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The Period of the Coefficients of the Gaussian Polynomial[N+33]

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THE PERIOD OF THE COEFFICIENTS OF

THE GAUSSIAN POLYNOMIAL $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$

A Thesis

by

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ABSTRACT

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Definition 1. *For any N, the central coefficient(s) of* $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ *is denoted by* $C_0(N)$ *and the coefficient that is x"away" from the central coefficient(s) of* $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ *is denoted by* $C_x(N)$.

In [1] the following result is proved:

Theorem 2. *The central coefficient(s) of the Gaussian polynomial* $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ *are described by the generating function*

$$\sum_{N=0}^{\infty} C_0(N) = \frac{1+q^3}{(1-q)(1-q^2)(1-q^4)}. \tag{0.1}$$

This generating function has period 4.

The main goal of this thesis is to generalize Theorem 0.2 by way of proving the following conjecture:

Conjecture 3. *For any x the generating function for* $C_x(N)$ *has period 4 and is given in three cases:*

$$\sum_{N=0}^{\infty} C_{3a}(N)q^N = \frac{1+q^2+q^3-q^{4a+2}}{(1-q)(1-q^2)(1-q^4)}, \tag{0.2}$$

$$\sum_{N=0}^{\infty} C_{3a+1}(N)q^N = \frac{1+q+q^3-q^{4a+3}}{(1-q)(1-q^2)(1-q^4)} \tag{0.3}$$

$$\sum_{N=0}^{\infty} C_{3a+2}(N)q^N = \frac{1+q+q^2-q^{4a+4}}{(1-q)(1-q^2)(1-q^4)} \tag{0.4}$$

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CHAPTER I

INTRODUCTION

Partition functions have always been an interesting area of research. The work shown here is only an extension of some results found in [1]. This thesis will look at the coefficients of Gaussian polynomials in an attempt at proving that the period of all coefficients in the Gaussian polynomial $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ is 4.

1.1 Example of Some Results

Because of [1], we know that the central coefficients of Gaussian polynomials have period $\frac{2lcm([m])}{m}$. If we were to focus on $m = 3$, then we would get that the period of the central coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ is 4. We hope to proof that the coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ can be described by three rational functions. We can construct quasipolynomials from these, where the period for the coefficients will be 4. An example of a Gaussian polynomail is

Example 4.

$$\sum_{n=0}^{15} p(n,3,5)q^n = \begin{bmatrix} 5+3 \\ 3 \end{bmatrix} = \frac{(q;q)_8}{(q;q)_3(q;q)_5} = \frac{(1-q^6)(1-q^7)(1-q^8)}{(1-q)(1-q^2)(1-q^3)}$$

$$= p(0,3,5) + p(1,3,5)q + p(2,3,5)q^2 + p(3,3,5)q^3 + p(4,3,5)q^4 + p(5,3,5)q^5$$

$$+ p(6,3,5)q^6 + p(7,3,5)q^7 + p(8,3,5)q^8 + p(9,3,5)q^9 + p(10,3,5)q^{10}$$

$$+ p(11,3,5)q^{11} + p(12,3,5)q^{12} + p(13,3,5)q^{13} + p(14,3,5)q^{14} + p(15,3,5)q^{15}$$

$$= 1 + q + q^2 + 3q^3 + 4q^4 + 5q^5 + 6q^6 + 6q^7 + 6q^8 + 6q^9 + 5q^{10} + 4q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15}.$$

Let $C_j(N)$ be the coefficient that is j-away from the central coefficient if the Gaussian polynomial $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$. Then we can define the generating function

$$\sum_{N=0}^{\infty} C_j(N)q^N$$

to be the formal power series whose coefficients are defined by $C_j(N)$. The rational functions that we can derive from this generation function are

$$\sum_{N=0}^{\infty} C_{3a}(N)q^N = \frac{1+q^2+q^3-q^{4k+2}}{(1-q)(1-q^2)(1-q^4)}, \tag{1.1}$$

$$\sum_{N=0}^{\infty} C_{3a+1}(N)q^N = \frac{1+q+q^3-q^{4k+3}}{(1-q)(1-q^2)(1-q^4)} \tag{1.2}$$

$$\sum_{N=0}^{\infty} C_{3a+2}(N)q^N = \frac{1+q+q^2-q^{4k+4}}{(1-q)(1-q^2)(1-q^4)} \tag{1.3}$$

This thesis will go over how these functions were attained. The remainder of this chapter will focus on presenting background material. In chapter 2, we prove that the period of the central coefficients is 4. Along with this, the motivation behind the results in this thesis will be made clear. The chapters that follow will showcase how we arrive at the functions above.

1.2 Background Material

Gaussian polynomials are the q-analogue of binomial coefficients. They are defined as follows.

Definition 5.

$$\sum_{n=0}^{Nm} p(n,m,N)q^n = \begin{bmatrix} N+m \\ m \end{bmatrix} = \frac{(q;q)_{N+m}}{(q;q)_m(q;q)_N} = \frac{(1-q)\cdots(1-q^{N+m})}{(1-q)\cdots(1-q^m)(1-q)\cdots(1-q^N)}. \tag{1.4}$$

We give a quick proof as to why this has to simplify to a polynomial. We first define a cyclotomic polynomial.

Definition 6. *Define the n^{th} cyclotomic polynomial $\Phi_n(x)$ to be the polynomial whose roots are the* *primitive n^{th} roots of unity:*

$$\Phi_n(x) = \prod_{\zeta \ primitive \ \in \mu_n} (x - \zeta) = \prod_{\substack{1 \le a < n \\ (a,n)=1}} (x - \zeta_n^a)$$

where μ_n is the group of n^{th} roots of unity of $\mathbb{Q}$.

We can define the gaussian polynomial $\begin{bmatrix} N+m \\ m \end{bmatrix}$ as follows

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \prod_{d=1}^{N+m} (\Phi_d(q))^{\lfloor \frac{N+m}{d} \rfloor - \lfloor \frac{N}{d} \rfloor - \lfloor \frac{m}{d} \rfloor}.$$

Since $\lfloor a+b \rfloor \ge \lfloor a \rfloor + \lfloor b \rfloor$ for all $a, b \ge 0$, we have for all $N, m, d > 0$

$$\left\lfloor \frac{N+m}{d} \right\rfloor = \left\lfloor \frac{N}{d} + \frac{m}{d} \right\rfloor \ge \left\lfloor \frac{N}{d} \right\rfloor + \left\lfloor \frac{m}{d} \right\rfloor.$$

and consequently

$$\left\lfloor \frac{N+m}{d} \right\rfloor - \left\lfloor \frac{N}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor \ge 0.$$

Since $\Phi_d(q)$ is a polynomial for all $d \in \mathbb{N}$, this implies that

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \prod_{d=1}^{N+m} (\Phi_d(q))^{\lfloor \frac{N+m}{d} \rfloor - \lfloor \frac{N}{d} \rfloor - \lfloor \frac{m}{d} \rfloor}$$

is a polynomial.

