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Pipe Flow of Newtonian and Non-Newtonian Fluids

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PIPE FLOW OF NEWTONIAN AND NON-NEWTONIAN FLUIDS

A Thesis

by

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NON-NEWTONIAN FLUIDS

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ABSTRACT

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We consider an incompressible, viscous fluid in a cylindrical pipe. We obtain velocity profile for both Newtonian fluid and non-Newtonian fluids such as shear-thinning, shear-thickening and Bingham plastic fluids. The flow is governed by the equation of continuity (conservation of mass) and the momentum equation. After presenting the governing system in the cylindrical coordinate system and assuming that the flow is due to the pressure drop and wall shear stress, we derive the expressions for the velocity component in the axial direction for these cases. Some computational results of the velocity profiles for various cases are presented. We will observe that and analyze the velocity profile for Newtonian fluids, as well as for shear-thinning, shear-thickening and Bingham fluids.

DEDICATION

This thesis is dedicated to my lovely family especially my parents Jaime Sanchez and Marisela Garcia. It was their unconditional love and support that motivated me to continue my education. I also dedicate this thesis to my closest friends: Angela, GJ, Nyla, Arturo, Ricky, and Nemo. They always made sure to push and motivate me to succeed in college. It is thanks to them I was able to make it this far. Thank you all.

ACKNOWLEDGMENTS

I would like to express my sincere thanks to the committee members especially my advisor Dr. Dambaru Bhatta. I would like to thank him for his motivation, support, and most importantly, his infinite patience.

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CHAPTER I

INTRODUCTION

Sir Isaac Newton, born January 4th, 1643 and deceased March 31st, 1727. Globally recognized for key developments in various fields, prominently mathematics, astronomy, and physics. These developments include the derivation of the laws of motion and universal gravitation. Apart from these achievements, Newton studied the way fluids behave, observing them and noting different factors, such as viscosity, their consistency, and the way their flow changed depending on their temperature or pressure. He discovered that for most liquids, their flow behavior or viscosity only changes depending exclusively on those two factors. And so, today we refer to the liquids that behave as such as Newtonian fluids.

However, this does not apply to every fluid. There exist fluids in which, assuming constant viscosity, one cannot find a linear relationship between the stress applied to the liquid and rate at which the fluid deforms, in which case we refer to them as Non-Newtonian fluids. There are several different forms of non-Newtonian fluids, some of the most important include shear-thinning (pseudoplastic), shear thickening (dilatant), and Bingham plastics, which we will analyze more in depth on a later chapter.

Some Definitions

In this section, we present the definitions of important terms that will be used throughout this thesis.

Matter*:* Matter is a material substance that occupies space, has mass, and is composed of atoms consisting of protons, neutrons, and electrons. It is classified according to chemical and physical properties. Matter can exist in three different states, solid, liquid, and gas, based on the way the atoms and molecules are arranged inside them. Solids are relatively rigid and have fixed shapes and volumes. In contrast, liquids have fixed volumes but flow to assume the shape of their containers. Gases have neither fixed shapes nor fixed volumes and expand to completely fill their containers. Matter can often change from one physical state to another in a process called a physical change. Water can be heated to form a gas called steam, or steam can be cooled to form liquid water. However, such changes of state do not affect the chemical composition of the substance. It is important to note that both liquids and gases may be classified as fluids.

Body: A body is a portion of a fluid. It has a volume V and a surface A at any time t.

Particle: A particle is a material point in the body. We introduce a reference frame in order to localize particles to describe their motions. Here, we use Cartesian coordinate system O_{xyz} with O as the origin and $O_x,\ O_y,\ O_z$ as three mutually perpendicular axes.

Place: A place **x** in 3-D space is localized by three coordinate values $x,\ y,\ z$, i.e. we use following convention

$$place \quad = \quad (x,\, y,\, z) \equiv \mathbf{x} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \{x\ y\ z\}$$

We use the expression *the place* **x**.

At an arbitrarily chosen reference time t_0 the place of a particle in the fluid body is given by a set of coordinates $X,\, Y,\, Z$. We attach the coordinate set **X** to the particle and use as an identification of the particle.

$$particle = (X,\, Y,\, Z) \equiv \mathbf{X} \equiv \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \equiv \{X\; Y\; Z\}$$

In continuum mechanics, we work with functions of place and time:

$$f = f\,(x, y, z, t) = f\,(\mathbf{x}, t)$$

Some examples are pressure p, density ρ (mass per unit volume), velocity **v** and temperature T. These quantities are either expressed as functions of the particle coordinates coordinates $X,\, Y,\, Z$, and present time t or by the place coordinates $x,\; y,\; z$, and present time t.

Polymer: A polymer is a substance that has a molecular structure that is composed of long chains of repeated units known as monomers that are strung together to yield giant macro-molecules.

Lagrangian and Eulerian Systems.

