displacement, $\langle n \rangle_j \ll N$, whereas extended states have rather large mean displacements. The averaged mean displacement $\overline{\langle n \rangle_j}$ is more useful in our analysis to determine the phase of the system. For the skin phase with NHSE, an extensive number

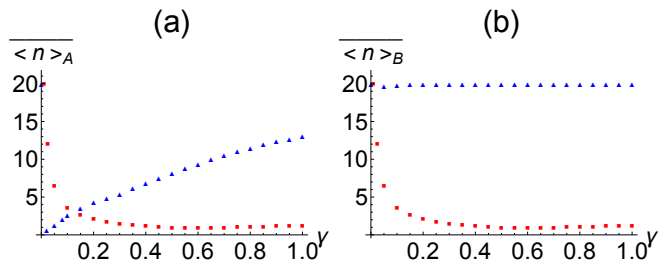


FIG. 4: The averaged mean displacements $\overline{\langle n \rangle_A}$ and $\overline{\langle n \rangle_B}$ as a function of γ for the A and B sublattices, respectively. They decrease for the open lattice with the uniform loss rates, $\gamma_n = \gamma$ (in red) and become very small unless the loss rate is small, indicating that NHSE occurs. However, an extensive number of eigenstate localization doesn't occur for the open lattice with the nonuniform loss rates, $\gamma_n = \gamma n$ (in blue) since the averaged mean displacements are not small for both sublattices. The parameters are $t_1 = 0.3$ and $t_2 = 0.5$, $N = 40$.

of eigenstate localization occurs at the left edge. This implies that the averaged mean displacement is small, $\overline{\langle n \rangle_j} \ll N$, and does not change considerably with system size. On the other hand, for the system without NHSE, there can be both extended states and localized states with various localization centers, implying that the averaged mean displacement is a large number, and increases considerably with the system size. Fig. 4 (a) and (b) show us how the averaged mean displacements change with γ for A and B sublattices, respectively. We plot them for the uniform (in red) and nonuniform (in blue) loss rates. As can be seen, the behaviors are drastically different from each other. In the case of the uniform loss rate $\gamma_n = \gamma_1$, averaged mean displacements are quite small in both sublattices, indicating that skin localization occurs in both sublattices. However, in the case of the nonuniform loss rate (3), the averaged mean displacements are small for small values of γ and increase with γ in A sublattice, whereas they are large and don't change considerably with γ in B sublattice. This indicates that skin localization can occur only in A sublattice for small values of γ , whereas no skin localizations occur in B sublattice for any value of γ . In fact, there exist states with imaginary eigenvalues localized around each site in B sublattice, and they become more tightly localized as γ is increased (like localized Wannier-Stark states in a single-band Hermitian tight-binding lattice with electric field). On the other hand, the states with real eigenvalues occupies mostly in A sublattice and they become more and more extended as γ is increased.

We next discuss the reason why the left-right asymmetry of P_n appears even in the absence of NHSE. We begin to note that the A and B sublattices favor opposite propagations even in the Hermitian case ($\gamma = 0$), implying that the asymmetry has nothing to do with the non-Hermiticity directly. This is because of the fact that the magnitude of the forward and backward couplings ($it_2/2$ and $-it_2/2$) in A sublattice is reversed in B sublattice.

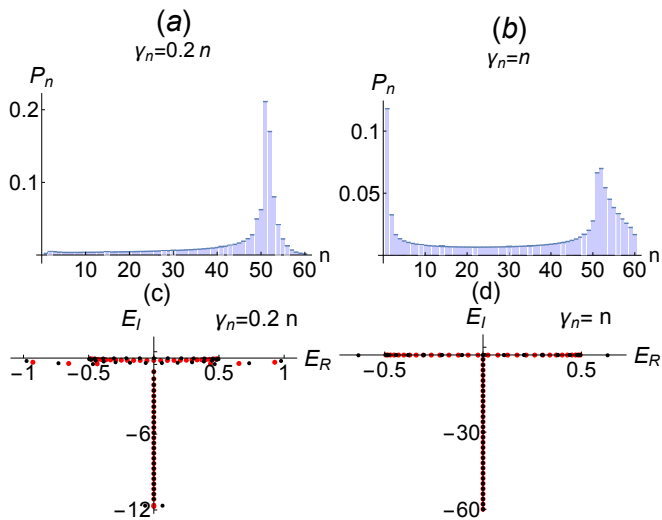


FIG. 5: P_n for two different γ_n (a,b). The corresponding $P_1/P_{min} = 1$ at $\gamma = 0.2n$ and $P_1/P_{min} = 24$ at $\gamma = n$. The corresponding spectra in the complex plane have T -shape structure (c,d) The ring (in black) and open (in red) lattices have almost the same spectra. Both systems satisfy the imaginary gaplessness condition. The parameters are $t_1 = 0.7$ and $t_2 = 0.5$ and $N = 60$.