Now, what we are interested in is the coefficients of Gaussian polynomials. These coefficients can be interpreted by partitions.

Definition 7. *For integers $n, m, N > 0$, the function that enumerates the partitions of n into at most m parts, no part larger than N is denoted by $p(n,m,N)$. For $n < 0$ and $n > Nm$, we agree that $p(n,m,N) = 0$.*

Gaussian polynomials are the generating functions for $p(n,m,N)$. The main Gaussian polynomial we will look at is $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$.

Example 8.

$$\sum_{n=0}^{15} p(n,3,5)q^n = \begin{bmatrix} 5+3 \\ 3 \end{bmatrix} = \frac{(q;q)_8}{(q;q)_3(q;q)_5} = \frac{(1-q^6)(1-q^7)(1-q^8)}{(1-q)(1-q^2)(1-q^3)}$$

$$= p(0,3,5) + p(1,3,5)q + p(2,3,5)q^2 + p(3,3,5)q^3 + p(4,3,5)q^4 + p(5,3,5)q^5$$

$$+ p(6,3,5)q^6 + p(7,3,5)q^7 + p(8,3,5)q^8 + p(9,3,5)q^9 + p(10,3,5)q^{10}$$

$$+ p(11,3,5)q^{11} + p(12,3,5)q^{12} + p(13,3,5)q^{13} + p(14,3,5)q^{14} + p(15,3,5)q^{15}$$

$$= 1 + q + q^2 + 3q^3 + 4q^4 + 5q^5 + 6q^6 + 6q^7 + 6q^8 + 6q^9 + 5q^{10} + 4q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15}.$$

A thing to note is that if we do not define N explicitly, then our approach to compute a polynomial differs from the example above. This is because we now to take into account the possible variables at play. Hence we need to generate a quasipolynomial with constituents representing all possibilities for N.

Definition 9. *A function $f(k)$ is a quasipolynomial if there exist polynomials, called constituents, $f_0(k),\ f_1(k),\dots,f_{d-1}(k)$ such that for all $k \in \mathbb{Z}$ one has*

$$f(k) = \begin{cases} f_0(k) & \text{if } k \equiv 0 \pmod{d} \\ f_1(k) & \text{if } k \equiv 1 \pmod{d} \\ \vdots & \\ f_{d-1}(k) & \text{if } k \equiv d-1 \pmod{d}. \end{cases}$$

The period d of the quasipolynomial is the number of constituents.

When N is defined, we use the constituents of the quasipolynomial formed from $\begin{bmatrix} N+m \\ m \end{bmatrix}$ to compute the coefficients.

With this we can now explain the difference between a central coefficient and a maximal coefficient. A central coefficient, as the name implies, is located at the center or middle of a

Gaussian polynomial and is computed using the equation.

$$C_0(N) = p\left(\lfloor\frac{nM}{2}\rfloor, m, N\right). \tag{1.5}$$

Notice that this will only give is the lefttmost central coefficient if there were to be two of them. Since Gaussian polynomials are unimodal and their coefficients symmetric, we need only work with the first half of the coefficients. The maximal coefficients are the largest coefficients in a Gaussian polynomial. In Example 4 the central coefficients are

$$p(7,3,5) = 6 \text{ and } p(8,3,5) = 6$$

while the maximal coefficients are

$$p(6,3,5) = 6, p(7,3,5) = 6, p(8,3,5) = 6 \text{ and } p(9,3,5) = 6.$$

Notice that all central coefficients are maximal, but the converse may not always be true.

CHAPTER II

THE PERIOD OF THE CENTRAL COEFFICIENTS OF $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$

In section 3 of [1], it is proved that the period of the maximal coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ is 4. We will show why this is the case by using the method we will use later to prove that the period of $C_j(N)$ is 4. Let $M_3(N)$ be the set of maximal coefficients, then we have the following theorem.

Theorem 10. *For $\ell \geq 0$ the quasipolynomial for $M_3(N)$ has period 4 and is given by:*

$$M_3(4\ell-2) = \left\{ \begin{array}{l} p(6\ell-4,3,4\ell-2) \\ p(6\ell-3,3,4\ell-2) \\ p(6\ell-2,3,4\ell-2) \end{array} \right\} = 2\binom{\ell+1}{2} + 2\binom{\ell}{2} = 2\ell^2. \tag{2.1}$$

$$M_3(4\ell-1) = \left\{ \begin{array}{l} p(6\ell-3,3,4\ell-1) \\ p(6\ell-2,3,4\ell-1) \\ p(6\ell-1,3,4\ell-1) \\ p(6\ell,3,4\ell-1) \end{array} \right\} = 3\binom{\ell+1}{2} + \binom{\ell}{2} = 2\ell^2 + \ell. \tag{2.2}$$

$$M_3(4\ell) = p(6\ell,3,4\ell) = \binom{\ell+2}{2} + 2\binom{\ell+1}{2} + \binom{\ell}{2} = 2\ell^2 + 2\ell + 1. \tag{2.3}$$

$$M_3(4\ell+1) = \left\{ \begin{array}{l} p(6\ell,3,4\ell+1) \\ p(6\ell+1,3,4\ell+1) \\ p(6\ell+2,3,4\ell+1) \\ p(6\ell+3,3,4\ell+1) \end{array} \right\} = \binom{\ell+2}{2} + 3\binom{\ell+1}{2} = 2\ell^2 + 3\ell + 1. \tag{2.4}$$

We know that the central coefficients are maximal, hence they also have period 4. This theorem will not proven in this thesis. Instead we show that the results of the theorem coincide with the proof that the central coefficients of $\left[{N+3 \atop 3}\right]$ is 4. Let $C_0(N)$ be the set of central coefficients. As stated before, there may be times where there are two central coefficients.

