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On Cubic Multisections

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ON CUBIC MULTISECTIONS

A Thesis

by

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ON CUBIC MULTISECTIONS

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ABSTRACT

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In this thesis, a systematic procedure is given for generating cubic multi-sections of Eisenstein series. The relevant series are determined from Fourier expansions for Eisenstein series by restricting the congruence class of the summation index modulo three. The resulting series are shown to be rational functions of the Dedekind eta function. A more general treatment of cubic dissection formulas is given by describing the dissection operators in terms of linear transformations.

DEDICATION

My educational endeavors could not have been attained without the unconditional love and encouragement that my parents have continuously provided throughout the years, nor without the watchful eye of my God, who gives meaning to all.

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I will be forever indebted to all my professors for their guidance during my graduate studies at UTPA. To Dr. Tim Huber, I would like to express my very great appreciation for countless hours of valuable advice which contributed to the completion of this thesis and to my understanding of mathematics.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In this thesis we significantly generalize an identity observed by F. Garvan and place it in a broader context. Garvan's identity was communicated by B.C. Berndt.

Theorem 1

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+2} d\Big)q^n = \frac{3(q^3;q^3)_\infty^6}{(q;q)_\infty^2}, \qquad |q|<1. \tag{1}$$

Here, and throughout this thesis we employ the usual notation for the q-expansion of the Dedekind eta function given in definition 1.

Definition 1 *Let $q = e^{2\pi i\tau}$, such that Im $\tau > 0$. We define the Dedekind eta function by*

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-q^n). \tag{2}$$

We use the notation,

$$(q;q)_\infty = q^{-1/24}\eta(\tau) = \prod_{n=1}^{\infty}(1-q^n). \tag{3}$$

We will prove Garvan's identity (1) and generalize it to several infinite classes of relations for dissections of Eisenstein series. For our discussion, two classes of Eisenstein series will be needed.

The first are normalized Eisenstein series of weight k,

$$E_{2k}(q) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}, \tag{4}$$

where ζ is the analytic continuation of the Riemann ζ-function. We will also refer to the Hecke Eisenstein series associated with the Dirichlet character χ modulo three,

$$E_{k,\chi}(q) = 1 + \frac{2}{L(1-k,\chi)} \sum_{n=1}^{\infty} \chi(n) \frac{n^{k-1} q^n}{1-q^n}, \tag{5}$$

where $L(1-k,\chi)$ denotes the analytic continuation of the associated Dirichlet L-series.

Some examples of dissections formulas for Eisenstein series that we derive are,

$$\sum_{n=0}^{\infty} \left(\sum_{d|3n+1} d^3 \right) q^n = (q;q)_\infty^8 + 3^4 q \frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4},$$

$$\sum_{n=0}^{\infty} \left(\sum_{d|3n+2} d^7 \right) q^n = 3 \cdot 43 (q;q)_\infty^{16} + 110 \cdot 3^6 q (q;q)_\infty^4 (q^3;q^3)_\infty^{12} + 41 \cdot 3^{10} q^2 \frac{(q^3;q^3)_\infty^{24}}{(q;q)_\infty^8},$$

and, if $\left(\frac{\cdot}{3}\right)$ denotes the Jacobi symbol modulo three,

$$\sum_{n=0}^{\infty} \left(\sum_{d|3n+1} \left(\frac{d}{3}\right) \right) q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{6}$$

$$\sum_{n=0}^{\infty} \left(\sum_{d|3n+2} \left(\frac{d}{3}\right) d^4 \right) q^n = -15 (q;q)_\infty^7 (q^3;q^3)_\infty^3 - 3^6 q \frac{(q^3;q^3)_\infty^{15}}{(q;q)_\infty^5}.$$

The preceding relations are similar to dissections for Eisenstein series of level five appearing in [2] involving Dirichlet characters modulo five. We have the following dissection formulas for the quintic Dirichlet character defined by $\langle \chi(n) \rangle_{n=0}^4 = \langle 0, 1, -i, i, -1 \rangle$.

Theorem 2

$$\sum_{n=0}^{\infty}\Big(\sum_{d|5n+4}\chi(d)\Big)q^n=\frac{i(q;q)_\infty(q^5;q^5)_\infty}{(q^2;q^5)_\infty^3(q^3;q^5)_\infty^3},\quad \sum_{n=0}^{\infty}\Big(\sum_{d|5n+3}\chi(d)\Big)q^n=\frac{(1+i)(q^5;q^5)_\infty^2}{(q^2;q^5)_\infty(q^3;q^5)_\infty},$$
$$\sum_{n=0}^{\infty}\Big(\sum_{d|5n+2}\chi(d)\Big)q^n=\frac{(1-i)(q^5;q^5)_\infty^2}{(q;q^5)_\infty(q^4;q^5)_\infty},\quad \sum_{n=0}^{\infty}\Big(\sum_{d|5n+1}\chi(d)\Big)q^n=\frac{(q;q)_\infty(q^5;q^5)_\infty}{(q;q^5)_\infty^3(q^4;q^5)_\infty^3}.$$

The fundamental building blocks for Eisenstein series in this thesis are the cubic theta functions, first studied by J. Borwein and P. Borwein [3, 2].

Definition 2 *The cubic theta functions $a(q)$, $b(q)$ and $c(q)$ are defined for $|q|<1$ by,*

$$a(q)=\sum_{m,n=-\infty}^{\infty}q^{n^2+nm+m^2},\quad b(q)=\sum_{m,n=-\infty}^{\infty}\omega^{n-m}q^{n^2+nm+m^2},\tag{7}$$
$$c(q)=\sum_{m,n=-\infty}^{\infty}q^{\left(n+\frac{1}{3}\right)^2+\left(n+\frac{1}{3}\right)\left(m+\frac{1}{3}\right)+\left(m+\frac{1}{3}\right)^2},\quad \omega=e^{2\pi i/3}.\tag{8}$$

The following two theorems are well known product expansions for $b(q)$ and $c(q)$ from [2]. The proof we give is based on that given in [2].

Theorem 3

$$b(q)=\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},\tag{9}$$
$$c(q)=3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.\tag{10}$$

Proof. For brevity we give a proof for $b(q)$ alone. Let $(a)_\infty := (a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$, $(a)_n := (a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1})$. Also, as usual $\omega=e^{2\pi i/3}$. We know from the Euler-Cauchy q-binomial theorem (see Theorem 14 of the Appendix), that for $|q|<1$

$$(-a;q)_\infty=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(q)_n}a^n.\tag{11}$$

Now applying (11) three times,

$$\begin{aligned}(-a^3;q^3)_\infty =&(-a;q)_\infty(-a\omega;q)_\infty(-a\omega^2;q)_\infty \\ \sum_{n=0}^{\infty}\frac{a^{3n}q^{3\binom{n}{2}}}{(q^3;q^3)_n} =& \sum_{n_1,n_2,n_3=0}^{\infty}\omega^{n_1+2n_2}a^{n_0+n_1+n_3} \\ &\times\frac{q^{\binom{n_0}{2}+\binom{n_1}{2}+\binom{n_2}{2}}}{(q)_{n_0}(q)_{n_1}(q)_{n_2}}.\end{aligned}$$

We can now equate coefficients of like powers of a to obtain,

$$\frac{1}{(q^3;q^3)_n} = \sum_{n_0+n_1+n_2=3n}\omega^{n_1-n_2}\frac{q^{\binom{n_0}{2}+\binom{n_1}{2}+\binom{n_2}{2}-3\binom{n}{2}}}{(q)_{n_0}(q)_{n_1}(q)_{n_2}}.$$

From here, let n_i be m_i+n for each i. We have,

$$\sum_{i=0}^{2}\binom{n_i}{2} = \frac{1}{2}\sum_{i=0}^{2}m_i^2+3\binom{k}{2}.$$

This follows since $\sum_{i=0}^{2}m_i = 0$. So,

$$\frac{1}{(q^3;q^3)_n} = \sum_{m_0+m_1+m_2=0}\omega^{m_1-m_2}\frac{q^{[m_0^2+m_1^2+m_2^2]/2}}{(q)_{m_0+n}(q)_{m_1+n}(q)_{m_2+n}}.$$

Letting n go to ∞, it follows that,

$$\frac{1}{(q^3;q^3)_\infty} = \sum_{m_0+m_1+m_2=0}\omega^{m_1-m_2}\frac{q^{[m_0^2+m_1^2+m_2^2]/2}}{(q)_\infty^3}.$$

Thus,

$$\sum_{m_1,m_2=-\infty}^{\infty}\omega^{m_1-m_2}q^{{m_1}^2+m_1m_2+{m_2}^2} = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}. \tag{12}$$

Therefore, we have shown

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}.$$

The product formula for $c(q)$ is similarly derived. □

The cubic theta functions satisfy the following relations (see [3]).

Theorem 4

$$a^3(q) = b^3(q) + c^3(q), \tag{13}$$

$$a(q) = a(q^3) + 2c(q^3), \qquad b(q) = a(q^3) - c(q^3). \tag{14}$$

We will not prove these identities here. Equivalent formulations of all three identities appear in Ramanujan's notebooks [5], but the first proof of (13) was given by J. Borwein and P. Borwein [2]. Proofs for (14) may be found in [3]. We will use these relations to demonstrate that dissections formulas like (1) result from parameterizations for Eisenstein series in terms of the cubic theta functions $a(q)$, $b(q)$, and $c(q)$. The combinatorial consequences of this work include several new results for twisted divisor sums and place a number of known results in context. The primary combinatorial consequences of this work are contained in Theorem 6. In order to discuss these consequences a number of definitions are needed.