The four coordinates $(\mathbf{X}, t)$ are called *Lagrangian coordinates.* They are named after Joseph Louis Lagrange [1736-1813]. The four coordinates $(\mathbf{x}, t)$ are called *Eulerian coordinates.* It is named after Leonhard Euler [1707-1783]. A function in Lagrangian coordinates $f\,(\mathbf{X}, t)$ is called a particle function. A function in Eulerian coordinates $f\,(\mathbf{x}, t)$ is called a place function. For the Lagrangian system, we consider one particular particle and study behavior of that particle passing through various places. For Eulerian system, we consider one place and study behavior of various particles passing through that place. In fluid mechanics, it is usually convenient to work with Eulerian coordinates $(\mathbf{x}, t)$. For a particular choice of place function $f\,(\mathbf{x}, t)$ is related to the place **x**. The particle velocity $\mathbf{v}\,(\mathbf{x}, t)$ represents the velocity of the particle **X** passing through the place **x** at time t.

Stress and Strain.

Stress means that a force is applied to a body. The result of that stress is described as Strain. When a load is applied on a body, it deforms. That deformation or change in length per original length is called strain. Newtonian fluids don't resist much stress that is applied on them like solids would do, so they don't show the signs of strain.

Stress is the amount of forces (strength or energy) that are being exerted on an object, divided by its cross-sectional area to account for size. Larger objects are able to withstand higher forces. By using stress instead of just force, we are able to use the same yield stress for the same material, regardless of how large the object actually is.

Strain refers to how much an object deforms when forces are applied to it. Most of the time this deformation will either cause the object to elongate or shorten, depending on how forces are applied. To compute strain, this change is divided by the object's original length, again to account for size. Larger objects will usually have a greater change in length than smaller objects, even though they experience the same forces acting on them.

Formula for Stress is

$$Stress = \frac{Force}{Area} \qquad i.e., \qquad \sigma = \frac{F}{A}$$

where

- σ represents the stress
- F represents the force
- A represents the area

Formula for Strain is

$$Strain = \frac{Change\ in\ Length}{Original\ Length} \qquad i.e., \qquad \gamma = \frac{\triangle L}{L}$$

where

- γ represents the strain
- $\triangle L$ represents the change in length
- L represents the original length

Deformation.

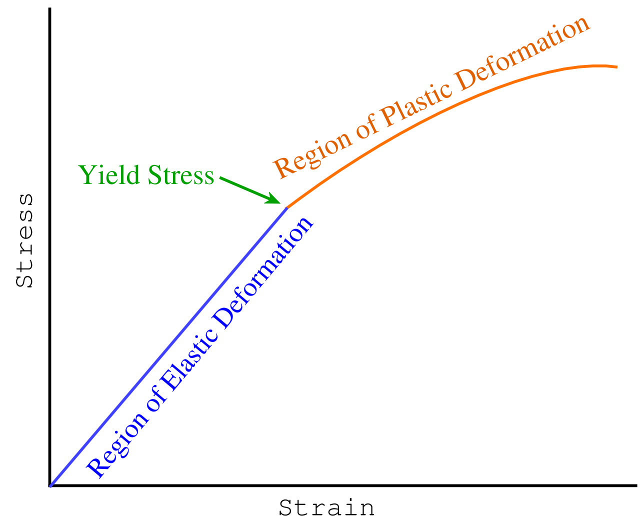
There are two main different types of deformation that we will consider: elastic and plastic

Elastic deformation: Elastic deformation occurs when stress is applied to an object and the deformation will automatically reverse itself when the external forces are removed.

Plastic deformation: Plastic deformation is a permanent deformation. To reverse it, an additional external force needs to be applied to return the object to its original shape.

Yield Stress: Stress and strain are directly related to each other: as one increases, the other increases as well. So, the more stress that an object experiences, the more it deforms until the object fails. All objects will begin experiencing elastic deformation at first, but once the stress on the object exceeds a certain amount, it will experience plastic deformation. When that switch happens, the object has reached its yield stress. Typically, every material has the same stress-strain relationship, though the size of each portion may be different. Elastic deformation is linear. The slope of the line is dependent on the material the object is made out of. Plastic deformation is not linear, making it more difficult to model. In materials science and engineering, the yield point is the point on a stress-strain curve that indicates the limit of elastic behavior and the beginning of plastic behavior. Prior to the yield point, a material will deform elastically and will return to its original shape when the applied stress is removed. We can see these concepts represented in Figure 1.

Figure 1: Stress and Strain



CHAPTER II

NEWTONIAN AND NON-NEWTONIAN FLUIDS

In this chapter, we compare between Newtonian and non-Newtonian fluids, noting the main differences and what defines each of them. We will also provide some examples of each type.

Newtonian Fluids

As noted by their given name, one of the first persons to study Newtonian fluids was Sir Isaac Newton. In addition to his many other discoveries, he did some revolutionary work with fluids. He discovered that the viscosity of most fluids is only affected by temperature or pressure. Viscosity is a measure of a fluid's ability or resistance resistance to flow. A fluid that resists flow is said to have a higher viscosity, or to be more viscous, than a fluid a that flows more easily. i.e., the more viscous a substance is, the longer it will take to pour than a less viscous fluid.

Furthermore, a Newtonian fluid is one that is defined by a linear relationship between the shear stress and the shear rate. Essentially, in a plot of shear stress versus shear rate, assuming a constant temperature, we obtain a constant slope that is completely independent of the shear rate. Such slope is what we refer to as the viscosity of the fluid. It is important to note that every gas is Newtonian.

Most of everyday fluids, such as water and oil, are also considered to be Newtonian. Their viscosity is mostly only affected by their temperature, therefore, if their temperature doesn't change, neither does their viscosity. Behavior of Newtonian fluids like water can be described exclusively by temperature and pressure.