This phase difference generates counterclockwise motions in the unit cell such that the A and B sublattices favor motions to the left and right, respectively. In the Hermitian lattice, the quantum walker moves to the left in A sublattice and right in B sublattice in such a way that no net motion is generated. Fortunately, the motion towards right in B sublattice is suppressed if we introduce losses in B sublattice. This leads to asymmetric behav-

ior of the quantum walker in our system, meaning that NHSE is not necessary for the edge burst. Note that this asymmetric behavior disappears if the loop in the unit cell is broken at $t_1 = 0$.

As mentioned above, the edge burst for the uniform lattice is thought to be originated jointly from NHSE and the imaginary gaplessness (the spectrum under periodic boundary conditions touches the real axis and closes the imaginary gap) [39]. The T -shaped spectra in the complex plane seem to satisfy the imaginary gaplessness condition for the appearance of the edge bursts in our specific examples. However close inspections reveal that there may be some other examples with T -shaped spectra but without the edge burst effect. We show two such examples in Fig. 5 with T -shaped spectra. However, the edge burst appear only for $\gamma_n = n$.

A relatively huge peak at the edge in the distribution of the decay probabilities in the bipartite lossy lattices is the evidence of the so-called edge burst. In this Letter, we consider such lattices with the nonuniform loss rate and show that edge burst occur even in the absence of NHSE. We discuss that the left-right asymmetry of the decay probabilities can be due to the phase difference of the couplings in the unit cell. We also show that the edge burst may not appear even if the spectrum closes imaginary gap in the complex energy plane. Our paper shows interesting dynamics and can stimulate other researchers to study dynamics of non-Hermitian systems in order to find the real source of the edge burst.

H. R. acknowledges the support by the Army Research Office Grant No. W911NF-20-1-0276, NSF Grant No. PHY-2012172 and OMA-2231387.

-
- [1] Shunyu Yao, Fei Song, and Zhong Wang, “Non-Hermitian Chern Bands, Phys. Rev. Lett. **121**, 136802 (2018).
 - [2] Shunyu Yao and Zhong Wang, “Edge States and Topological Invariants of Non-Hermitian Systems, Phys. Rev. Lett. **121**, 086803 (2018).
 - [3] Huitao Shen, Bo Zhen, and Liang Fu, “Topological Band Theory for Non-Hermitian Hamiltonians, Phys. Rev. Lett. **120**, 146402 (2018).
 - [4] Hui Jiang, Li-Jun Lang, Chao Yang, Shi-Liang Zhu, and Shu Chen, “Interplay of non-Hermitian skin effects and Anderson localization in nonreciprocal quasiperiodic lattices, Phys. Rev. B **100**, 054301 (2019).
 - [5] C Yuce, “Anomalous features of non-Hermitian topological states, Ann. Phys. **415**, 168098 (2020).
 - [6] Yanxia Liu, Qi Zhou, and Shu Chen, “Localization transition, spectrum structure, and winding numbers for one-dimensional non-Hermitian quasicrystals, Phys. Rev. B **104**, 024201 (2021).
 - [7] C Yuce, “PT symmetric Aubry-Andre model, Phys. Lett. A **378**, 2024 (2014).
 - [8] Jahan Claes and Taylor L. Hughes, “Skin effect and winding number in disordered non-Hermitian systems, Phys. Rev. B **103**, L140201 (2021).
 - [9] Ling-Zhi Tang, Guo-Qing Zhang, Ling-Feng Zhang, and Dan-Wei Zhang, “Localization and topological transitions in non-Hermitian quasiperiodic lattices, Phys. Rev. A **103**, 033325 (2021).
 - [10] Qi-Bo Zeng and Yong Xu, “Winding numbers and generalized mobility edges in non-Hermitian systems, Phys. Rev. Research **2**, 033052 (2020).
 - [11] C Yuce, “Nonlinear non-Hermitian skin effect, Phys. Lett. A **408**, 127484 (2021).
 - [12] Yifei Yi and Zhesen Yang, “Non-Hermitian Skin Modes Induced by On-Site Dissipations and Chiral Tunneling Effect, Phys. Rev. Lett. **125**, 186802 (2020).
 - [13] S. Longhi, “Topological Phase Transition in non-Hermitian Quasicrystals, Phys. Rev. Lett. **122**, 237601 (2019).
 - [14] Sebastian Schiffer, Xia-Ji Liu, Hui Hu, and Jia Wang, “Anderson localization transition in a robust PT-symmetric phase of a generalized Aubry-Andre model, Phys. Rev. A **103**, L011302 (2021).
 - [15] Z. Ozcakmakli Turker and C. Yuce, “Open and closed