Definition 3 *A partition of a positive integer n is a sequence* $a_1, a_2, \cdots, a_k$ *where*

$$a_1 \geq a_2 \geq \cdots \geq a_k,$$

where a_i are positive integers for all i, such that

$$n = a_1 + a_2 + \cdots + a_k. \tag{15}$$

Definition 4 *The Ferrers diagram of a partition of n is an array of dots in rows. The number of dots in each row represent a corresponding positive integer in that partition of n.*

Example 1 *A Ferrers diagram of the partition* $7+4+2$ *of* 13 *with enumerated rows and columns is*

	1	2	3	4	5	6	7
1	•	•	•	•	•	•	•
2	•	•	•	•			
3	•	•					

Naturally, we may assign a coordinate to each dot by enumerating said rows and columns. Therefore, by the (i, j) entry, we mean the dot in the ith row and jth column.

Definition 5 *The* (i, j) *hook is the set* $H_{i,j}$ *of dots under and to the right of* (i, j) *along with the dot* (i, j).

Definition 6 *The hook number is the total number of dots in the hook* $H_{i,j}$*, which we denote as* $|H_{i,j}|$.

Definition 7 *A* t*-core partition of* n *is a partition of* n *in which none of the hook numbers are divisible by* t.

The following result may be found in [5, Eq. (3)].

Theorem 5 *The generating function for* $c_t(n)$ *is given by:*

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{16}$$

Definition 8 *: The Legendre Symbol* $\left(\frac{a}{p}\right)$ *is defined for an integer a and an odd prime p, as*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if there is an integer } x \text{ such that, } x^2 \equiv a \pmod{p}, \\ 0 & \text{if } p \mid a, \\ -1 & \text{otherwise.} \end{cases} \tag{17}$$

Definition 9 *The Jacobi symbol* $\left(\frac{a}{n}\right)$ *for any integer a and odd integer n is the product of the Legendre symbols corresponding to the prime factors of n. So for* $n = p_1 \cdot p_2 \cdots p_k$,

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_k}\right) \tag{18}$$

Theorem 6 *Suppose* $\ell \in \mathbb{N}$ *is odd and s is the largest integer such that* $3^s | l$ *. Then for* $k \in \mathbb{N}$,

$$\sum_{d|3k+2} \left(\frac{d}{3}\right) d^{2\ell} \equiv 0 \pmod{3^{s+1}}, \qquad \sum_{d|3k+1} \left(\frac{d}{3}\right) d^{2\ell} \equiv c_3(k) \pmod{3^{s+1}}, \tag{19}$$

$$\sum_{d|3k+2} d^{\ell} \equiv 0 \pmod{3^{s+1}}, \qquad \sum_{d|3k+1} d^{\ell} \equiv c_3(k) \pmod{3^{s+1}}, \tag{20}$$

where $c_3(k)$ *is the number of* 3*-core partitions of k.*

These congruences will be proven in chapter 3. In Lemmas 1 and 2 we will show, via Fermat's little theorem, that each of the generating functions for the expressions in (20) reduce modulo 3^{s+1} to dissected Eisenstein series of low weight. If $\ell = s = 0$ in the latter congruence of (19), we obtain an equality corresponding to (6) appearing in Granville and Ono's work [5]. The left congruences in (19) and (20) are proven without reference to Eisenstein series in [7]. The remaining congruences from Theorem 6 appear to be new. Moreover, our work results in explicit eta function expansions for the generating functions corresponding to the trisected divisor sums in Theorem 6.

In the final chapter, chapter 5, a more general treatment of cubic multi-sections is presented. We study the dissections in terms of operators on subspaces of homogenous polynomials in the cubic theta functions. The matrix representations for these operators reveal some interesting congruences.

The results of this thesis comprise the paper [1], currently under review as of the submission date of this thesis.

CHAPTER II

PRINCIPAL DISSECTIONS

We begin with a theorem demonstrating that Garvan's formula (1) arises from a multi-section of the Hecke Eisenstein series of weight two and trivial character **1**, $E_{2k,\mathbf{1}}(q)$. The techniques we apply in the subsequent proof are of particular importance and are used in the rest of the thesis.

Theorem 7 *Let $\left(\frac{\cdot}{3}\right)$ denote the Jacobi symbol modulo three. Then*

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+2} d\Big)q^n = \frac{3(q^3;q^3)_\infty^6}{(q;q)_\infty^2}, \tag{21}$$

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+1}\left(\frac{d}{3}\right)\right)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \qquad \textit{and} \qquad \sum_{n=0}^{\infty}\left(\sum_{d|3n+2}\left(\frac{d}{3}\right)\right)q^n = 0. \tag{22}$$

Proof. To prove (21), we start with the cubic parameterization for the Eisenstein series of weight two [5, Theorem 11,10].

$$\frac{3}{2}E_2(q^3) - \frac{1}{2}E_2(q) = a^2(q), \tag{23}$$

where

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

and

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2} := \sum_{n=0}^{\infty} k_n q^n.$$

Replace q with $q^{1/3}$, and we have

$$\frac{3}{2}E_2(q) - \frac{1}{2}E_2(q^{1/3}) = a^2(q^{1/3}). \tag{24}$$

On the right hand side we employ (14) from Theorem 4 to get

$$\begin{aligned} a^2(q^{1/3}) &= (a(q) + 2c(q))^2 \\ &= a^2(q) + 4a(q)c(q) + 4c^2(q). \end{aligned}$$

Substituting into (23) we arrive at

$$\frac{3}{2}E_2(q) - \frac{1}{2}E_2(q^{1/3}) = a^2(q) + 4a(q)c(q) + 4c^2(q). \tag{25}$$

The infinite series on both sides of (23) have a cubic multi-section in the variable $q^{\frac{1}{3}}$. The right hand side of (25) may be expressed formally as

$$\sum_{n=0}^{\infty} v_n q^{n/3} = \sum_{n=0}^{\infty} v_{3n} q^n + \sum_{n=0}^{\infty} v_{3n+1} q^{(3n+1)/3} + \sum_{n=0}^{\infty} v_{3n+2} q^{(3n+2)/3}. \tag{26}$$

Recall the product expansion for $c(q)$,

$$c(q) = 3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

By comparing the right hand sides of (24) and (25) we may identify the contributions to each sum in the multi-section. Within the expression

$$a^2(q) + 4a(q)c(q) + 4c^2(q),$$

we look for the contribution to

$$\sum_{n=0}^{\infty} v_{3n} q^n,$$

by looking for the terms with integer powers on q, and we see they come entirely from

$$a^2(q) = \left(\sum_{n=1}^{\infty} k_n q^n \right)^2,$$

On the other hand, $4a(q)c(q)$ is $4\times$(power series in q)$\times 3q^{1/3}\times$(power series in q), and $4c^2(q)$ is $q^{2/3}\times$(power series in q)^2. Therefore, by uniqueness of power series expansions, the only way we get terms of the form

$$\sum_{n=0}^{\infty} v_{3n+1} q^{(3n+1)/3} = \sum_{n=0}^{\infty} v_{3n+1} q^{n+1/3} = q^{1/3} \sum_{n=0}^{\infty} v_{3n+1} q^n$$

from $a^2(q) + 4a(q)c(q) + 4c^2(q)$, is from the term $4a(q)c(q)$. Likewise, the only way we get terms of the form

$$\sum_{n=0}^{\infty} v_{3n+2} q^{(3n+2)/3} = \sum_{n=0}^{\infty} v_{3n+2} q^{n+2/3} = q^{2/3} \sum_{n=0}^{\infty} v_{3n+2} q^n$$

is from $4c^2(q)$.

Now for the left hand side of (23),

$$\frac{3}{2} E_2(q) - \frac{1}{2} E_2(q^{1/3}) \tag{27}$$

we proceed in similar fashion and identify terms

$$\frac{3}{2} E_2(q) = \frac{3}{2} \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right)$$

that contribute to each part of the mutli-section. It is advantageous for us to find the MacLaurin series expansion for the series

$$1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Since

$$\frac{q^n}{1-q^n}=\sum_{m=1}^{\infty}(q^n)^m=\sum_{m=1}^{\infty}q^{nm},$$

we may write $E_2(q)$ as

$$1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=1-24\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{nm}=1-24\sum_{k=1}^{\infty}w_kq^k.$$

For each fixed $k=nm$, in the sum above, the contribution to q^{nm}equals the *sum of divisors function* of k,

$$\sum_{d|k}d.$$

Therefore,

$$w_k=\sum_{d|k}d,$$

where

$$1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=1-24\sum_{k=1}^{\infty}w_kq^k=1-24\sum_{k=1}^{\infty}\left(\sum_{d|k}d\right)q^k.$$

We see that

$$\frac{3}{2}E_2(q)=\frac{3}{2}\left(1-24\sum_{k=1}^{\infty}\left(\sum_{d|k}d\right)q^k\right)=\frac{3}{2}-36\sum_{k=1}^{\infty}\left(\sum_{d|k}d\right)q^k.$$

Thus, since $\frac{3}{2}E_2(q)$ is a power series in q, in the Maclaurin expansion for

$$\frac{3}{2}E_2(q)-\frac{1}{2}E_2(q^{1/3}),$$

the first term, $\frac{3}{2}E_2(q)$, contributes only to the sum with integer powers in the multi-section. That is, it contributes only to the first sum in the series

$$\sum_{n=0}^{\infty} v_n q^{n/3} = \sum_{n=0}^{\infty} v_{3n} q^n + \sum_{n=0}^{\infty} v_{3n+1} q^{(3n+1)/3} + \sum_{n=0}^{\infty} v_{3n+2} q^{(3n+2)/3}.$$