Non-Newtonian Fluids

Unlike Newtonian fluids, the viscosity of some fluids is affected by factors other than temperature. Therefore, we refer to these fluids as non-Newtonian fluids. Non-Newtonian fluids are polymers.

Apart from temperature, the viscosity of a non-Newtonian fluid may depend also on the pressure applied to such fluid, also known as shear stress. As mentioned in the previous section, a Newtonian fluid's viscosity will not be affected by shear rate. When is comes to non-Newtonian fluids, shear stress and the shear rate follow a non-linear relationship. In non-Newtonian fluids, viscosity can either increase or decrease if a sheer stress is applied to it, i.e., it can either become more solid or more liquid depending on the type of non-Newtonian fluid it is. And so, a non-Newtonian fluid is a fluid whose viscosity is variable based on applied stress or force acting on it from second to second.

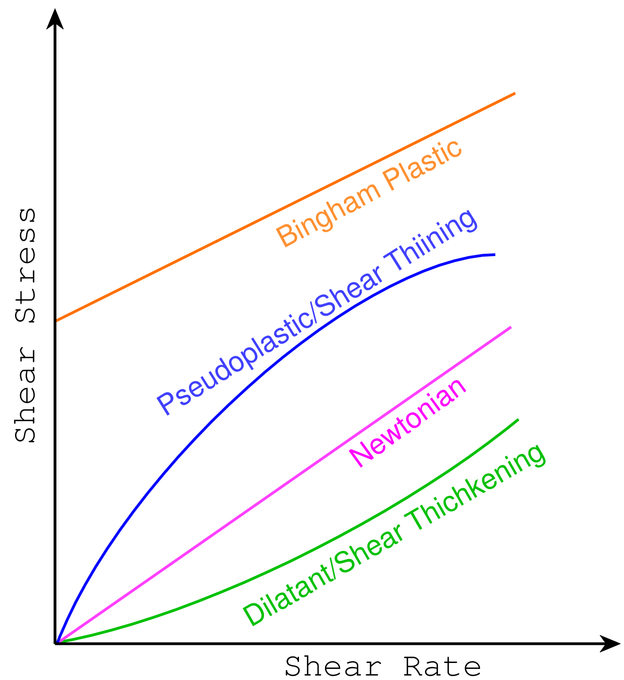
Non-Newtonian fluids are divided into various other subcategories. The main types of Non-Newtonian fluids that we will discuss are shear-thinning, or pseudoplastic; shear-thickening, or dilatant; and Bingham plastics. Fluids whose viscosity decreases when shear pressure is applied are known as shear-thinning fluids. Examples include blood, toothpaste, or tomato sauce. On the other hand, there are other types of fluids for which the effect of shear stress is exactly the opposite. If a shear stress is applied, their viscosity increases. The way we refer to these fluids is shear-thickening. A good example is a mixture of corn starch and water; since, when squeezed, it becomes significantly more dense, which results in the fluid appearing to solidify.

The reason this happens, is because when a stress is applied slowly to a shear-thickening fluid, the polymer chains are able to move out and rearrange themselves, leading to viscosity being mostly unaffected. However, if a higher stress acts or is physically applied to the fluid, the polymer chains do not have time to move. Instead of rearranging, the polymers intertwine themselves and result in the fluid becoming more solid-like, as the viscosity greatly increases.

Different types of non-Newtonian fluids.

As previously mentioned, not every non-Newtonian fluid behaves in the same way when stress is applied – some become more solid, others more fluid. Some non-Newtonian fluids react as a result of the amount of stress applied, while others react as a result of the length of time that stress is applied. We can see Newtonian fluids and the different types of non-Newtonian fluids in figure 2.

Figure 2: Newtonian and Non-Newtonian Flow Curves

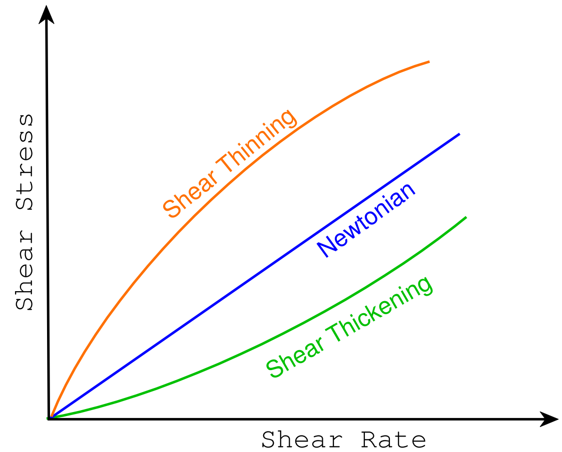


Thixotropic and Rheopectic fluids: Some fluids behave differently when shear stress is applied over time. Rheopectic fluids refer to those that become more viscous as a shear stress is applied over time. Thixotropic fluids are those on the opposite end of the spectrum, for which their viscosity decreases as as a shear stress is applied over time.

Shear thinning and shear thickening fluids: Some fluids behave differently when stress is applied altogether. Shear thickening liquids increase in viscosity as stress increases. Shear thinning liquids decrease in viscosity as stress increases.

We can see in Figure 3 the relationship of shear rate versus shear stress of Newtonian, shear thinning and shear thickening fluids.