Theorem 11. *The central coefficients of the Gaussian polynomial* $\left[{N+3 \atop 3}\right]$ *have period 4 and can be described by the rational function*

$$\frac{1+q^3}{(1-q)(1-q^2)(1-q^4)}. \tag{2.5}$$

We will use two methods to prove this theorem. One will involve Theorem 6, since this method was already used in [1]. The second method will utilize the work this thesis will use to prove Conjecture 10.

$Proof(1)$: From [1], we know that the quasipolynomial is computed from the following expression

$$(q^2+q^3+2q^4+3q^5+2q^6+3q^7+2q^8+q^9+q^{10}) \times \sum_{\ell=0}^{\infty} \binom{\ell+2}{2} q^{4\ell} \tag{2.6}$$

which is equal to this expression

$$\frac{q^2+q^3+2q^4+3q^5+2q^6+3q^7+2q^8+q^9+q^{10}}{(1-q^4)^3}. \tag{2.7}$$

Computing further, we arrive at

$$\frac{q^2+q^3+2q^4+3q^5+2q^6+3q^7+2q^8+q^9+q^{10}}{(1-q^4)^3} = \frac{q^2(1+q+q^2+q^3)(1+q^2)(1+q^3)}{(1-q)(1+q+q^2+q^3)(1-q^2)(1+q^2)(1-q^4)} \tag{2.8}$$

$$= \frac{q^2(1+q^3)}{(1-q)(1-q^2)(1-q^4)}. \tag{2.9}$$

Since (2.9) comes from $\sum_{N=0}^{\infty} M_3(N-2)q^N$, by [1], and we are looking at $\sum_{N=0}^{\infty} C_0(N)q^N$, the q^2 in (2.9) is not necessary. Thus we have proved Theorem 7. With this done, the next method can now hold more validity as with the rest of the work in this thesis.

$Proof(2)$: Let $N = 0,1,2,3,\ldots$. The set of central coefficients is

$$C_0(N) = 1,1,2,3,5,6,8,10,13,15,19,21,25,28,32,36,41,45,\ldots.$$

We know that these coefficients are partitions of the form $p(n,3,N)$. In particular, the even terms are of the form $p(3k,3,2k)$, and the odd terms are of the form $p(3k+1,3,2k+1)$. Hence we can make define the generating function $\sum_{N=0}^{\infty} C_0(N)q^N$ as follows

$$\sum_{k=0}^{\infty} p(3k,3,2k)q^{2k} + p(3k+1,3,2k+1)q^{2k+1}. \tag{2.10}$$

Here the even terms are

$$\sum_{k=0}^{\infty} p(3k,3,2k)q^{2k} = 1+2q^2+5q^4+8q^6+13q^8+19q^{10}+25q^{12}+32q^{14}+41q^{16}+\cdots. \tag{2.11}$$

By inputting our coefficients into OEIS, we find the sequence $A000982$ with generating function

$$\frac{q(1+q^2)}{(1+q)(1-q)^3} \tag{2.12}$$

We fix this generating function to get our desired exponents and remove the zero term. To do this we change q to q^2 and eliminate the first q in the numerator. This gives us

$$\frac{1+q^4}{(1+q^2)(1-q^2)^3}. \tag{2.13}$$

Now we do the same for the odd terms. These are defined by

$$\sum_{k=0}^{\infty} p(3k+1,3,2k+1)q^{2k+1} = q+3q^3+6q^5+10q^7+15q^9+21q^{11}+28q^{13}+36q^{15}+45q^{17}+\cdots.$$

Using OEIS again we find the sequence $A000217$ with generating function

$$\frac{q}{(1-q)^3}. \tag{2.14}$$

Here, in order to get our desired exponets, we may only need to change the q in the denominator to q^2. This gives us

$$\frac{q}{(1-q^2)^3}. \tag{2.15}$$

By adding both rational functions, we get

$$\frac{1+q^4}{(1+q^2)(1-q^2)^3}+\frac{q}{(1-q^2)^3}=\frac{1+q^4+q(1+q^2)}{(1+q^2)(1-q^2)^3} \tag{2.16}$$

$$=\frac{1+q+q^3+q^4}{(1+q)(1-q)(1-q^2)(1-q^4)} \tag{2.17}$$

$$=\frac{1+q^3}{(1-q)(1-q^2)(1-q^4)}, \tag{2.18}$$

which is exactly the rational function (2.5). We have now proven one part of the theorem. To prove the second part we express the rational function (2.5) as a quasipolynomial. We begin by multiplying by 1 in a special way.

$$\frac{1+q^3}{(1-q)(1-q^2)(1-q^4)}\times\frac{\frac{(1-q^4)^3}{(1-q)(1-q^2)(1-q^4)}}{\frac{(1-q^4)^3}{(1-q)(1-q^2)(1-q^4)}}=\frac{1+q+2q^2+3q^3+2q^4+3q^5+2q^6+q^7+q^8}{(1-q^4)^3}. \tag{2.19}$$

We call the numerator of this new rational function the Earhart numerator. Now Remark 2.7 in [1] tells us that we have to "re-cast", with respect to our current work, the generating function from an index of k to an index of ℓ. We can see this happen from (2.5) through (2.8). This is because the final denominator in (2.7) is equal to

$$\sum_{\ell=0}^{\infty}\binom{\ell+2}{2}q^{4\ell}, \tag{2.20}$$

where we define the binomial coefficient as follows.

Definition 12.