We now need to determine the contribution from $\frac{1}{2}E_2(q^{1/3})$ to the sums

$$\sum_{n=0}^{\infty} v_{3n} q^n + \sum_{n=0}^{\infty} v_{3n+1} q^{(3n+1)/3} + \sum_{n=0}^{\infty} v_{3n+2} q^{(3n+2)/3}.$$

From the MacLaurin expansion for $E_2(q)$ that we just derived, we see

$$-\frac{1}{2}E_2(q^{1/3}) = -\frac{1}{2}\left(1 - 24\sum_{k=1}^{\infty}\left(\sum_{d|k} d\right) q^{k/3}\right) = -\frac{1}{2} + 12\sum_{k=1}^{\infty}\left(\sum_{d|k} d\right) q^{k/3}.$$

Therefore, the cubic multi-section of $\frac{1}{2}E_2(q^{1/3})$ equals

$$-\frac{1}{2} + 12\sum_{k=0}^{\infty}\left(\sum_{d|3k+3} d\right) q^{(3k+3)/3} + 12\sum_{k=0}^{\infty}\left(\sum_{d|3k+1} d\right) q^{(3k+1)/3}$$

$$+12\sum_{k=0}^{\infty}\left(\sum_{d|3k+2} d\right) q^{(3k+2)/3}.$$

Since the terms in the series expansions are uniquely determined, we can equate corresponding terms from the left and right hand sides of (24), namely,

$$\frac{3}{2}E_2(q) - \frac{1}{2}E_2(q^{1/3}) = a^2(q) + 4a(q)c(q) + 4c^2(q).$$

Let us determine which terms correspond to

$$\sum_{n=0}^{\infty} v_{3n+2} q^{(3n+2)/3}.$$

On the left, we obtain

$$12\sum_{k=0}^{\infty}\left(\sum_{d|k}d\right)q^{(3k+2)/3},$$

and on the right, the only contributions come from $4c^2(q)$. Therefore,

$$\begin{aligned}12\sum_{k=0}^{\infty}\left(\sum_{d|3k+2}d\right)q^{(3k+2)/3} &= 4c^2(q)\\ &= 4\left(3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^2.\end{aligned}$$

Simplify both sides to get

$$12q^{2/3}\sum_{k=0}^{\infty}\left(\sum_{d|3k+2}d\right)q^k = 36q^{2/3}\left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^2.$$

Moreover,

$$\sum_{k=1}^{\infty}\left(\sum_{d|3k+2}d\right)q^k = 3\left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^2.$$

This is equivalent to (20) as needed.

A proof of (22) is likewise obtained by applying Theorem 4 to an identity which gives a cubic theta parameterization of the eisenstein series with jacobi symbol

$$E_{1,\left(\frac{\cdot}{3}\right)}(q) = 1+6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n} = a(q). \tag{28}$$

After replacing q by $q^{1/3}$, as before, we use the MacLaurin expansion and find the left hand side equals

$$E_{1,\left(\frac{\cdot}{3}\right)}(q^{1/3}) = 1+6\sum_{k=1}^{\infty}\left(\sum_{d|k}\left(\frac{d}{3}\right)\right)q^{k/3}.$$

We find the cubic multi-section for $E_{1,\left(\frac{\cdot}{3}\right)}(q^{1/3})$ is,

$$1+6\sum_{k=0}^{\infty}\left(\sum_{d|3k+3}\left(\frac{d}{3}\right)\right)q^{(3k+3)/3}+6\sum_{k=0}^{\infty}\left(\sum_{d|3k+2}\left(\frac{d}{3}\right)\right)q^{(3k+2)/3}$$
$$+6\sum_{k=0}^{\infty}\left(\sum_{d|3k+1}\left(\frac{d}{3}\right)\right)q^{(3k+1)/3}.$$

On the right hand side we apply theorem 4 to $a(q^{1/3})$ and substitute into (27) to arrive at

$$1+6\sum_{k=0}^{\infty}\left(\sum_{d|3k+3}\left(\frac{d}{3}\right)\right)q^{(3k+3)/3}+6\sum_{k=0}^{\infty}\left(\sum_{d|3k+2}\left(\frac{d}{3}\right)\right)q^{(3k+2)/3}$$
$$+6\sum_{k=0}^{\infty}\left(\sum_{d|3k+1}\left(\frac{d}{3}\right)\right)q^{(3k+1)/3}$$

$$=a(q)+2c(q). \tag{29}$$

Equating the terms which contribute to indices $3k+1$ in the multi-section gives us

$$6\sum_{k=0}^{\infty}\left(\sum_{d|3k+1}\left(\frac{d}{3}\right)\right)q^{(3k+1)/3}=2c(q)=2\left(3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right).$$

Simplifying, we have

$$\sum_{k=0}^{\infty}\left(\sum_{d|3k+1}\left(\frac{d}{3}\right)\right)q^{k}=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

For the latter identity of (21) we note that the $a(q)$ from (28) will contribute only integer power exponents to its multi-section. Therefore, we conclude

$$\sum_{n=0}^{\infty}\left(\sum_{d|3k+2}\left(\frac{d}{3}\right)\right)q^{k}=0$$

CHAPTER III

COMBINATORIAL CONSEQUENCES

In this chapter we will prove the congruences from Theorem 6. Recall, from Theorem 5,

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

Therefore the generating function for $c_3(n)$ is given by

$$\sum_{n=0}^{\infty} c_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{30}$$

So, by (22), of Theorem 7, we have,

$$c_3(n) = \sum_{d|3n+1} \left(\frac{d}{3}\right). \tag{31}$$

We prove theorem 5 in two lemmas.

Lemma 1 *Suppose ℓ is an odd integer and s is the largest integer such that $3^s|\ell$. Then for $k \in \mathbb{N}$,*

$$\sum_{d|3k+2} \left(\frac{d}{3}\right) d^{2\ell} \equiv 0 \pmod{3^{s+1}}, \qquad \sum_{d|3k+1} \left(\frac{d}{3}\right) d^{2\ell} \equiv c_3(k) \pmod{3^{s+1}}, \tag{32}$$

where $c_3(k)$ is the number of 3-core partitions of k.

Proof. Let ℓ be odd and s be the largest integer such that $3^s|\ell$. First note that for both congruences in (32), $gcd(d,3^{s+1}) = 1$. We wish to evaluate

$$\sum_{d|3k+2} \left(\frac{d}{3}\right) d^{2\ell} \tag{33}$$

modulo 3^{s+1}. Let us state some well known identities. Using Euler's formula we have $\varphi(p^k) = p^{k-1}(p-1)$ for prime p, therefore observe that

$$\varphi(3^{s+1}) = 3^s(3-1) = 3^s \cdot 2, \tag{34}$$

where $\varphi(n)$ is Euler's totient function. Also, Euler's theorem tells us for a and n such that $gcd(a,n) = 1$, we have,

$$a^{\varphi(n)} \equiv 1 \pmod{n}. \tag{35}$$

To reduce

$$\sum_{d|3n+2} \left(\frac{d}{3}\right) d^{2\ell}, \tag{36}$$

observe that,since $3^s|\ell$, we have $\ell = 3^s \cdot m$ for some integer m. Thus (36) may be expressed as

$$\sum_{d|3n+2} \left(\frac{d}{3}\right) d^{2(3^s \cdot m)}.$$

Applying (34) and (35) we see that,

$$\begin{aligned}\sum_{d|3n+2} \left(\frac{d}{3}\right) d^{2(3^s \cdot m)} = \sum_{d|3n+2} \left(\frac{d}{3}\right) d^{\varphi(3^{s+1})m} &\equiv \sum_{d|3n+2} \left(\frac{d}{3}\right) 1^m \pmod{3^{s+1}} \\ &\equiv \sum_{d|3n+2} \left(\frac{d}{3}\right) \pmod{3^{s+1}}.\end{aligned}$$

To arrive at the desired congruence, we note that by (22) from Theorem 7, in the MacLaurin expansion, the sum of the terms with index congruent to 2 modulo 3 equal 0. Therefore, since the coefficients in the series are uniquely determined, we have shown the desired congruence.

For the second congruence on (32), the argument may be mirrored from the previous, but instead, we apply our line of reasoning to

$$\sum_{d|3k+1}\left(\frac{d}{3}\right)d^{2\ell} \tag{37}$$

rather than (32). The congruence follows. ∎

Lemma 2 *Suppose ℓ is an odd integer and s is the largest integer such that $3^s|\ell$. Then for $k\in\mathbb{N}$,*

$$\sum_{d|3k+2} d^{\ell} \equiv 0 \pmod{3^{s+1}}, \qquad \sum_{d|3k+1} d^{\ell} \equiv c_3(k) \pmod{3^{s+1}}, \tag{38}$$

where $c_3(k)$ is the number of 3-core partitions of k.

Proof. For this lemma we make use the another lemma, which is proven in [6].