- boundaries in non-Hermitian topological systems, *Phys. Rev. A* **99**, 022127 (2019).
- [16] C Yuce, “Non-Hermitian anomalous skin effect”, *Phys. Lett. A* **384**, 126094 (2020).
- [17] Xiaoming Cai, “Boundary-dependent self-dualities, winding numbers, and asymmetrical localization in non-Hermitian aperiodic one-dimensional models, *Phys. Rev. B* **103**, 014201 (2021).
- [18] Ching Hua Lee and Ronny Thomale, Ching Hua Lee, Ronny Thomale, “Anatomy of skin modes and topology in non-Hermitian systems, *Phys. Rev. B* **99**, 201103(R) (2019).
- [19] P. Wang, L. Jin, and Z. Song, “Non-Hermitian phase transition and eigenstate localization induced by asymmetric coupling, *Phys. Rev. A* **99**, 062112 (2019).
- [20] Motohiko Ezawa, “Non-Hermitian boundary and interface states in nonreciprocal higher-order topological metals and electrical circuits, *Phys. Rev. B* **99**, 121411(R) (2019).
- [21] Ching Hua Lee, Linhu Li, and Jiangbin Gong, “Hybrid Higher-Order Skin-Topological Modes in Nonreciprocal Systems, *Phys. Rev. Lett.* **123**, 016805 (2019).
- [22] Linhu Li, Ching Hua Lee, Sen Mu, Jiangbin Gong, “Critical non-Hermitian skin effect, *Nat. Commun.* **12**, 5491 (2020).
- [23] Flore K. Kunst, Guido van Miert, and Emil J. Bergholtz, “Extended Bloch theorem for topological lattice models with open boundaries, *Phys. Rev. B* **99**, 085427 (2019).
- [24] C. Yuce, H. Ramezani, “Coexistence of extended and localized states in one-dimensional non-Hermitian Anderson model, *Phys. Rev. B* **106**, 024202 (2022).
- [25] Hui Jiang, Rong Lü, Shu Chen, “Topological invariants, zero mode edge states and finite size effect for a generalized non-reciprocal Su-Schrieffer-Heeger model, *Eur. Phys. J. B* **93**, 125 (2020).
- [26] C. Yuce, H. Ramezani, “Robust Exceptional Points in Disordered Systems, *EPL* **126**, 17002 (2019).
- [27] Jong Yeon Lee, Junyeong Ahn, Hengyun Zhou, and Ashvin Vishwanath, “Topological Correspondence between Hermitian and Non-Hermitian Systems: Anomalous Dynamics, *Phys. Rev. Lett.* **123**, 206404 (2019).
- [28] Yang Cao, Yang Li, and Xiaosen Yang, “Non-Hermitian bulk-boundary correspondence in a periodically driven system, *Phys. Rev. B* **103**, 075126 (2021).
- [29] C. Yuce, “Quasi-stationary solutions in 1D non-Hermitian systems, *Phys. Lett. A* **403**, 127384 (2021).
- [30] C Yuce, “Spontaneous topological pumping in non-Hermitian systems, *Phys. Rev. A* **99**, 032109 (2019).
- [31] Janet Zhong, Kai Wang, Yubin Park, Viktor Asadchy, Charles C. Wojcik, Avik Dutt, and Shanhui Fan, “Non-trivial point-gap topology and non-Hermitian skin effect in photonic crystals, *Phys. Rev. B* **104**, 125416 (2021).
- [32] Ken-Ichiro Imura and Yositake Takane, “Generalized bulk-edge correspondence for non-Hermitian topological systems, *Phys. Rev. B* **100**, 165430 (2019).
- [33] Nobuyuki Okuma, Kohei Kawabata, Ken Shiozaki, and Masatoshi Sato, “Topological Origin of Non-Hermitian Skin Effects, *Phys. Rev. Lett.* **124**, 086801(2020).
- [34] Kai Zhang, Zhesen Yang, and Chen Fang, “Correspondence between Winding Numbers and Skin Modes in Non-Hermitian Systems, *Phys. Rev. Lett.* **125**, 126402 (2020).
- [35] Z. Yang, K. Zhang, C. Fang, J. Hu, “Non-Hermitian Bulk-Boundary Correspondence and Auxiliary Generalized Brillouin Zone Theory, *Phys. Rev. Lett.* **125**, 226402 (2020).
- [36] K. Wang, A. Dutt, K. Y. Yang, C. C. Wojcik, J. Vuckovic, S. Fan, “Generating arbitrary topological windings of a non-Hermitian band, *Science* **371**, 1240 (2021).
- [37] L Xiao, et. al., “Observation of topological edge states in parity-time-symmetric quantum walks, *Nature Physics* **13**, 1117 (2017).
- [38] Li Wang, Qing Liu, and Yunbo Zhang, “Quantum dynamics on a lossy non-hermitian lattice, *Chinese Physics B* **30**, 020506 (2021).
- [39] Wen-Tan Xue, Yu-Min Hu, Fei Song, and Zhong Wang, “Non-Hermitian edge burst, *Phys. Rev. Lett.* **128**, 120401 (2022).