- *Whenever* $a < b$ *then* $\binom{a}{b} = 0$.
- *Whenever* $a \geq b$ *we use the normal translation to the monomial basis:* $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

Now let $\ell \geq 0$. Then we have

$$\frac{1+q+2q^2+3q^3+2q^4+3q^5+2q^6+q^7+q^8}{(1-q^4)^3}$$
$$= (1+q+2q^2+3q^3+2q^4+3q^5+2q^6+q^7+q^8) \times \sum_{\ell=0}^{\infty} \binom{\ell+2}{2} q^{4\ell}. \tag{2.21}$$

Notice that $(1-q^4)^3$ and $q^{4\ell}$ automatically imply that the period is 4. Nevertheless we construct the quasipolynomail in order to relate this back to Theorem 6. We begin by grouping the coefficients of $C_0(N)$ in groups of 4, sequentially. This gives us the sets

$$\{1,1,2,3\},\ \{5,6,8,10\}\ \{13,15,19,21\},\ldots$$

Remember that every coefficient in $C_0(N)$ is represented by a partition function of the form $p(n,3,N)$ with $N = 0,1,2,3,4,5,6,7,\ldots$ and $n = 0,1,3,4,6,7,9,10\ldots$. We can see that the first element in every set is of the form $p(6\ell,3,4\ell)$ with the second, third, and fourth elements being of the forms $p(6\ell+1,3,4\ell+1), p(6\ell+3,3,4\ell+2), p(6\ell+4,3,4\ell+3)$ respectively. By computing the product in (2.6), we have that the quasipolynomial for $\sum_{N=0}^{\infty} C_0(N)$ is

$$p(6\ell,3,4\ell) = \binom{\ell+2}{2} + 2\binom{\ell+1}{2} + \binom{\ell}{2} = 2\ell^2 + 2\ell + 1 \tag{2.22}$$

$$p(6\ell+1,3,4\ell+1) = \binom{\ell+2}{2} + 3\binom{\ell+1}{2} = 2\ell^2 + 3\ell + 1 \tag{2.23}$$

$$p(6\ell+2,3,4\ell+2) = 2\binom{\ell+1}{2} + 2\binom{\ell}{2} = 2\ell^2 \tag{2.24}$$

$$p(6\ell+3,3,4\ell+3) = 3\binom{\ell+2}{2} + 1\binom{\ell+1}{2} = 2\ell^2 + \ell \tag{2.25}$$

CHAPTER III

EXPANDING ON THEOREM 9

3.1 Arranging the Coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$

The previous chapter proved that the central coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ have period 4. However [1] proves this for maximal coefficients.Hence other coefficients aside from the central must also be of period 4. With this in mind we have the following conjecture.

Conjecture 13. *The coefficients of the family of Gaussian polynomials that are j-away from the center of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$, have period 4.*

As stated before, since the coefficients of Gaussian polynomials are symmetric and unimodal, we can focus on the first half of the coefficients. Using the same methods as in the previous chapter we show that the coefficients in C_1 have period 4. From Table 3.1 we see that C_1 is $1,2,3,4,6,8,10,12,15,\ldots$. By the sequence $A130519$ from OEIS, a generating function for these coefficients is

$$\frac{q^4}{(1-q)^2(1-q^4)}. \tag{3.1}$$

Since we want the denominator to be $(1-q)(1-q^2)(1-q^4)$, we multiply by $\frac{1+q}{1+q}$. Hence we get

$$\frac{q^4}{(1-q)^2(1-q^4)} \times \frac{1+q}{1+q} = \frac{q^4(1+q)}{(1-q)(1-q^2)(1-q^4)}. \tag{3.2}$$

Since the sequence A130519 starts with 4 zeros, we compensate by removing q^4 from the generating function. Thus

$$\frac{1+q}{(1-q)(1-q^2)(1-q^4)} \tag{3.3}$$

is the generating function for the coefficients of C_1. Note that this method only works if a generating

Table 3.1:

···	C_{14}	C_{13}	C_{12}	C_{11}	C_{10}	C_9	C_8	C_7	C_6	C_5	C_4	C_3	C_2	C_1	C_0	
															1	$:\begin{bmatrix}0+3\\3\end{bmatrix}$
														1	1	$:\begin{bmatrix}1+3\\3\end{bmatrix}$
												1	1	2	2	$:\begin{bmatrix}2+3\\3\end{bmatrix}$
											1	1	2	3	3	$:\begin{bmatrix}3+3\\3\end{bmatrix}$
									1	1	2	3	4	4	5	$:\begin{bmatrix}4+3\\3\end{bmatrix}$
								1	1	2	3	4	5	6	6	$:\begin{bmatrix}5+3\\3\end{bmatrix}$
						1	1	2	3	4	5	7	7	8	8	$:\begin{bmatrix}6+3\\3\end{bmatrix}$
					1	1	2	3	4	5	7	8	9	10	10	$:\begin{bmatrix}7+3\\3\end{bmatrix}$
			1	1	2	3	4	5	7	8	10	11	12	12	13	$:\begin{bmatrix}8+3\\3\end{bmatrix}$
		1	1	2	3	4	5	7	8	10	12	13	14	15	15	$:\begin{bmatrix}9+3\\3\end{bmatrix}$
1	1	2	3	4	5	7	8	10	12	14	15	17	17	18	18	$:\begin{bmatrix}10+3\\3\end{bmatrix}$
																$\vdots$

function for any of the C_i exists. An example of it not working is the coefficients of C_3. Here we have the sequence $1,1,3,4,7,8,11,13,\ldots$. OEIS gives us nothing, hence we take the same method as for C_0. We look at the even and odd terms and we get the sequences A047838 and A034856 respectively from OEIS. The generating functions for each are

$$\frac{q^2(1+q+q^2-q^3)}{(1-q)^3(1+q)} \text{ and } \frac{q(1+q-q^2)}{(1-q)^3} \tag{3.4}$$

respectively. Just like the others, we change these generating functions to accommodate. Hence we arrive at

$$\frac{q(1+q^2-q^4)}{(1-q^2)^3} \text{ and } \frac{1+q^2+q^4-q^6}{(1-q^2)^3(1+q^2)} \tag{3.5}$$

for the odd and even terms of C_3 respectively. To get a better understanding of what these rational functions represent, we look at C_3 and notice that even terms are of the form $p(3k,3,2(k+1)$ and odd terms are of the form $p(3k+1,3,2(k+1)+1)$. We can use [1] to compute

$$\sum_{N=0}^{\infty} C_3(N)q^N = \sum_{k=0}^{\infty} \left[p(3k,3,2(k+1))q^{2k} + p(3k+1,3,2(k+1)+1)q^{2k+1}\right] \tag{3.6}$$

$$= \frac{1+q^2+q^4-q^6}{(1-q^2)^3(1+q^2)} + \frac{q(1+q^2-q^4)}{(1-q^2)^3} \tag{3.7}$$