Lemma 3 *Let λ and a be integers, and p be an odd prime such that $gcd(n,p^{\lambda})=1$, then*

$$n^{\frac{1}{2}\varphi(p^{\lambda})} \equiv \left(\frac{n}{p}\right) \pmod{p^{\lambda}}. \tag{39}$$

Again we wish to evaluate

$$\sum_{d|3k+1} d^{\ell} \tag{40}$$

modulo 3^{s+1}. By assumption we have

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+1} d^{l}\right)q^{n} = \sum_{n=0}^{\infty}\left(\sum_{d|3n+1} d^{3^{s}\cdot m}\right)q^{n}. \tag{41}$$

Use lemma 3; let $\lambda = s+1$ and $p = 3$ then, $d^{\frac{1}{2}\varphi(3^{s+1})} = d^{\frac{1}{2}2\cdot(3^s)} = d^{3^s} \equiv \left(\frac{d}{3}\right) \pmod{3^{s+1}}$. Hence,

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+1} d^{3^s\cdot m}\right)q^n \equiv \sum_{n=0}^{\infty}\left(\sum_{d|3n+1}\left(\frac{d}{3}\right)^m\right)q^n \pmod{3^{s+1}} \tag{42}$$

Note that since ℓ was odd and $\ell = 3^s \cdot m$, m is odd, and $\left(\frac{d}{3}\right) \in \{-1,0,1\}$, therefore,

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+1}\left(\frac{d}{3}\right)^m\right)q^n \pmod{3^{s+1}} = \sum_{n=0}^{\infty}\left(\sum_{d|3n+1}\left(\frac{d}{3}\right)\right)q^n \pmod{3^{s+1}} \tag{43}$$

$$= \sum_{n=0}^{\infty} c_3(n)q^n \pmod{3^{s+1}}. \tag{44}$$

Equation (42) follows from (29). For the left congruence on line (36) the proof is straightforward by applying the same reasoning to the series

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+2} d^{\ell}\right)q^n.$$

□

CHAPTER IV

EXPANSIONS FOR GENERATING FUNCTIONS OF HIGHER WIEGHT

In this chapter we use recursion formulas for Eisenstein series to derive eta function expansions for multi-sections of Eisenstein series of higher weight. For generating functions in terms of cubic theta functions, we use the two classes of Eisenstein that series we have utilized throughout the thesis. The normalized Eisenstein series of weight k

$$E_{2k}(q) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}, \tag{45}$$

and the Hecke Eisenstein series associated with the Dirichlet character χ modulo three

$$E_{k,\chi}(q) = 1 + \frac{2}{L(1-k,\chi)} \sum_{n=1}^{\infty} \chi(n) \frac{n^{k-1} q^n}{1-q^n}, \tag{46}$$

where $L(1-k,\chi)$ denotes associated Dirichlet L-series. The Eisenstein series on the full modular group are related to the Eisenstein series of trivial character $\mathbf{1}$ through the identity given in the following lemma. The straightforward derivation is omitted.

Lemma 4

$$E_{2k,\mathbf{1}}(q) = \frac{3^{2k-1} E_{2k}(q^3) - E_{2k}(q)}{3^{2k-1} - 1}.$$

Since we build cubic Eisenstein expansions inductively, we require representations for low weight series. From [5, 10]:

$$E_4(q) = a^4(q) + 8ac^3(q), \tag{47}$$

$$E_6(q) = a^6(q) - 20a^3(q)c^3(q) - 8c^6(q), \tag{48}$$

$$E_{1,\left(\frac{\cdot}{3}\right)} = a(q), \tag{49}$$

$$E_{3,\left(\frac{\cdot}{3}\right)}(q) = b^3(q). \tag{50}$$

To express the Hecke Eisenstein series of arbitrary weight as polynomials in the cubic theta functions, we employ recursion formulas for cubic Eisenstein series derived by S. Cooper and others [5]. To avoid conflict of notation, we adopt the conventions

$$G_{2k,\mathbf{1}}(q) = \frac{L(1-2k,\mathbf{1})}{2}E_{2k,\mathbf{1}}(q), \qquad G_{2k,\left(\frac{\cdot}{3}\right)}(q) = \frac{L\left(-2k,\left(\frac{\cdot}{3}\right)\right)}{2}E_{2k+1,\left(\frac{\cdot}{3}\right)}(q), \tag{51}$$

$$G_{2k}(q) = \frac{\zeta(1-2k)}{2}E_{2k}(q). \tag{52}$$

Lemma 5 *Let χ and $\mathbf{1}$ denote, respectively, the Jacobi symbol and the principal character modulo three. Then, for each integer $n \geq 1$,*

$$\begin{aligned} G_{2n+2,\chi}(q) = {} & -9(2n+1)(2n+2)G_{0,\chi}^2(q)G_{2n,\chi}(q) \\ & - 2(2n+1)(2n+2)\sum_{j=1}^{n-1}\binom{2n}{2j}G_{2j,\chi}(q)G_{2n+2-2j}(q), \\ G_{2n+2,\mathbf{1}}(q) = {} & 18G_{0,\chi}(q)G_{2n,\chi}(q) + 6\sum_{j=1}^{n-1}\binom{2n}{2j}G_{2j,\chi}(q)G_{2n-2j,\chi}(q). \end{aligned}$$

To implement this recursion, we also need a well-known recursion formula for the Eisenstein series on the full modular group [5, p. 30].

Lemma 6 *Let ζ denote the Riemann zeta function and*

$$d_0 = 6\zeta(4)E_4(q), \qquad d_1 = 10\zeta(6)E_6(q). \tag{53}$$

For $n \geq 2$, the normalized Eisenstein series $E_{2n}(q)$ on the full modular group satisfy

$$E_{2n}(q) = \frac{d_{n-2}}{(n-2)!(4n-2)\zeta(2n)}, \quad d_n = \frac{3n}{5+2n}\sum_{k=0}^{n-2}\binom{2n-2}{k}d_k d_{2n-2-k}. \tag{54}$$

Using these recursion formulas, we list the first few cases of $G_{2k,\left(\frac{\cdot}{3}\right)}(q)$ as follows:

$$\begin{aligned}
G_{2,\left(\frac{\cdot}{3}\right)}(q) &= -\frac{b^3}{9}, \qquad \text{in accordance with (50),}\\
G_{4,\left(\frac{\cdot}{3}\right)}(q) &= \frac{a^2b^3}{3},\\
G_{6,\left(\frac{\cdot}{3}\right)}(q) &= \frac{1}{3}ab^3\left(-7a^3+4c^3\right),\\
G_{8,\left(\frac{\cdot}{3}\right)}(q) &= \frac{1}{27}b^3\left(809a^6-808a^3c^3+80c^6\right),\\
G_{10,\left(\frac{\cdot}{3}\right)}(q) &= -\frac{1}{3}a^2b^3\left(1847a^6-2668a^3c^3+848c^6\right),\\
G_{12,\left(\frac{\cdot}{3}\right)}(q) &= \frac{1}{3}ab^3\left(55601a^9-105024a^6c^3+55584a^3c^6-6080c^9\right).
\end{aligned}$$

Similarly, for $G_{2n+2,\mathbf{1}}(q)$ we have:

$$
\begin{aligned}
G_{2,\mathbf{1}}(q) &= \frac{a^2}{2}, \\
G_{4,\mathbf{1}}(q) &= \frac{a^2b^3}{3}, \\
G_{6,\mathbf{1}}(q) &= \frac{1}{3}ab^3\left(4c^3-7a^3\right), \\
G_{8,\mathbf{1}}(q) &= \frac{1}{27}b^3\left(809a^6-808a^3c^3+80c^6\right), \\
G_{10,\mathbf{1}}(q) &= -\frac{1}{3}a^2b^3\left(1847a^6-2668a^3c^3+848c^6\right), \\
G_{12,\mathbf{1}}(q) &= \frac{1}{3}ab^3\left(55601a^9-105024a^6c^3+55584a^3c^6-6080c^9\right).
\end{aligned}
$$

These formulas were obtained using Mathematica code included at the end of the Appendix.

4.1 Eisentstein Series of Higher Weight

To obtain the claimed eta-function expansions, we characterize the Eisenstein series whose multi-sections are homogeneous polynomials in $c(q)$ and $b(q)$.

Lemma 7 *If $\overline{n}$ denotes the least positive residue class of n modulo 3, then the following series are homogeneous polynomials in $c(q)$ and $b(q)$ over $\mathbb{Q}$:*

$$\sum_{k=0}^{\infty}\left(\sum_{d|3k+\overline{2n+1}}\left(\frac{d}{3}\right)d^{2n}\right)q^{(3k+\overline{2n+1})/3}, \tag{55}$$

$$\sum_{k=0}^{\infty}\left(\sum_{d|3k+\overline{2n}}d^{2n-1}\right)q^{(3k+\overline{2n})/3}. \tag{56}$$

Proof. From (47) - (50), observe that the lower weight Eisenstein expansions are homogenous polynomials in $a(q)$ and $c(q)$. Using an identity from Theorem 4, namely, $b^3(q) = a^3(q) - c^3(q)$, (50) is also a homogenous polynomial in $a(q)$ and $c(q)$. Since the lower weight series are the base cases of the recursion formulas from Lemma 4, we calculate the coefficients in the base case polynomials and find they are rational. Therefore, we may calculate higher weight polynomials using the recursion formulas with rational coefficient values for each homogeneous polynomial . Now since the powers on $c(q)$ in the base case are always congruent to 0 (mod 3) we may substitute $c^3(q) = a^3(q) - b^3(q)$. We obtain,

$$G_{2n,\left(\frac{\cdot}{3}\right)}(q) = f_{2n+1}\Big(a(q), b(q)\Big) \qquad \text{and} \qquad G_{2n,\mathbf{1}}(q) = g_{2n}\Big(a(q), b(q)\Big) \tag{57}$$

for homogeneous polynomials f_{2n+1} and g_{2n} in $a(q)$ and $b(q)$ of degrees at most $2n+1$ and $2n$, respectively. Note, we are using the fact that products of homogenous polynomials are homogeneous polynomials. By replacing q by $q^{1/3}$ and applying Theorem 4, we derive expansions for $f_{2n+1}(a(q^{1/3}), b(q^{1/3}))$ and $g_{2n}(a(q^{1/3}), b(q^{1/3}))$ as homogeneous polynomials in $a(q)$ and $c(q)$ with rational coefficients. From (13), the $\overline{2n+1}$-dissection of $f_{2n+1}(a(q^{1/3}), b(q^{1/3}))$ equals, for $\lambda_k \in \mathbb{Q}$,