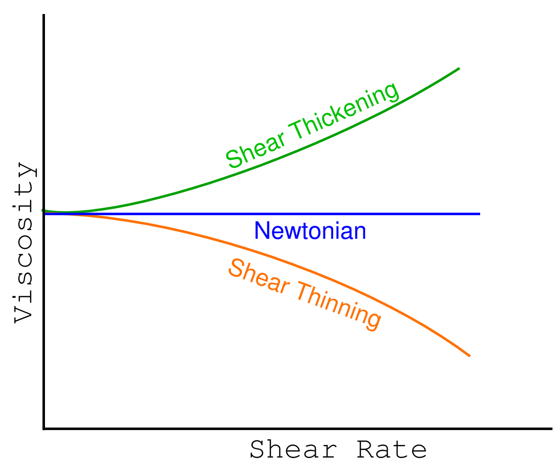
Figure 3: Shear Thinning and Shear Thickening



Shear rate and viscosity

In Figure 4 we can see the relationship of shear rate versus density of Newtonian, shear thinning and shear thickening fluids.

Figure 4: Shear rate and viscosity



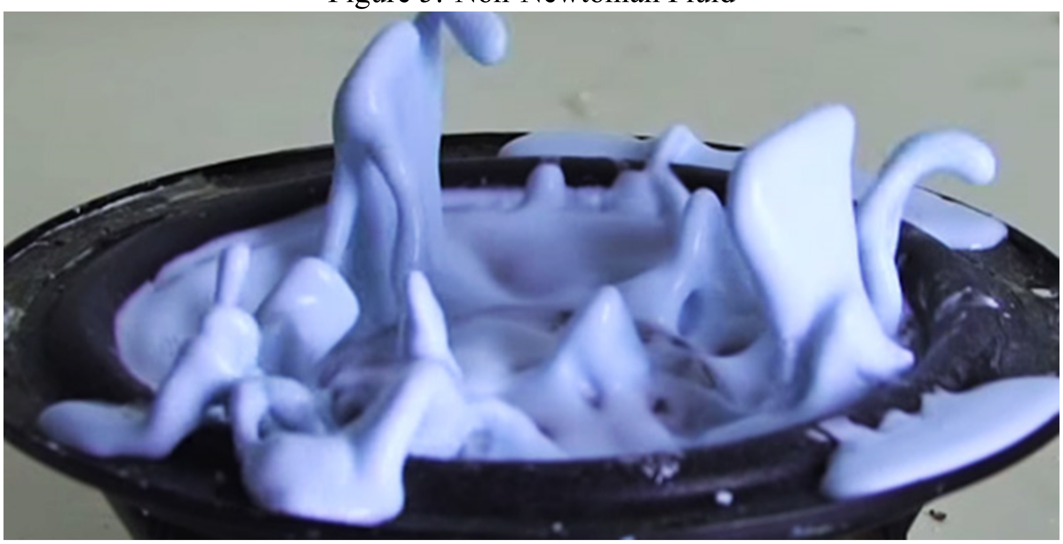
Examples of substances displaying Newtonian and non-Newtonian fluid characteristics.

Here we provide a list of some examples of fluids that display Newtonian and non-Newtonian properties and their classifications.

Table 1: Fluid examples

Newtonian	Water, oil, gases
Bingham Plastic	Mayonnaise, mustard, chocolate
Pseudoplastic	Blood, toothpaste, tomato sauce
Dilatant	Cornstarch with water, wet sand

Figure 5: Non-Newtonian Fluid



CHAPTER III

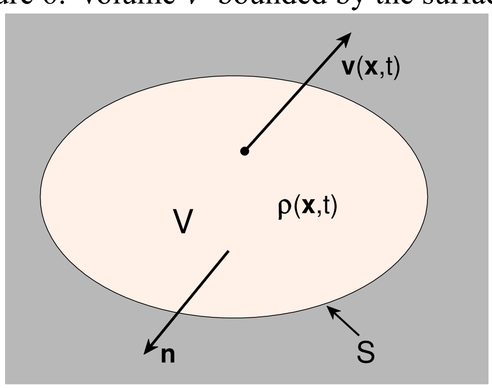
DERIVATION OF MATHEMATICAL EQUATIONS

Main Equations

Equation of Continuity.

We consider a fluid occupying a volume V enclosed by a surface S. Let $\mathbf{v}(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$, respectively, denote the velocity and density at a point in the fluid. We take **n** is the outward normal on S.

Figure 6: Volume V bounded by the surface S



The mass of the fluid is given by

$$M = \int_V \rho(\mathbf{x}, t)\, dV$$

Now, considering the rate of change, we can write

$$\begin{aligned}
\frac{dM}{dt} &= \frac{d}{dt}\int_V \rho\left(\mathbf{x},\, t\right) dV \\
&= \lim_{\triangle t \to 0} \frac{\int_{V+\triangle V} \rho\left(\mathbf{x}, t+\triangle t\right) dV - \int_V \rho\left(\mathbf{x}, t\right) dV}{\triangle t}
\end{aligned}$$

Now, we have

$$\int_{V+\triangle V} \rho\left(\mathbf{x}, t+\triangle t\right) dV = \int_V \rho\left(\mathbf{x},\, t+\triangle t\right) dV + \int_{\triangle V} \rho\left(\mathbf{x},\, t+\triangle t\right) dV$$