$$= \frac{q(1+q^2)(1+q^2-q^4)+1+q^2+q^4-q^6}{(1+q^2)(1-q^2)^3} \tag{3.8}$$

$$= \frac{(1+q)(1-q+2q^2-q^3+q^4-q^5)}{(1-q)(1-q^2)(1-q^4)} \tag{3.9}$$

$$= \frac{1+q^2+q^3-q^6}{(1-q)(1-q^2)(1-q^4)}. \tag{3.10}$$

Using these techniques we are able to compute the following

$$\sum_{N=0}^{\infty} C_0(N)q^N = \frac{1+q^3}{(1-q)(1-q^2)(1-q^4)} \tag{3.11}$$

$$\sum_{N=0}^{\infty} C_1(N)q^N = \frac{1+q}{(1-q)(1-q^2)(1-q^4)} \tag{3.12}$$

$$\sum_{N=0}^{\infty} C_2(N)q^N = \frac{1+q+q^2-q^4}{(1-q)(1-q^2)(1-q^4)} \tag{3.13}$$

$$\sum_{N=0}^{\infty} C_3(N)q^N = \frac{1+q^2+q^3-q^6}{(1-q)(1-q^2)(1-q^4)} \tag{3.14}$$

$$\vdots$$

Along with their respective quasipolynomial. The work in chapter 2 suggests that these rational functions have period 4 due to the denominator $(1-q)(1-q^2)(1-q^4)$. Since there are an infinite amount of C_i, we cannot possible prove that each individual set of coefficients is of period 4. Furthermore, if there is a certain number of coefficients whose period is 4, the number would these would increase as N increases for $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$. This motivates the idea of generalizing the C_i.

3.2 Generalizations

We now formally introduce the three functions at the beginning of this thesis with a theorem.

Theorem 14. *Let C_j be the set of coefficients $j-$away from the central coefficient of the Gaussian polynomial $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$, then we have the following.*

$$\sum_{N=0}^{\infty} C_{3a}(N)q^N = \frac{1+q^2+q^3-q^{4a+2}}{(1-q)(1-q^2)(1-q^4)}, \tag{3.15}$$

$$\sum_{N=0}^{\infty} C_{3a+1}(N)q^N = \frac{1+q+q^3-q^{4a+3}}{(1-q)(1-q^2)(1-q^4)} \tag{3.16}$$

$$\sum_{N=0}^{\infty} C_{3a+2}(N)q^N = \frac{1+q+q^2-q^{4a+4}}{(1-q)(1-q^2)(1-q^4)} \tag{3.17}$$

In the last section it was mentioned that it would be difficult to generalize every C_i with a single generating function. With this in mind, we shift our attention to C_0 and C_3. The generating functions

$$\sum_{k=0}^{\infty} \left[p(3k,3,2k)q^{2k} + p(3k+1,3,2k+1)q^{2k+1}\right] \text{ and } \sum_{k=0}^{\infty} \left[p(3k,3,2(k+1))q^{2k} + p(3k+1,3,2(k+1)+1)q^{2k+1}\right]$$

only differ from the change from $2k+1$ to $2(k+1)+1$. We show this is actually a natural occurrence by proving equation. (3.15). From the table we have that

$$C_0 : 1,1,2,3,5,6,8,10,13,15,\ldots \tag{3.18}$$

$$C_3 : 1,1,3,4,6,8,11,13,\ldots \tag{3.19}$$

$$C_6 : 1,1,3,4,7,8,\ldots \tag{3.20}$$

The generating functions for each are

$$\sum_{N=0}^{\infty} C_0(N)q^N = \sum_{k=0}^{\infty} \left[p(3k,3,2k)q^{2k} + p(3k+1,3,2k+1)q^{2k+1}\right] \tag{3.21}$$

$$\sum_{N=0}^{\infty} C_3(N)q^N = \sum_{k=0}^{\infty} \left[p(3k,3,2(k+1))q^{2k} + p(3k+1,3,2(k+1)+1)q^{2k+1}\right] \tag{3.22}$$

$$\sum_{N=0}^{\infty} C_6(N)q^N = \sum_{k=0}^{\infty} \left[p(3k,3,2(k+2))q^{2k} + p(3k+1,3,2(k+2)+1)q^{2k+1}\right]. \tag{3.23}$$

Following the work from chapter 2 we know that the even and odd terms of C_0 are represented by equations (2.8) and (2.10). For C_3, we use OEIS again and find that the even terms coincide with sequence $A047838$ which has generating function

$$\frac{q^2(1+q1+q^2-q^3)}{(1-q)(1+q)^3}. \tag{3.24}$$

We change the function to fit our desired exponents and get

$$\frac{1+q^2+q^4-q^6}{(1+q^2)(1-q^2)^3}. \tag{3.25}$$

For the odd terms we have sequence $A034856$ with generating function

$$\frac{q(1+q-q^2)}{(1-q)^3} \tag{3.26}$$

which when changed to fit the needed exponents equals

$$\frac{q(1+q^2-q^4)}{(1-q^2)^3}. \tag{3.27}$$

We add (3.26) and (3.25)

$$\frac{1+q^2+q^4-q^6}{(1+q^2)(1-q^2)^3}+\frac{1+q^2-q^4}{(1-q^2)^3} = \frac{(1+q^+q^4-q^6)+q(1+q^2)(1+q^2-q^4)}{(1+q^2)(1-q^2)^3} \tag{3.28}$$

$$= \frac{(1+q)(1-q+2q^2-q^3+q^4-q^5)}{(1-q)(1-q^2)(1-q^4)} \tag{3.29}$$

$$= \frac{1+q^2+q^3-q^6}{(1-q)(1-q^2)(1-q^4)}. \tag{3.30}$$

For C_6 we have that the even and odd terms have generating functions

$$\sum_{k=0}^{\infty} p(3k,3,2(k+2))q^{2k} = \frac{1+q^2+q^4-q^{10}}{(1+q^2)(1-q^2)^3} \tag{3.31}$$

and

$$\sum_{k=0}^{\infty} p(3k+1,3,2(k+2)+1)q^{2k+1} = \frac{q(1+q^2-q^4+q^6-q^8)}{(1-q^2)^3}. \tag{3.32}$$

With the sum being

$$\frac{1+q^2+q^4-q^{10}}{(1+q^2)(1-q^2)^3}+\frac{q(1+q^2-q^4+q^6-q^8)}{(1-q^2)^3} = \frac{q(1-q+2q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9}{(1-q)(1-q^2)(1-q^4)} \tag{3.33}$$

$$= \frac{1+q^2+q^3-q^{10}}{(1-q)(1-q^2)(1-q^4)}. \tag{3.34}$$

Table 3.2:

	$a=0$	$a=3$	$a=6$
$\sum_{k=0}^{\infty} p(3k,3,2(k+a))q^{2k}$	$\frac{1+q^4}{(1+q^2)(1-q^2)^3}$	$\frac{1+q^2+q^4-q^6}{(1+q^2)(1-q^2)^3}$	$\frac{1+q^2+q^4-q^{10}}{(1+q^2)(1-q^2)^3}$
$\sum_{k=0}^{\infty} p(3k+1,3,2(k+a)+1)q^{2k+1}$	$\frac{q}{(1-q^2)^3}$	$\frac{q(1+q^2-q^4)}{(1-q^2)^3}$	$\frac{q(1+q^2-q^4+q^6-q^8)}{(1-q^2)^3}$
$\sum_{N=0}^{\infty} C_{3a}(N)q^N$	$\frac{1+q^3}{(1-q)(1-q^2)(1-q^4)}$	$\frac{1+q^2+q^3-q^6}{(1-q)(1-q^2)(1-q^4)}$	$\frac{1+q^2+q^3-q^{10}}{(1-q)(1-q^2)(1-q^4)}$

With these results we can now arrange the equations like in Table 3.2. Notice that the third row of Table 3.2 already hints at (3.15). For even terms we have that as a increases, the general rational function becomes

$$\frac{1+q^2+q^4-q^{4a+2}}{(1+q^2)(1-q^2)^3}. \tag{3.35}$$

In order to prove this, we look at the cases where $a=3x$, $a=3x+1$, and $a=3x+2$.These cases render (3.35) into

$$\frac{1+q^2+q^4-q^{12x+2}}{(1+q^2)(1-q^2)^3}, \tag{3.36}$$

$$\frac{1+q^2+q^4-q^{12x+6}}{(1+q^2)(1-q^2)^3}, \tag{3.37}$$

$$\text{and } \frac{1+q^2+q^4-q^{12x+10}}{(1+q^2)(1-q^2)^3} \tag{3.38}$$

respectively. In this thesis, (3.36) will be proven. The even terms of any C_{3a} can be described by

$$\begin{array}{lll} p(6(3k),3,6(2k+x)), & p(6(3k),3,6(2k+x)+4), & p(6(3k),3,6(2k+x+1)+2) \\ p(6(3k)+3,3,6(2k+x)+2), & p(6(3k)+3,3,6(2k+x+1)), & p(6(3k)+3,3,6(2k+1+x)+4) \end{array}$$

for $a=3x$

$$\begin{array}{lll} p(6(3k),3,6(2k+x)+2), & p(6(3k),3,6(2k+x+1)), & p(6(3k),3,6(2k+x+1)+4) \\ p(6(3k)+3,3,6(2k+x)+4), & p(6(3k)+3,3,6(2k+x+1)+2), & p(6(3k)+3,3,6(2k+2+x)) \end{array}$$

for $a=3x+1$, and

$$\begin{array}{lll} p(6(3k),3,6(2k+x)+4), & p(6(3k),3,6(2k+x+1)+2), & p(6(3k),3,6(2k+x+2)) \\ p(6(3k)+3,3,6(2k+x+1)), & p(6(3k)+3,3,6(2k+x+1)+4), & p(6(3k)+3,3,6(2k+2+x)+2) \end{array}$$

for $a = 3x+2$. We use the following constituents from appendix A form [1]

$$-12\sum_{k=x-1}^{\infty} q^{12k}\binom{k-x+1}{2} - 6\sum_{k=x}^{\infty} q^{12k}\binom{k-x}{2} + 4\sum_{k=0}^{\infty}\binom{3k+1}{2}q^{12k} + \sum_{k=0}^{\infty}\binom{3k+2}{2}q^{12k} + \sum_{k=0}^{\infty}\binom{3k}{2}q^{12k} \tag{3.39}$$

$$= -\frac{12\left(q^{12}+2\right)q^{12}}{\left(q^{12}-1\right)^3} - \frac{3\left(2q^{12}+1\right)q^{12}}{\left(q^{12}-1\right)^3} + \frac{12q^{12x+12}}{\left(q^{12}-1\right)^3} + \frac{6q^{12x+24}}{\left(q^{12}-1\right)^3} - \frac{q^{24}+7q^{12}+1}{\left(q^{12}-1\right)^3} \tag{3.40}$$

$$= \frac{12q^{12(x+1)} + 6q^{12(x+2)} - 19q^{24} - 34q^{12} - 1}{\left(q^{12}-1\right)^3} \tag{3.41}$$

$$-13\sum_{k=x-1}^{\infty} q^{12k+2}\binom{k-x+1}{2} - \sum_{k=x-2}^{\infty} q^{12k+2}\binom{k-x+2}{2} - 4\sum_{k=x}^{\infty} q^{12k+2}\binom{k-x}{2}$$
$$+3\sum_{k=0}^{\infty}\binom{3k+1}{2}q^{12k+2} + 3\sum_{k=0}^{\infty}\binom{3k+2}{2}q^{12k+2} \tag{3.42}$$

$$= \frac{q^{12x+2}}{\left(q^{12}-1\right)^3} + \frac{13q^{12x+14}}{\left(q^{12}-1\right)^3} + \frac{4q^{12x+26}}{\left(q^{12}-1\right)^3} - \frac{9\left(q^{12}+2\right)q^{14}}{\left(q^{12}-1\right)^3} - \frac{3\left(q^{24}+7q^{12}+1\right)q^2}{\left(q^{12}-1\right)^3} \tag{3.43}$$

$$= \frac{q^2\left(4q^{24}+13q^{12}+1\right)\left(q^{12x}-3\right)}{\left(q^{12}-1\right)^3} \tag{3.44}$$

$$-14\sum_{k=x-1}^{\infty} q^{12k+4}\binom{k-x+1}{2} - 2\sum_{k=x-2}^{\infty} q^{12k+4}\binom{k-x+2}{2} - 2\sum_{k=x}^{\infty} q^{12k+4}\binom{k-x}{2} +$$
$$\sum_{k=0}^{\infty}\binom{3k+1}{2}q^{12k+4} + 4\sum_{k=0}^{\infty}\binom{3k+2}{2}q^{12k+4} + \sum_{k=0}^{\infty}\binom{3k+3}{2}q^{12k+4} \tag{3.45}$$