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\sum_{d|3k+\overline{2n+1}}\left(\frac{d}{3}\right)d^{2n}\right)q^{\frac{3k+\overline{2n+1}}{3}} &= \sum_{k=0}^{2n+1}\lambda_k a^{2n+1-\overline{2n+1}-3k}(q)c^{\overline{2n+1}+3k}(q),\\ &= \sum_{k=0}^{2n+1}\lambda_k\Big(b^3(q)+c^3(q)\Big)^{\frac{2n+1-\overline{2n+1}-3k}{3}}c^{\overline{2n+1}+3k}(q).\end{aligned}$$

Since $m - \overline{m} \equiv 0 \pmod 3$, each exponent of $(b^3(q) + c^3(q))$ in the last summand is a natural number. Therefore, the $\overline{2n+1}$-dissection of $f_{2n+1}(a(q^{1/3}), b(q^{1/3}))$ is a homogeneous polynomial in $b(q)$ and $c(q)$ of degree at most $2n+1$. A similar calculation shows the $\overline{2n}$-dissection of the polynomial $g_{2n}(a(q^{1/3}), b(q^{1/3}))$ is of the required form. ∎

4.2 Multisections of Eisenstein Series of Higher Weight

By Lemma 7 we now have representations for higher weight Eisenstein series as polynomials in

$b(q)$ and $c(q)$. Recall, by Theorem 3, the product expansions

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},$$
$$c(q) = 3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

Substituting these values, we may use Lemma 5 to generate each expansion displayed in the introduction and derive corresponding multisections for Eisenstein series of higher weight:

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+1} d^3\right)q^n = (q;q)_\infty^8 + 3^4 q\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4},$$

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+2} d^7\right)q^n = 3\cdot 43(q;q)_\infty^{16} + 110\cdot 3^6 q(q;q)_\infty^4(q^3;q^3)_\infty^{12} + 41\cdot 3^{10}q^2\frac{(q^3;q^3)_\infty^{24}}{(q;q)_\infty^8},$$

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+1} d^9\Big)q^n = \frac{(q;q)_\infty^{26}}{(q^3;q^3)_\infty^6} + 3^5\cdot 23\cdot 47q(q;q)_\infty^{14}(q^3;q^3)_\infty^6 + 3^9\cdot 2237q^2(q;q)_\infty^2(q^3;q^3)_\infty^{18} + \frac{3^{13}\cdot 11\cdot 61q^3(q^3;q^3)_\infty^{30}}{(q;q)_\infty^{10}},$$

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+2} d^{13}\Big)q^n = \frac{3\cdot 2731(q;q)_\infty^{34}}{(q^3;q^3)_\infty^6} + 3^6\cdot 1674872q(q;q)_\infty^{22}(q^3;q^3)_\infty^6 + 3^{10}\cdot 9766130q^2(q;q)_\infty^{10}(q^3;q^3)_\infty^{18} + \frac{3^{15}\cdot 2790064q^3(q^3;q^3)_\infty^{30}}{(q;q)_\infty^2} + \frac{3^{19}\cdot 597871q^4(q^3;q^3)_\infty^{42}}{(q;q)_\infty^{14}},$$

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+1} d^{15}\Big)q^n = \frac{(q;q)_\infty^{44}}{(q^3;q^3)_\infty^{12}} + 3^4\cdot 13\cdot 1019729q(q;q)_\infty^{32}$$
$$+3^{10}\cdot 80982274q^2(q;q)_\infty^{20}(q^3;q^3)_\infty^{12} + 3^{14}\cdot 228972994q^3(q;q)_\infty^{8}(q^3;q^3)_\infty^{24}$$
$$+\frac{3^{18}\cdot 152647045q^4(q^3;q^3)_\infty^{36}}{(q;q)_\infty^{4}} + \frac{3^{22}\cdot 28621321q^5(q^3;q^3)_\infty^{48}}{(q;q)_\infty^{16}},$$

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+1}\left(\frac{d}{3}\right)\right)q^n = \frac{(q^3;q^3)_\infty^{3}}{(q;q)_\infty},$$

$$\sum_{n=0}^{\infty}\left(\sum_{d|3n+2}\left(\frac{d}{3}\right)d^4\right)q^n = -15(q;q)_\infty^{7}(q^3;q^3)_\infty^{3} - 3^6q\frac{(q^3;q^3)_\infty^{15}}{(q;q)_\infty^{5}},$$

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+1}\left(\frac{d}{3}\right)d^6\Big)q^n = \frac{(q;q)_\infty^{17}}{(q^3;q^3)_\infty^{3}} + 3^4\cdot 50q(q;q)_\infty^{5}(q^3;q^3)_\infty^{9} + \frac{3^9\cdot 7q^2(q^3;q^3)_\infty^{21}}{(q;q)_\infty^{7}},$$

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+2}\left(\frac{d}{3}\right)d^{10}\Big)q^n = -\frac{3\cdot 341(q;q)_\infty^{25}}{(q^3;q^3)_\infty^{3}} - 3^7\cdot 4477q(q;q)_\infty^{13}(q^3;q^3)_\infty^{9}$$
$$-3^{10}\cdot 20317q^2(q;q)_\infty(q^3;q^3)_\infty^{21} - \frac{3^{15}\cdot 1847q^3(q^3;q^3)_\infty^{33}}{(q;q)_\infty^{11}},$$

$$\sum_{n=0}^{\infty}\Big(\sum_{d|3n+1}\left(\frac{d}{3}\right)d^{12}\Big)q^n = \frac{(q;q)_\infty^{35}}{(q^3;q^3)_\infty^{9}} + 3^4\cdot 207076q(q;q)_\infty^{23}(q^3;q^3)_\infty^{3}$$
$$+3^9\cdot 722810q^2(q;q)_\infty^{11}(q^3;q^3)_\infty^{15} + \frac{3^{13}\cdot 722812q^3(q^3;q^3)_\infty^{27}}{(q;q)_\infty} + \frac{3^{18}\cdot 55601q^4(q^3;q^3)_\infty^{39}}{(q;q)_\infty^{13}}.$$

CHAPTER V

MORE GENERAL MULTISECTIONS OPERATORS

We now consider cubic multisections of modular forms beyond the Eisenstein series studied in the previous chapter. The purpose here is to study the multisection operators

$$\Omega_{3,k}\left(\sum_{n=0}^{\infty} v_n q^n\right) = \sum_{n=0}^{\infty} v_{3n+k} q^n, \qquad \pi\left(\sum_{n=0}^{\infty} v_n q^n\right) = \sum_{n=0}^{\infty} v_n q^{n/3} \tag{58}$$

$k = 0,1,2$, on subspaces of homogeneous polynomials in the cubic theta functions. We construct explicit matrix representations for these operators and discuss their spectral properties. Some of the integer eigenvalue and eigenvector pairs induce interesting congruences for the coefficients of corresponding eigenforms modulo powers of three.

Theorem 8 *Let*

$$\frac{3^5}{504}(E_6(q^3) - E_6(q)) = \sum_{n=0}^{\infty} u_n q^n, \qquad 27q(q^3;q^3)_\infty^6 (q;q)_\infty^6 = \sum_{n=0}^{\infty} v_n q^n, \tag{59}$$

then

$$u_{3^\ell n} \equiv 0 \ (mod\ 3^{5\ell}), \qquad v_{3^\ell n} \equiv 0 \ (mod\ 3^{2\ell}). \tag{60}$$

To prove this theorem, we first derive expansions for more general operators.

Theorem 9 *let $\mathfrak{O}_d(x,y)$ denote the complex span of homogeneous polynomials in x and y of degree* *d and let B_d be the $(d+1)\times(3d+1)$ matrix whose (r,k)th entry equals*

$$\sum_{j=0}^{k}\sum_{\ell=0}^{3d-3j}\binom{k}{j}\binom{3d-3j}{\ell}\binom{3j}{r-\ell}2^{\ell}(-1)^{3j+\ell-r}. \tag{61}$$

Suppose that $f(q)\in\mathfrak{O}_d(a^3,c^3)$ with

$$f(q)=\sum_{n=0}^{\infty}v_nq^n=\sum_{k=0}^{d}\alpha_k a^{3(d-k)}(q)c^{3k}(q), \qquad \alpha_n\in\mathbb{C}, \tag{62}$$

$$B_d(\alpha_0,\alpha_1,\ldots,\alpha_d)^T=(\beta_0,\beta_1,\ldots,\beta_{3d})^T. \tag{63}$$

Then for $m=0,1,2$,

$$\Omega_{3,m}(f)=\sum_{n=0}^{\infty}k_{3n+m}q^n=q^{-m/3}\sum_{k=0}^{d-\mathbf{1}(m)}\beta_{3k+m}a^{3d-3k-m}(q)c^{3k+m}(q). \tag{64}$$

Before we prove Theorem 9 we provide an example.