But, using Taylor's expansion, we write

$$\int_V \rho\left(\mathbf{x}, t+\triangle t\right) dV = \int_V \rho\left(\mathbf{x}, t\right) dV + \int_V \frac{\partial \rho}{\partial t} \triangle t \, dV + O\left((\triangle t)^2\right)$$

and

$$\int_{\triangle V} \rho\left(\mathbf{x}, t+\triangle t\right) dV = \int_{\triangle V} \rho\left(\mathbf{x}, t\right) dV + \int_{\triangle V} \frac{\partial \rho}{\partial t} \triangle t \, dV + O\left((\triangle t)^2\right)$$

Hence,

$$\begin{aligned}
&\int_{V+\triangle V} \rho\left(\mathbf{x}, t+\triangle t\right) dV - \int_V \rho\left(\mathbf{x},\, t\right) dV \\
&\qquad = \int_V \frac{\partial \rho}{\partial t} \triangle t \, dV + \int_{\triangle V} \rho\left(\mathbf{x}, t\right) dV + \int_{\triangle V} \frac{\partial \rho}{\partial t} \triangle t \, dV + O\left((\triangle t)^2\right)
\end{aligned}$$

which yields us

$$\begin{aligned}
&\frac{\int_{V+\triangle V} \rho\left(\mathbf{x}, t+\triangle t\right) dV - \int_V \rho\left(\mathbf{x}, t\right) dV}{\triangle t} \\
&= \int_V \frac{\partial \rho}{\partial t}\, dV + \frac{\int_{\triangle V} \rho\, dV}{\triangle t} \\
&\quad + \int_{\triangle V} \frac{\partial \rho}{\partial t}\, dV + O\left(\triangle t\right)
\end{aligned}$$

As $\triangle t \to 0$, $\triangle V \to 0$. and so the last to terms of the above expression are zero. And for the second integral, $\triangle V = \mathbf{v} \cdot \mathbf{n}\, \triangle t\, \triangle S$, so we have

$$\begin{aligned}
\lim_{\triangle t \to 0} \frac{\int_{\triangle V} \rho\, dV}{\triangle t} &= \int_S (\rho \mathbf{v}) \cdot \mathbf{n}\, dS \\
&= \int_V div(\rho \mathbf{v})\, dV
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\frac{dM}{dt} &= \lim_{\triangle t \to 0} \frac{\int_{V+\triangle V} \rho\left(\mathbf{x}, t+\triangle t\right) dV - \int_V \rho\left(\mathbf{x}, t\right) dV}{\triangle t} \\
&= \int_V \frac{\partial \rho}{\partial t}\, dV + \int_V div(\rho \mathbf{v})\, dV \\
&= \int_V \left[\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v})\right] dV
\end{aligned}$$

Since there is no source or sink (no change in mass), we have

$$\frac{dM}{dt} = 0$$

i.e.,

$$\int_V \left[\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v})\right] dV = 0$$

But V is any arbitrary volume, thus, we must have

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0$$

This is known as the Equation of Continuity or Mass Conservation Equation.

Since

$$\begin{aligned}\nabla \cdot (\rho \mathbf{v}) &= (\nabla \rho) \cdot \mathbf{v} + \rho (\nabla \cdot \mathbf{v}) \\ &= \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}\end{aligned}$$

continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0$$
$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

Continuity Equation for Incompressible Fluid.

If the fluid is incompressible, density is treated as constant, in that case, the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

becomes

$$\nabla \cdot \mathbf{v} = 0$$

Continuity Equation in Cartesian Coordinate System

Let

$$\mathbf{v} \;=\; (u,\, v,\, w)$$

where $u,\ v,\ w$ represent the $x,\ y,\ z$ components of the velocity $\mathbf{v}(x, y, z, t)$.

In 3-D Cartesian co-ordinates, we have

$$\nabla \;=\; \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

so the continuity equation for incompressible fluid is

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Continuity Equation in Cylindrical Coordinate System

Let $u,\ v,\ w$ represent the $r,\ \theta,\ z$ components of the velocity $\mathbf{v}(r, \theta, z, t)$.

In Cylindrical co-ordinates, we have

$$\nabla \;=\; \left(\frac{\partial}{\partial r}, \frac{\partial}{r\partial \theta}, \frac{\partial}{\partial z}\right)$$

so the continuity equation for incompressible fluid is

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

Continuity Equation in Spherical Co-ordinate System

Let $u,\ v,\ w$ represent the $r,\ \theta,\ \varphi$ components of the velocity $\mathbf{v}(r, \theta, \varphi, t)$. In Spherical co-ordinates, we have

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{r\partial\theta}, \frac{\partial}{r\partial\varphi}\right)$$

so the continuity equation for incompressible fluid is

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \varphi} = 0$$

Momentum Equation (Conservation of Momentum)

Law of mechanics states that mass times acceleration is equal to the sum of forces that act on a volume unit. Total acceleration is composed of the local and the convective acceleration:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v}$$

The momentum equation is

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v}\right] = \mathbf{F} + \nabla \cdot \sigma$$

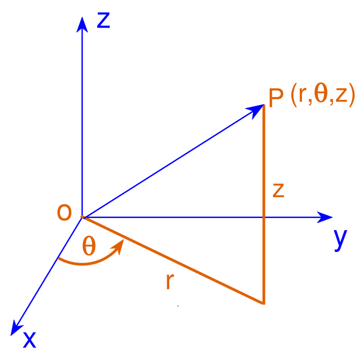
where **F** represents external force (like gravity) and σ is the stress tensor.