$$= \frac{2q^{12x+4}}{\left(q^{12}-1\right)^3} + \frac{14q^{12x+16}}{\left(q^{12}-1\right)^3} + \frac{2q^{12x+28}}{\left(q^{12}-1\right)^3} - \frac{3\left(q^{12}+2\right)q^{16}}{\left(q^{12}-1\right)^3} - \frac{3\left(2q^{12}+1\right)q^4}{\left(q^{12}-1\right)^3} - \frac{4\left(q^{24}+7q^{12}+1\right)q^4}{\left(q^{12}-1\right)^3} \tag{3.46}$$

$$= \frac{q^4\left(2q^{12x} + 14q^{12(x+1)} + 2q^{12(x+2)} - 7q^{24} - 40q^{12} - 7\right)}{\left(q^{12}-1\right)^3} \tag{3.47}$$

$$= -13\sum_{k=x-1}^{\infty} q^{12k+6}\binom{k-x+1}{2} - 4\sum_{k=x-2}^{\infty} q^{12k+6}\binom{k-x+2}{2} - \sum_{k=x}^{\infty} q^{12k+6}\binom{k-x}{2}$$
$$+3\sum_{k=0}^{\infty}\binom{3k+2}{2}q^{12k+6} + 3\sum_{k=0}^{\infty}\binom{3k+3}{2}q^{12k+6} \tag{3.48}$$

$$= \frac{4q^{12x+6}}{\left(q^{12}-1\right)^3} + \frac{13q^{12x+18}}{\left(q^{12}-1\right)^3} + \frac{q^{12x+30}}{\left(q^{12}-1\right)^3} - \frac{9\left(2q^{12}+1\right)q^6}{\left(q^{12}-1\right)^3} - \frac{3\left(q^{24}+7q^{12}+1\right)q^6}{\left(q^{12}-1\right)^3} \tag{3.49}$$

$$= \frac{q^6\left(q^{24}+13q^{12}+4\right)\left(q^{12x}-3\right)}{\left(q^{12}-1\right)^3} \tag{3.50}$$

$$-12\sum_{k=x-1}^{\infty}q^{12k+8}\binom{k-x+1}{2}-6\sum_{k=x-2}^{\infty}q^{12k+8}\binom{k-x+2}{2}+\sum_{k=0}^{\infty}\binom{3k+2}{2}q^{12k+8}+4\sum_{k=0}^{\infty}\binom{3k+3}{2}q^{12k+8}$$
$$+\sum_{k=0}^{\infty}\binom{3k+4}{2}q^{12k+8} \tag{3.51}$$
$$=\frac{6q^{12x+8}}{\left(q^{12}-1\right)^3}+\frac{12q^{12x+20}}{\left(q^{12}-1\right)^3}-\frac{3\left(q^{12}+2\right)q^8}{\left(q^{12}-1\right)^3}-\frac{12\left(2q^{12}+1\right)q^8}{\left(q^{12}-1\right)^3}-\frac{\left(q^{24}+7q^{12}+1\right)q^8}{\left(q^{12}-1\right)^3} \tag{3.52}$$
$$=-\frac{q^8\left(-6q^{12x}-12q^{12(x+1)}+q^{24}+34q^{12}+19\right)}{\left(q^{12}-1\right)^3} \tag{3.53}$$
$$-9\sum_{k=x-1}^{\infty}q^{12k+10}\binom{k-x+1}{2}-9\sum_{k=x-2}^{\infty}q^{12k+10}\binom{k-x+2}{2}+3\sum_{k=0}^{\infty}\binom{3k+3}{2}q^{12k+10}$$
$$+3\sum_{k=0}^{\infty}\binom{3k+4}{2}q^{12k+10} \tag{3.54}$$
$$=\frac{9q^{12x+10}}{\left(q^{12}-1\right)^3}+\frac{9q^{12x+22}}{\left(q^{12}-1\right)^3}-\frac{9\left(q^{12}+2\right)q^{10}}{\left(q^{12}-1\right)^3}-\frac{9\left(2q^{12}+1\right)q^{10}}{\left(q^{12}-1\right)^3} \tag{3.55}$$
$$=\frac{9q^{10}\left(q^{12}+1\right)\left(q^{12x}-3\right)}{\left(q^{12}-1\right)^3}. \tag{3.56}$$

Now we add (3.41), (3.44), (3.47), (3.50), (3.53), (3.56).

$$\frac{12q^{12(x+1)}+6q^{12(x+2)}-19q^{24}-34q^{12}-1}{\left(q^{12}-1\right)^3}+\frac{q^2\left(4q^{24}+13q^{12}+1\right)\left(q^{12x}-3\right)}{\left(q^{12}-1\right)^3}$$
$$+\frac{q^4\left(2q^{12x}+14q^{12(x+1)}+2q^{12(x+2)}-7q^{24}-40q^{12}-7\right)}{\left(q^{12}-1\right)^3}+\frac{q^6\left(q^{24}+13q^{12}+4\right)\left(q^{12x}-3\right)}{\left(q^{12}-1\right)^3}$$
$$-\frac{q^8\left(-6q^{12x}-12q^{12(x+1)}+q^{24}+34q^{12}+19\right)}{\left(q^{12}-1\right)^3}+\frac{9q^{10}\left(q^{12}+1\right)\left(q^{12x}-3\right)}{\left(q^{12}-1\right)^3} \tag{3.57}$$
$$=\frac{1+q^2+q^4-q^{12x+2}}{\left(1-q^2\right)^3\left(q^2+1\right)}. \tag{3.58}$$

Thus we have shown that (3.36) is true. The proof for odd terms follows the same logic. This also hold for all other rational functions introduced after this. For odd terms, the general rational function is

$$\frac{q(1+\sum_{\beta=1}^{a}(q^{4(\beta+1)+2}-q^{4(\beta+1)+4}))}{(1-q^2)^3}=\frac{q(1+2q^2-q^{4a+2})}{(1+q^2)(1-q^2)^3} \tag{3.59}$$