Example 2 *Let $d=1$ and $f(q)$ be a homogenous polynomial in $a^3(q)$ and $c^3(q)$,*

$$f(q)=\alpha_0a^3(q)+\alpha_1b^3(q), \qquad \alpha_k\in\mathbb{C}. \tag{65}$$

Applying $\pi f(q)=f(q^{1/3})$ along with identities from Theorem 4 we have,

$$\begin{aligned}\pi(f(q))&=\alpha_0a^3(q^{1/3})+\alpha_1c^3(q^{1/3})\\&=\alpha_0(a(q)+2c(q))^3+\alpha_1[(a(q)+2c(q))^3-(a(q)-c(q))^3]\\&=a^3(q)\alpha_0+6a^2(q)c(q)\alpha_0+9a^2(q)c(q)\alpha_1+12a(q)c^2(q)\alpha_0\\&\quad+9a(q)c^2(q)\alpha_1+8c^3(q)\alpha_0+9c^3(q)\alpha_1.\end{aligned} \tag{66}$$

To form our matrix B_d, for the (r,k) entries, r corresponds to the power on $c(q)$ and k corresponds to the index on the coefficients α, where the indexing of the rows and columns start from 0 rather than 1, we obtain

$$\begin{pmatrix} 1 & 0 \\ 6 & 9 \\ 12 & 9 \\ 8 & 9 \end{pmatrix}. \tag{67}$$

In the notation of (63),

$$\begin{pmatrix} 1 & 0 \\ 6 & 9 \\ 12 & 9 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ 6\alpha_0 + 9\alpha_1 \\ 12\alpha_0 + 9\alpha_1 \\ 8\alpha_0 + 9\alpha_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}. \tag{68}$$

From (65) we may write,

$$\pi(f(q)) = \beta_0 a^3(q) + \beta_1 a^2(q)c(q) + \beta_2 a(q)c^2(q) + \beta_3 c^3(q). \tag{69}$$

Thus,

$$\begin{aligned} \Omega_{3,0}(f(q)) &= \beta_0 a^3(q) + \beta_3 c^3(q) \\ \Omega_{3,1}(f(q)) &= q^{-1/3}\beta_1 a^2(q)c(q) \\ \Omega_{3,2}(f(q)) &= q^{-2/3}\beta_2 a(q)c^2(q), \end{aligned}$$

in particular,

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}. \tag{70}$$

These square matrices arising from $m \equiv 0 \pmod 3$ *dissections do provide utility in the proof of Theorem 8.*

Using this method, the first few matrices B_d for $d = 1,2,3$ by Theorem 9 we can give explicitly by

$$\begin{pmatrix} 1 & 6 & 12 & 8 \\ 0 & 9 & 9 & 9 \end{pmatrix}^T, \quad \begin{pmatrix} 1 & 12 & 60 & 160 & 240 & 192 & 64 \\ 0 & 9 & 63 & 171 & 234 & 180 & 72 \\ 0 & 0 & 81 & 162 & 243 & 162 & 81 \end{pmatrix}^T, \tag{71}$$

$$\begin{pmatrix} 1 & 18 & 144 & 672 & 2016 & 4032 & 5376 & 4608 & 2304 & 512 \\ 0 & 9 & 117 & 657 & 2088 & 4140 & 5328 & 4464 & 2304 & 576 \\ 0 & 0 & 81 & 648 & 2187 & 4212 & 5265 & 4374 & 2268 & 648 \\ 0 & 0 & 0 & 729 & 2187 & 4374 & 5103 & 4374 & 2187 & 729 \end{pmatrix}^T. \tag{72}$$

Proof. [Proof of Theorem 9] Suppose that $f(q) \in \mathfrak{O}_n(a^3, c^3)$ has has the expansions given by (62). Then, from the Borwein's identity (13) and the binomial theorem we deduce

$$f(q) = \sum_{k=0}^{d} \alpha_k a^{3(d-k)} (a^3 - b^3)^k = \sum_{k=0}^{d} \alpha_k \sum_{j=0}^{k} \binom{k}{j} (-1)^{3j} a^{3d-3j} b^{3j}. \tag{73}$$

Applying Theorem 4 and the binomial theorem, we see that $\pi(f) = f(q^{1/3})$ equals

$$\begin{aligned}
&\sum_{k=0}^{d} \alpha_k \sum_{j=0}^{k} \binom{k}{j} (-1)^{3j} (a+2c)^{3d-3j} (a-c)^{3j} \\
&= \sum_{k=0}^{d} \alpha_k \sum_{j=0}^{k} \binom{k}{j} (-1)^{3j} \sum_{l=0}^{3d-3j} \binom{3d-3j}{l} a^{3d-3j-l} (2c)^l (a-c)^{3j} && (74)\\
&= \sum_{k=0}^{d} \sum_{j=0}^{k} \sum_{\ell=0}^{3d-3j} \sum_{i=0}^{3j} \binom{k}{j} \binom{3d-3j}{l} \binom{3j}{i} (-1)^{3j-i} 2^l a^{3d-l-i} c^{l+i} \alpha_k. && (75)
\end{aligned}$$

Therefore, for $0 \leq k \leq d$ and $0 \leq r \leq 3d$, the coefficient of $\alpha_k a^{3d-r} c^r$ equals the expression on line (61). Since the contribution to the m-dissection $\sum_{n=0}^{\infty} v_{3n+m} q^{(3n+m)/3}$ of $\pi(f) = f(q^{1/3})$ arises entirely from terms in (75) with $r \equiv m \ (mod\ 3)$ we have shown that the matrices defined by (61) generate the requisite expansions for $\Omega_{3,m}(f)$. □

Proof. [Proof of Theorem 8] From Theorem 9, we deduce that the operator $\Omega_{3,0}$ over $\mathfrak{O}_d(a^3, c^3)$ corresponds to a $(d+1) \times (d+1)$ matrix whose entries come from rows $1, 4, \ldots, 3d+1$ of B_d. The $d = 1$ case is shown in (70). In particular, the respective matrix representations for $\Omega_{3,0}$ on $\mathfrak{O}_d(a^3, c^3)$, $d = 1, 2, 3$ are

$$\begin{pmatrix} 1 & 0 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 160 & 171 & 162 \\ 64 & 72 & 81 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 672 & 657 & 648 & 729 \\ 5376 & 5328 & 5265 & 5103 \\ 512 & 576 & 648 & 729 \end{pmatrix}. \qquad (76)$$

The eigenvectors x_i and eigenvalues λ_i for the matrix corresponding to $\mathfrak{O}_2(a^3, c^3)$ are

$$x_i = \begin{pmatrix} 0 & 9 & 4 \end{pmatrix}^T, \quad \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T, \quad \begin{pmatrix} 121 & -152 & 40 \end{pmatrix}^T, \quad \lambda_i = 243, 9, 1, \quad i = 1, 2, 3.$$

We may apply the cubic theta function parameterizations for Eisenstein series from [6] to deduce that the eigenform for x_1 equals the Fourier expansion at $\tau = 0$ for the Hecke Eisenstein series

$$9a^3(q)c^3(q) + 4c^6(q) = \sum_{n=1}^{\infty} \frac{n^5(q^n + q^{2n})}{1 - q^{3n}} = \frac{3^5}{504}(E_6(q^3) - E_6(q)), \tag{77}$$

while x_2 and x_3 correspond, respectively, to the eigenforms

$$a^3(q)c^3(q) - c^6(q) = c^3(q)b^3(q), \qquad 121a^6(q) - 152a^3(q)c^3(q) + 40c^6(q) = 121E_{5,\left(\frac{\cdot}{3}\right)}(q)$$

The congruences on (60) follow from iterating the operator $\Omega_{3,0}$ ℓ times. ∎

The matrices corresponding to $\Omega_{3,0}$ on $\mathfrak{O}_{3j}(a^3, c^3)$ have other interesting properties which seem to be similar to those of corresponding quintic operators from [8]. We conjecture that, up to sign, the determinants are powers of three. A proof of the conjecture may follow from an analysis of the spectral structure of $\Omega_{3,0}$ by way of classical Hecke operators.

Conjecture 1 *Let C_d denote the matrix representation for $\Omega_{3,0}$ on the vector space $\mathfrak{O}_{3d}(a^3, c^3)$ of homogeneous polynomials of degree $3d$ in $a^3(q)$ and $c^3(q)$ over $\mathbb{C}$. Then $\det C_n = \pm 3^{w(n)}$ for some $w(n) \in \mathbb{N}$. For even indices, we have $w(2n) = 4n(6n-2)$.*

Similar formulas may be obtained for the determinants of a class of matrix representations for $\pi(f)$ corresponding to the embedding $\pi(\mathfrak{O}_n(a,b))$ in $\mathfrak{O}_n(a,c)$. We give a precise construction for these trimidiation arrays in Theorem 10. Our proof of the theorem is similar to that of Theorem 9, and depends primarily on the Borwein's identity (13), Theorem 4, and the binomial theorem.