Consider Cartesian coordinates (x, y, z). Writing the velocity as $\mathbf{v} = (u, v, w)$ and external force as $\mathbf{f} = (f_x, f_y, f_z)$, momentum equations becomes

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial P}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+\rho f_x$$
$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial P}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+\rho f_y$$
$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial P}{\partial z}+\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+\rho f_z$$

Consider cylindrical coordinates (r, θ, z) as shown in Figure 7.

Figure 7: Cylindrical Coordinate System



writing the velocity as $\mathbf{v} = (v_r, v_\theta, v_z)$ and external force as $\mathbf{f} = (f_r, f_\theta, f_z)$, momentum equations becomes

$$\rho\left[\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}-\frac{v_\theta^2}{r}\right]$$
$$=-\frac{\partial P}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right)+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}-\frac{1}{r}\tau_{\theta\theta}+\rho f_r \quad (1)$$

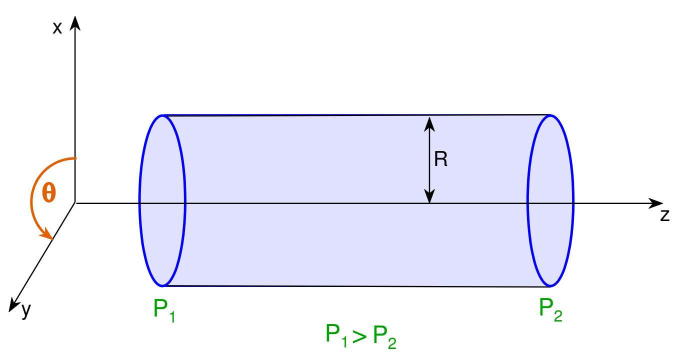
$$
\begin{aligned}
&\rho\left[\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right]\\
&\qquad=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{\theta r}\right)+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\rho f_\theta
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\rho\left[\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right]\\
&\qquad=-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{zr}\right)+\frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}+\rho f_z
\end{aligned}
\tag{3}
$$

Cylindrical Pipe

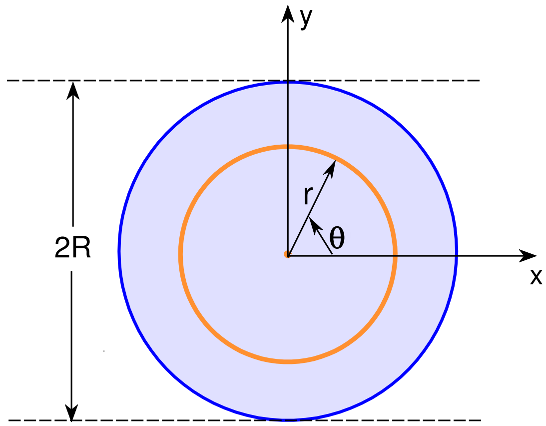
We consider a liquid flowing through a cylindrical pipe of constant radius R. A geometrical sketch in cylindrical coordinates (r, θ, z) is shown in Figure 8. The flow is driven by the pressure gradient, i.e., there is flow from left to right due to pressure difference as $P_1 > P_2$. Our aim is to obtain the velocity profile over a cross section of the pipe and the volumetric flow through the pipe. We will consider Newtonian and non-Newtonian fluids.

Figure 8: Cylindrical Pipe



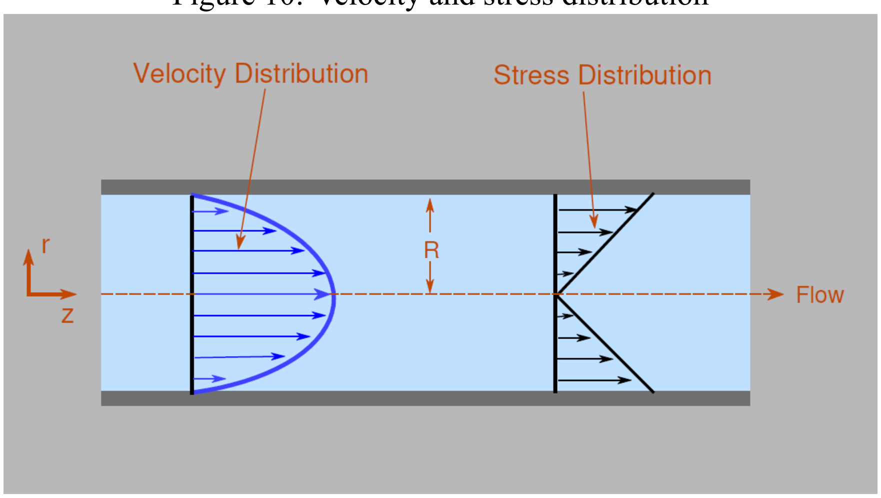
A cross section of the cylinder is displayed in Figure 9.

Figure 9: Cross Section of the Cylinder



Velocity and stress distribution is shown in Figure 10.