Just like before, we add both functions

$$\frac{1+q^2+q^4-q^{4a+2}}{(1+q^2)(1-q^2)^3}+\frac{q(1+2q^2-q^{4a+2})}{(1+q^2)(1-q^2)^3}=\frac{(q(1+2q^2-q^{4a+2})+1+q^2+q^4-q^{4a+2})}{(1+q^2)(1-q^2)^3} \tag{3.60}$$

$$=\frac{1+q^2+q^3-q^{4a+2}}{(1-q)^3(1+q)^2(1+q^2)} \tag{3.61}$$

$$=\frac{1+q^2+q^3-q^{4a+2}}{(1-q)(1-q^2)(1-q^4)}. \tag{3.62}$$

With this, we have proven equation (3.15) of Theorem 11. We are able to prove that (3.16) and (3.17) also hold by following the same logic as (3.35).Similarly we have a table for C_{3a+1}. We can

Table 3.3:

	$a=1$	$a=2$	$a=3$
$\sum_{k=0}^{\infty} p(3k,3,2(k+a)+1)q^{2k}$	$\frac{1}{(1-q^2)^3}$	$\frac{1+q^4-q^6}{(1-q^2)^3}$	$\frac{1+q^4-q^6+q^8-q^{10}}{(1-q^2)^3}$
$\sum_{k=0}^{\infty} p(3k+2,3,2(k+a+1))q^{2k+1}$	$\frac{2q}{(1+q^2)(1-q^2)^3}$	$\frac{q(2+q^2-q^6)}{(1+q^2)(1-q^2)^3}$	$\frac{q(2+q^2-q^{10})}{(1+q^2)(1-q^2)^3}$
$\sum_{N=0}^{\infty} C_{3a+1}(N)q^N$	$\frac{1+q}{(1-q)(1-q^2)(1-q^4)}$	$\frac{1+q+q^3-q^7}{(1-q)(1-q^2)(1-q^4)}$	$\frac{1+q+q^3-q^{11}}{(1-q)(1-q^2)(1-q^4)}$

see that in Table 3.3, the even terms have a pattern described by

$$\frac{1+\sum_{\beta=1}^{a}(q^{4\beta}-q^{4\beta+2})}{(1-q^2)^3}=1+\frac{q^4(1-q^{4a})}{(1+q^2)^3} \tag{3.63}$$

$$=\frac{1+q^2+q^4-q^{4(a+1)}}{(1+q^2)(1-q^2)^3} \tag{3.64}$$

while the odds are described by

$$\frac{2q+q^3-q^{4a+3}}{(1+q^2)(1-q^2)^3}. \tag{3.65}$$

The sum of (3.41) and (3.42) is

$$\frac{1+q^2+q^4-q^{4(a+1)}}{(1+q^2)(1-q^2)^3}+\frac{2q+q^3-q^{4a+3}}{(1+q^2)(1-q^2)^3}=\frac{1+q+q^3-q^{4a+3}}{(1-q)^3(1+q)^2(1+q^2)} \tag{3.66}$$

$$=\frac{1+q+q^3-q^{4a+3}}{(1-q)(1-q^2)(1-q^4)}. \tag{3.67}$$

This shows (3.16) is true for all nonnegative integers a. Now we have the table for C_{3a+2} Again we

Table 3.4:

	$a=1$	$a=2$	$a=3$
$\sum_{k=0}^{\infty} p(3k+1,3,2(k+1+a))q^{2k}$	$\frac{1+2q^2-q^4}{(1+q^2)(1-q^2)^3}$	$\frac{1+2q^2-q^8}{(1+q^2)(1-q^2)^3}$	$\frac{1+2q^2-q^{12}}{(1+q^2)(1-q^2)^3}$
$\sum_{k=0}^{\infty} p(3k+2,3,2(k+a+1)+1)q^{2k+1}$	$\frac{q(2-q^2)}{(1-q^2)^3}$	$\frac{q(2-q^2+q^4-q^6)}{(1-q^2)^3}$	$\frac{q(2-q^2+q^4-q^6+q^8-q^{10})}{(1-q^2)^3}$
$\sum_{N=0}^{\infty} C_{3a+2}(N)q^N$	$\frac{1+q+q^2-q^4}{(1-q)(1-q^2)(1-q^4)}$	$\frac{1+q+q^2-q^8}{(1-q)(1-q^2)(1-q^4)}$	$\frac{1+q+q^2-q^{12}}{(1-q)(1-q^2)(1-q^4)}$

look at the even terms and their generating functions and notice that they are described by

$$\frac{1+2q^2-q^{4(a+1)}}{(1+q^2)(1-q^2)^3}. \tag{3.68}$$

With the odds being described by

$$\frac{q(2-q^2+\sum_{\beta=1}^{a}(q^{4\beta-q^{4\beta+2}}))}{(1-q^2)^3}=\frac{q(2+q^2-q^{4a+4})}{(1+q^2)(1-q^2)^3} \tag{3.69}$$

When (3.45) and (3.46) are added, we arrive at

$$\frac{1+2q^2-q^{4(a+1)}}{(1+q^2)(1-q^2)^3}+\frac{q(2+q^2-q^{4a+4})}{(1+q^2)(1-q^2)^3}=\frac{1+q+q^2-q^{4a+4}}{(1-q)^3(1+q)^2(1+q^2)} \tag{3.70}$$

$$=\frac{1+q+q^2-q^{4(a+1)}}{(1-q)(1-q^2)(1-q^4)}. \tag{3.71}$$

This result along with (3.39) and (3.44) prove Theorem 10.

CHAPTER IV

CONCLUSION

This thesis presents an ”orthogonal" view of the coefficients of the Gaussian polynomial $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ and it implies that these coefficients,when viewed in this orthogonal way, can be expressed as one of three rational functions in Theorem 14. The obvious step is to explore other values for m starting with 4. From [1], we know that the central coefficients can be expressed by a rational function with period

$$\frac{2\text{lcm}(m)}{m} \tag{4.1}$$

where $\text{lcm}(m)$ is defined to be the least common multiple of the numbers in $\{1,2,3,\ldots,m\}$. If we can fully generalize this approach it may be useful in proving the non-negativity of coefficients of the series expansion of certain rational functions.

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BIOGRAPHICAL SKETCH

Arturo J. Martinez was born in Mcallen, Texas. After receiving his high school diploma from Valley View High School in Pharr, Texas in 2012, Arturo attended The University of Texas Pan-American, whose name changed to The University of Texas Rio Grande valley half way into his degree. Arturo received a Bachelor of Science with a major in mathematics from The University of Texas Rio Grande valley in May 2018. he then continued his education and in the following two years he received a Master of Science from The University of Texas Rio Grande Valley in August 2020. For contact purposes, his email is arturo.j.martinez01@utrgv.edu.