Theorem 10 *Let $f(q) = \sum_{n=0}^{\infty} \nu_n q^n$ be a homogeneous polynomial in $a(q)$ and $b(q)$,*

$$f(q) = \sum_{n=0}^{d} \alpha_n a^n(q) b^{(d-n)}(q), \tag{78}$$

where $\alpha_n \in \mathbb{C}$. Then there exists a $(d+1)\times(d+1)$ matrix $\mathscr{B}_d$ over $\mathbb{Z}$ such that if

$$\mathscr{B}_d(\alpha_0,\alpha_1,\ldots,\alpha_d)^T = (\beta_0,\beta_1,\ldots,\beta_d)^T \tag{79}$$

then

$$\pi(f) = f(q^{1/3}) = \sum_{n=0}^{d} \beta_n a^n(q) c^{(d-n)}(q). \tag{80}$$

Moreover, the (r,n)th entry of $\mathscr{B}_d$ equals

$$\sum_{k=0}^{d-n} 2^{(d-n)-k}(-1)^{n-(r-k)} \binom{d-n}{k}\binom{n}{r-k}. \tag{81}$$

The first few matrices $\mathscr{B}_d$ defined in Theorem 10 are

$$\mathscr{B}_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathscr{B}_2 = \begin{pmatrix} 4 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathscr{B}_3 = \begin{pmatrix} 8 & -4 & 2 & -1 \\ 12 & 0 & -3 & 3 \\ 6 & 3 & 0 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

As with the matrices in Theorem 9, we observe that the determinants of the matrices from Theorem 10 are certain powers of three. This leads to a more general conjecture.

Conjecture 2 *Let B_d be the trimidiation matrix defined in Theorem 10. Then*

$$\det B_n = 3^{n(n+1)/2}.$$

BIBLIOGRAPHY

[1] A. Alaniz and T. Huber. On cubic multi sections of Eisenstein series. *Preprint,* 2013.

[2] George E. Andrews. *Number Theory.* Dover Publications Inc., New York, 1994. Corrected reprint of the 1971 original. Dover, New York; MR0309838 (468943)

[3] J.M. Borwein and P.B. Borwein. A cubic counterpart of Jacobi's identity and the AGM. *Trans. Amer. Math. Soc.,* 323(2):691-701, 1991.

[4] J.M. Borwein, P.B. Borwein, and F.G. Garvan. Some cubis modular identities of Ramanujan. *Trans. Amer. Math. Soc.*, 343(1):35-47, 1994.

[5] K. Chandrasekharan. *Elliptic functions*, volume 281 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin,
1985

[6] S. Cooper. Cubic elliptic functions. *Ramanujan J.*, 11(3):355-397, 2006.

[7] A. Granville and K. Ono. Defect zero p-blocks for finite simple groups. *Trans. Amer. Math. Soc.*, 348(1):331-347, 1996.

[8] T. Huber. A theory of theta functions to the quintic base. *J. of Number Theory*, To Appear.

[9] D.b.Lahari. Some congruences for the elementary divisor functions. *Amer. Math. Monthly,* 76:395-397, 1969.

[10] S. Ramanujan. *Notebooks. Vols. !, 2*. Tata Institute of the Fundamental Research, Bombbay, 1957.

[11] L.C. Shen. On some cubic modular identities. *Proc. Amer. Math. Soc.*, 119(1):203-208, 1993.

APPENDIX

APPENDIX

In this Appendix we address several important underlying details which are needed to justify results in the thesis. First we prove several theorems concerning the convergence of infinite products and series. These results and their proofs are paraphrased from [2].

Theorem 11 *If $a_n \geq 0$, then $\prod_{n=1}^{\infty}(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ are both convergent or both divergent.*

Proof.

We see that $g(x) = e^x - x - 1$ has an absolute minimum at $x = 0$ and $g(0) = 0$, and $1 + x \leq e^x$ for all x. Therefore,

$$1 + a_1 + a_2 + \cdots + a_N \leq \prod_{n=1}^{\infty}(1+a_n) \leq e^{a_1+a_2+\cdots+a_N}.$$

This implies that if either the sequence of partial sums or the sequence of partial products converges, then the other is bounded and so must converge because both are nondecreasing. So, $\prod_{n=1}^{\infty}(1+a_n) \neq 0$ since we see that each of the partial products is at least 1. □

Theorem 12 *If $1 > a_n \geq 0$, then $\prod_{n=1}^{\infty}(1-a_n)$ and $\sum_{n=1}^{\infty} a_n$ are both convergent or both divergent.*

Proof.

Let $\sum_{n=1}^{\infty} a_n$ converge, then there is a number, N, such that $\sum_{n=1}^{\infty} a_N < \frac{1}{2}$. So,

$$\begin{aligned}(1-a_N)(1-a_{N+1}) &= 1 - a_N - a_{N+1} + a_N a_{N+1} \\ &\geq 1 - a_N - a_{N+1}\end{aligned}$$

and

$$
\begin{aligned}
(1-a_N)(1-a_{N+1})(1-a_{N+2}) &\geq (1-a_N-a_{N+1})(1-a_{N+2}) \\
&\geq 1-a_N-a_{N+1}-a_{N+2}.
\end{aligned}
$$

By induction we want to show that for $m \geq N$,

$$
(1-a_N)(1-a_{N+1})\dots(1-a_m) \geq 1-a_N-a_{N+1}\cdots-a_{m.}
$$

We have that if $p_m = \prod_{n=1}^{\infty}(1-a_m)$, then for $m \geq N$

$$
\begin{aligned}
\frac{p_m}{p_{N-1}} &= (1-a_N)(1-a_{N+1})\dots(1-a_m) \\
&\geq 1-a_N-a_{N+1}\dots a_m > 1-\frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

Now $p_m > 0$ for all m, since p_m is just a product of positive numbers. Moreover,

$$
p_m - p_{m+1} = p_m(1-(1-a_{m+1})) = a_{m+1}p_m \geq 0,
$$

and we can see that the p_m form a decreasing sequence. Also, $p_m > \frac{1}{2}p_{N-1} > 0$. So it follows that the sequence is bounded below by a positive number and therefore tends to a positive limit, so, $\lim_{m\to\infty} p_m$ exists and isn't zero. That is, $\prod_{n=1}^{\infty}(1-a_n)$ converges.

Now to show the convergence of the sum we begin by assuming that $\prod_{n=1}^{\infty}(1-a_n)$ converges. Then since $1-x \leq e^{-x}$ for all x, this comes from the fact that we established at the beginning of theorem (11) we have the following inequalities

$$
0 < c < (1-a_1)(1-a_2)\dots(1-a_m) \leq e^{-a_1-a_2\cdots-a_m},
$$

where $c < 1$, therefore,

$$\log c < -a_1 - a_2 - \cdots - a_m$$

equivalently

$$-\log c > a_1 + a_2 \cdots + a_m.$$

Hence, $\{\sum_{n=1}^{\infty} a_n\}_{m=1}^{\infty}$ forms a bounded increasing sequence, and therefore $\sum_{n=1}^{\infty} a_n$ converges. □

Definition 10 *The infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *is said to* ***converge absolutely*** *if* $\prod_{n=1}^{\infty}(1+\mid a_n \mid)$ *converges.*

Definition 11 $a_{n_{n=1}}^{\infty}$ *is a Cauchy sequence , provided that for every* ε *there exists an* M *such that* $|a_R - a_S| < \varepsilon$ *whenever* $R \geq S \geq M$.

Theorem 13 *If* $\mid a_n \mid < 1$, *and* $\prod_{n=1}^{\infty}(1+a_n)$ *converges absolutely, then* $\prod_{n=1}^{\infty}(1+a_n)$ *converges.*

Proof. We let $P_N = \prod_{n=1}^{\infty}(1+\mid a_n \mid)$ and $p_N = \prod_{n=1}^{\infty}(1+a_n)$. Then

$$\begin{aligned} \mid \quad & p_N - p_{N+1} \mid = \mid (1+a_1)(1+a_2)\ldots(1+a_{N-1})a_N \mid \\ \leq \quad & (1+\mid a_1 \mid)(1+\mid a_2 \mid)\ldots(1+\mid a_{N-1} \mid)\mid a_N \mid \\ = \quad & P_N - P_{N-1}. \end{aligned}$$

So, if $R > S$,

$$\begin{aligned} \mid \quad & p_R - p_S \mid = \mid p_R - p_{R-1} + p_{R-1} - p_{R-2} \ldots p_{S+1} - P_S \mid \\ \leq \quad & \mid p_R - p_{R-1} \mid + \mid p_{R-1} - p_{R-2} \mid + \cdots + \mid p_{S+1} - p_S \mid \\ \leq \quad & P_R - P_{R-1} + P_{R-1} - P_{R-2} + \cdots + P_{S+1} - P_S \\ = \quad & P_R - P_S. \end{aligned}$$

If $\lim_{n\to\infty} P_N$ exists we have that $\{P_N\}_{n=1}^{\infty}$ is a Cauchy sequence and is therefore convergent. □

With the proceeding theorems in mind, we may proceed throughout the thesis with the understanding that convergence and rearrangement of series are justified, provided the series and products converge absolutely.

The following is the q-binomial Theorem which is needed since we draw identities between products and sums throughout the thesis.