Figure 10: Velocity and stress distribution



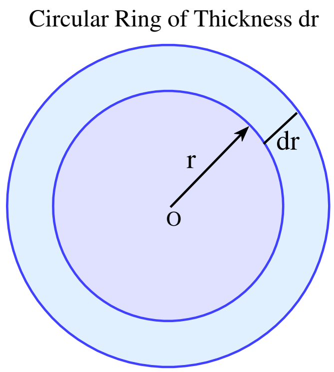
We consider a steady laminar flow with velocity $\mathbf{v}(r, \theta, z)$ with $\mathbf{v} = (v_r, v_\theta, v_z)$. Here v_r denotes the velocity component in radial direction, r; v_θ denotes the velocity component in θ direction, θ; v_z denotes the velocity component in the cylinder axis direction, z. Thus, we have

$$v_z = v_z(r), \quad v_r = 0, \quad v_\theta = 0, \quad v_z(R) = 0 \tag{4}$$

For volumetric flow, we consider the volume of fluid flowing through the ring element dA (shown in Figure 11).

$$dQ = v(r)\,dA = v(r) \times 2\pi r\,dr$$

Figure 11: Ring with area dA and thickness dr



Thus, integrating we can write

$$Q = 2\pi \int_0^R r\,v(r)\,dr \tag{5}$$

Due to the symmetry of the flow,

$$\tau_{\theta z} = \tau_{z\theta} = 0, \quad \tau_{r\theta} = \tau_{\theta r} = 0. \tag{6}$$

Since the velocity is a function of r only, the stresses are functions of r only. So, we have

$$\tau_{rr}(r), \;\; \tau_{\theta\theta}(r), \;\; \tau_{zz}(r), \;\; \tau_{zr}(r) = \tau_{rz}(r). \tag{7}$$

A cylindrical fluid body with radius r and length dz, and subject to stresses. Since τ_{zz} is independent of θ and z, i.e., it is just a function of the independent variable r.Thus, Euler's axiom allows us to write the equilibrium equation for the fluid body as

$$\tau_{rz} \times (2\pi r\,dz) \quad = \quad \left(\pi r^2\right) \frac{\partial P}{\partial z}\,dz$$

i.e.,

$$\tau_{rz} = -\frac{rK}{2} \tag{8}$$

For steady state case, using (1)-(3) and (6), (7), we obtain the following

$$-\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) - \frac{1}{r}\tau_{\theta\theta} = 0 \tag{9}$$

and

$$-\frac{1}{r}\frac{\partial P}{\partial \theta} = 0 \tag{10}$$

$$-\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) = 0 \tag{11}$$

Since flow in the positive z-direction due to pressure gradient and pressure is decreasing from left to right of the cylinder, we take

$$\frac{\partial P}{\partial z} = -K$$

where K is a constant. Equation (10) implies $P = P\left(r, z\right)$

Thus, from (11), we get

$$\frac{d}{dr}\left[r\,\tau_{rz}(r)\right] + rK = 0 \tag{12}$$

Integration yields

$$r\,\tau_{rz} = -\frac{r^2 K}{2} + C_1$$

where C_1 is a constant of integration.

The symmetry condition

$$\tau_{rz}(0) = 0$$

gives us $C_1 = 0$. Thus, we have

$$\tau_{rz} = -\frac{rK}{2} \tag{13}$$

Result given in (13) can be used to obtain the velocity profile $v(r)$, i.e., $\mathbf{v} = (v(r),\, 0,\, 0)$ and the volumetric flow Q through the pipe.

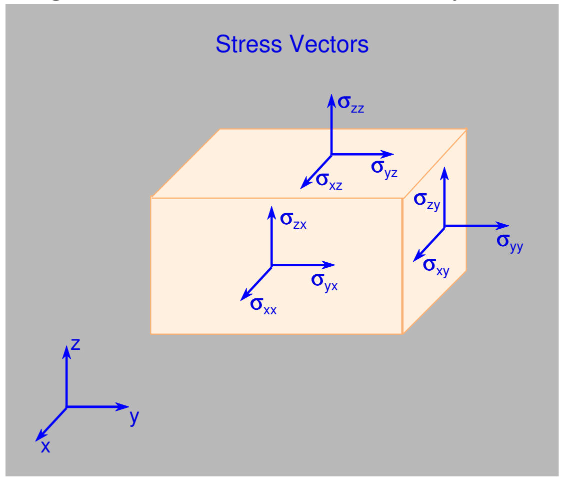
Navier Stokes Equation.

Stress tensor can be expressed as

$$\begin{aligned}
\sigma &= \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \\
&= -\begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \\
&= -P\mathbf{I} + \mathbf{T}
\end{aligned}$$

Stress vectors are shown in Figure 12.

Figure 12: Cross Section of the Cylinder



Thus, we have

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F} - \nabla P + \nabla \cdot \mathbf{T}$$

which is known as the Navier-Stokes Equation.