Theorem 14 *If* $|\, q \,| < 1$, *then*

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}z^n}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{n=0}^{\infty}(1+zq^n) \tag{82}$$

and

$$1+\sum_{n=1}^{\infty}\frac{z^n}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1+zq^n)} \tag{83}$$

with $|\, z \,| < 1$ *in (80).*

Proof. We see that infinite product in (80) must have a MacLaurin series expansion in z. So let,

$$f(z)=\prod_{n=0}^{\infty}(1+zq^n),$$

and we let the MacLaurin series expansion for $f(z)$ be

$$f(z)=\sum_{n=0}^{\infty}A_n z^n, \tag{84}$$

where the coefficients $A_n$depend on q. Proceeding, we now have

$$\begin{aligned} f(z) &= \prod_{n=0}^{\infty}(1+zq^n) \\ &= (1+z)\prod_{n=1}^{\infty}(1+zq^n) \\ &= (1+z)\prod_{n=0}^{\infty}(1+zq^{n+1}) \\ &= (1+z)\prod_{n=1}^{\infty}(1+zqq^n) \\ &= (1+z)f(zq). \end{aligned} \tag{85}$$

Substituting (84) into (85), we obtain the following relations

$$\begin{aligned} \sum_{n=0}^{\infty} A_n z^n &= (1+z)\sum_{n=0}^{\infty} A_n z^n q^n \\ &= \sum_{n=0}^{\infty} A_n z^n q^n + \sum_{n=0}^{\infty} A_n z^{n+1} q^n. \end{aligned} \tag{86}$$

From here we see that since A_0 is the constant term of the MacLaurin series, $A_0 = f(0) = 1$. Now, letting $N > 0$ we compare the coefficients of a general power, z^N, in the preceding equation. If we express the coefficient on the left side as A_N, then on the right side we have $A_N q^N + A_{N-1} q^{N-1}$. Since we know a function has a unique MacLaurin series expansion, we consider $f(z)$, we have,

$$A_N = A_N q^N + A_{N-1} q^{N-1}.$$

Therefore,

$$(1-q^N)A_N = A_{N-1} q^{N-1}.$$

Moreover,

$$A_N = \frac{A_{N-1}q^{N-1}}{(1-q^N)}. \tag{87}$$

If we repeat this process, we can express A_N in terms only of q since,

$$\begin{aligned}
A_N &= \frac{q^{N-1}}{(1-q^N)}A_{N-1} \\
&= \frac{q^{N-1}}{(1-q^N)}\frac{q^{N-2}}{(1-q^{N-1})}A_{N-2} \\
&= \frac{q^{N-1}}{(1-q^N)}\frac{q^{N-2}}{(1-q^{N-1})}\frac{q^{N-3}}{(1-q^{N-2})}A_{N-3} \\
&\dots \\
&= \frac{q^{(N-1)+(N-2)+\cdots+2+1}}{(1-q^N)(1-q^{N-1})\dots(1-q)}A_{0.}
\end{aligned}$$

Since the sum of the first $N-1$ positive integers is $(N^2-N)/2$, and $A_0 = 1$,we have the following formula

$$A_N = \frac{q^{(N^2-N)/2}}{(1-q^N)(1-q^{N-1})\dots(1-q)}.$$

Therefore,

$$\begin{aligned}
\prod_{n=0}^{\infty}(1+zq^n) &= f(z) \\
&= \sum_{n=0}^{\infty}A_n z^n \\
&= 1+\frac{q^{(n^2-n)/2}z^n}{(1-q^n)(1-q^{n-1})\dots(1-q)},
\end{aligned}$$

which shows (82)

For (83) the argument is similar. Let

$$g(z) = \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)},$$

and let the MacLaurin expansion for $g(z)$ be

$$g(z) = \sum_{n=0}^{\infty} B_n z^n.$$

Here again the coefficients B_n depend on q. So now,

$$\begin{aligned} g(z) &= \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)} \\ &= \frac{1}{(1-z)} \prod_{n=1}^{\infty} \frac{1}{(1-zq^n)} \\ &= \frac{1}{(1-z)} \prod_{n=0}^{\infty} \frac{1}{(1-zq^{n+1})} \\ &= \frac{1}{(1-z)} \prod_{n=0}^{\infty} \frac{1}{(1-zqq^n)} \\ &= \frac{1}{(1-z)} g(zq) \end{aligned}$$

which gives us

$$(1-z)g(z) = g(zq).$$

So, much like before, let $B_0 = 1$, the constant, term and from

$$(1-z)\sum_{n=0}^{\infty} B_n z^n = \sum_{n=0}^{\infty} B_n z^n q^n,$$

we can see

$$\sum_{n=0}^{\infty} B_n z^n - \sum_{n=0}^{\infty} B_n z^{n+1} = \sum_{n=0}^{\infty} B_n z^n q^n,$$

which implies

$$\sum_{n=0}^{\infty} B_n z^n - \sum_{n=0}^{\infty} B_N z^N q^N = \sum_{n=0}^{\infty} B_n z^n.$$

From here let B_N be, once again, the coefficient in the equations and we have,

$$(1-q^N)B_N = B_{N-1},$$

and we have an expression for B_N as,

$$\begin{aligned} B_N &= \frac{B_{N-1}}{(1-q^N)} \\ &= \frac{B_{N-2}}{(1-q^N)(1-q^{N-1})} \\ &\dots \\ &= \frac{B_0}{(1-q^N)(1-q^{N-1})\dots(1-q)}, \end{aligned}$$

and since $B_0 = 1$, we arrive at, for $n \geq 1$,

$$B_N = \frac{1}{(1-q^N)(1-q^{N-1})\dots(1-q)}.$$

From here since,

$$\sum_{n=0}^{\infty} B_n z^n = \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)},$$

we have,

$$B_0 z^0 + \sum_{n=1}^{\infty} B_n z^n = \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)},$$

and finally,

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)}.$$

□

We use the following Mathematica code to construct the matrices for the operators discussed in Chapter 5.

```
In[49]:= ff[r_,k_,n_]:=Sum[Binomial[k,j]Binomial[3 n - 3 j ,l]
Binomial[3 j,r-l](-1)^(3 j + l - r) 2^l,{j,0,k},{l,0,3 n - 3 j}];

g[r_,k_,n_]:=Sum[Binomial[k,j]Binomial[3((n-k)+j),l]
Binomial[3(k-j),(3n-r)-l](-1)^((3 n-r)-l) 2^(3(n-k)+3j-l)
,{j,0,k},{l,0,3((n-k)+j)}];

ArrM[d_] :=Table[g[r,k,d],{r,0,3*d},{k,0,d}]
ArrrM[d_] :=Table[ff[r,k,d],{r,0,3*d,3},{k,0,d}]

MatrixForm[ArrM[1]]
MatrixForm[ArrM[2]]
MatrixForm[ArrM[3]]
MatrixForm[ArrM[4]]
MatrixForm[ArrrM[1]]
MatrixForm[ArrrM[2]]
MatrixForm[ArrrM[3]]
MatrixForm[ArrrM[4]]
Out[53]//MatrixForm= (1 0
6 9
12 9
8 9


)
```

```
Out[54]//MatrixForm= (1 0 0
12 9 0
60 63 81
160 171 162
240 234 243
192 180 162
64 72 81

)
Out[55]//MatrixForm= (1 0 0 0
18 9 0 0
144 117 81 0
672 657 648 729
2016 2088 2187 2187
4032 4140 4212 4374
5376 5328 5265 5103
4608 4464 4374 4374
2304 2304 2268 2187
512 576 648 729

)
Out[56]//MatrixForm= (1 0 0 0 0
24 9 0 0 0
264 171 81 0 0
1760 1467 1134 729 0
7920 7506 7047 6561 6561
```

```
25344 25488 25758 26244 26244
59136 60480 61965 63423 65610
101376 102816 104004 104976 104976
126720 126144 125388 124659 124659
112640 110592 108864 107163 104976
67584 66816 66096 65610 65610
24576 25344 25920 26244 26244
4096 4608 5184 5832 6561

)
Out[57]//MatrixForm= (1 0
8 9

)
Out[58]//MatrixForm= (1 0 0
160 171 162
64 72 81

)
Out[59]//MatrixForm= (1 0 0 0
672 657 648 729
5376 5328 5265 5103
512 576 648 729

)
Out[60]//MatrixForm= (1 0 0 0 0
```

```
1760 1467 1134 729 0

59136 60480 61965 63423 65610

112640 110592 108864 107163 104976

4096 4608 5184 5832 6561


)
```

We conclude the Appendix with Mathematica code for implementing the recursions in Lemma 5 and Lemma 6.

```
EEC[0] = a/6;

EEC[2] = -b^3/9;

EEC[-2] = 0;

EEC[-4] = 0;


Q[q] = a^4 + 8 a c^3;

R[q] = a^6 - 20 a^3 c^3 - 8 c^6;


d[0] = 3 2 Zeta[4] Q[q];

d[1] = 5 2 Zeta[6] R[q];


EE[n_] := (Zeta[1 - ( n)]/2) d[
  n/2 - 2]/((2 (n/2 - 2) + 3) ((n/2 - 2)!) 2 Zeta[n]);

d[n_] := ((3 (n - 2) + 6)/(2 (n - 2) + 9)) Sum[
  Binomial[n - 2, k] d[k] d[n - 2 - k], {k, 0, n - 2}];


EEC[n_] := - 9 (n - 1) n

EEC[0]^2 EEC[n - 2] -
```

```
 2 (n - 1) n Sum[
   Binomial[n - 2, 2 j]
   EEC[2 j] EE[n - 2 j], {j, 1, (n/2) - 2}];

EE2[n_] := 18 EEC[0] EEC[n - 2] +
 6 Sum[Binomial[n - 2, 2 k]
 EEC[2 k] EEC[n - 2 - 2 k], {k, 1, (n/2) - 2}];
```

BIOGRAPHICAL SKETCH

Andrew Alaniz was born on August fourth 1985. He is the youngest of three boys born to his parents Mr. Daniel Alaniz and Mrs. Sonia Alaniz. He received his Bachelor of Arts degree in Political Science in May 2008 from UTPA, and his Masters of Science degree in Mathematics in August 2013, also from UTPA. For correspondence he may be reached at aoalaniz@icloud.com.