Simplified Version For Newtonian Fluid

For Newtonian fluid, the stress is proportional to the rate of deformation (the change in velocity in the directions of the stress), i.e.,

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F} - \nabla P + \nabla \cdot \mathbf{T}$$

which is known as the Navier-Stokes Equation.

$$\tau_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$

The proportionality constant μ is called the viscosity (kinematic) of the fluid. Viscosity defines how easily the fluid flows when subjected to body forces.

$$
\begin{aligned}
\nabla \cdot \mathbf{T} &= \mu \nabla \cdot \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{bmatrix} \\
&= \mu \begin{bmatrix} \frac{\partial}{\partial x}\left(2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(2\frac{\partial w}{\partial z}\right) \end{bmatrix}'
\end{aligned}
$$

Here $'$ denotes the transpose. Assuming incompressibility, we have

$$
\begin{aligned}
\nabla \cdot \mathbf{T} &= \mu \begin{bmatrix} \nabla^2 u + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ \nabla^2 v + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ \nabla^2 w + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \end{bmatrix}' = \mu\left(\nabla^2 u, \nabla^2 v, \nabla^2 w\right) \\
&= \mu \nabla^2 (u,\, v,\, w) = \mu \nabla^2 \mathbf{v}
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] &= -\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \rho f_x \\
\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] &= -\frac{\partial P}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \rho f_y \\
\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] &= -\frac{\partial P}{\partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \rho f_z
\end{aligned}
$$

Finally, momentum equation becomes

$$
\rho\frac{D\mathbf{v}}{Dt} = \mathbf{F} - \nabla P + \mu \nabla^2 \mathbf{v}
$$

Velocity Profile for Newtonian Fluid

Here we consider the flow of a Newtonian fluid through the cylinder. Deformation during a time increment dt of a small plane fluid element is shown in The shear rate $\frac{d\gamma}{dt}$ can be derived as

$$\frac{d\gamma}{dt} \equiv \frac{d}{dt}(\gamma_{rz}) = \frac{dv}{dr}$$

This is the only non-zero deformation rate. Here we can obtain

$$\tau_{rz} = \tau_{zr} = \mu\frac{dv}{dr} \tag{14}$$

and

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = 0$$
$$\tau_{r\theta} = \tau_{\theta r} = \tau_{\theta z} = \tau_{z\theta} = 0 \tag{15}$$

Now (13) and (14) yield us

$$\frac{dv}{dr} = -\frac{rK}{2\mu}$$

Integrating with respect to r, we have

$$v(r) = -\frac{r^2K}{4\mu} + C_2.$$

Here C_2 is a constant of integration. The sticking condition at the cylinder boundary

$$v(R) = 0$$

allows us to evaluate C_2 as

$$C_2 = \frac{R^2K}{4\mu}$$

Thus, finally we can express $v(r)$ as

$$\begin{aligned} v(r) &= \frac{R^2K}{4\mu} - \frac{r^2K}{4\mu} \\ &= \frac{K}{4\mu}\left(R^2 - r^2\right) \end{aligned}$$

Using the notation, v_0 for velocity at the center, i.e., at $r = 0$, we get $v_0 = \frac{R^2 K}{4\mu}$. Hence, we get the velocity profile for Newtonian fluid as

$$v(r) = v_0 \left[1 - \left(\frac{r}{R}\right)^2\right], \tag{16}$$

with

$$v_0 = \frac{R^2 K}{4\mu}. \tag{17}$$

Non-Newtonian fluids

Here we consider the flow of two non-Newtonian fluids through the cylinder:

- Power Law Fluid
- Bingham Fluid

Power Law Fluid

Here we can have

$$\tau_{rz} = \tau_{zr} = \eta \frac{dv}{dr} \tag{18}$$

where

$$\eta = \mu \left|\frac{dv}{dr}\right|^{n-1}$$

For Newtonian, $n = 1$, which yields $\eta = \mu$.

Also,

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{r\theta} = \tau_{\theta r} = 0$$

Now (13) and (18) yield us

$$\eta \frac{dv}{dr} = -\frac{rK}{2}$$

Thus, we obtain

$$\mu \left| \frac{dv}{dr} \right|^{n-1} \frac{dv}{dr} = -\frac{rK}{2}$$

i.e.,

$$\frac{dv}{dr} = -\left[\frac{rK}{2\mu} \right]^{1/n}$$

Integrating with respect to r, we get

$$v(r) = -\left[\frac{K}{2\mu} \right]^{1/n} \times \frac{r^{1+1/n}}{1+1/n} + C_3.$$

Here C_3 is a constant of integration. The sticking condition at the cylinder boundary

$$v(R) = 0$$

allows us to evaluate C_3 as

$$C_3 = \left[\frac{K}{2\mu} \right]^{1/n} \times \frac{R^{1+1/n}}{1+1/n}$$

Thus, finally we can express $v(r)$ as

$$\begin{aligned} v(r) &= \left[\frac{K}{2\mu} \right]^{1/n} \times \frac{R^{1+1/n}}{1+1/n} - \left[\frac{K}{2\mu} \right]^{1/n} \times \frac{r^{1+1/n}}{1+1/n} \\ &= \frac{n}{n+1} \left[\frac{K}{2\mu} \right]^{1/n} \left(R^{1+1/n} - r^{1+1/n} \right) \end{aligned}$$

Using the notation v_0 for velocity at the center, i.e., at $r = 0$, we get $v_0 = \frac{n}{n+1}\left[\frac{K}{2\mu}\right]^{1/n} R^{1+1/n}$. Hence, we get the velocity profile as

$$v(r) = v_0\left[1 - \left(\frac{r}{R}\right)^{1+1/n}\right], \tag{19}$$

with

$$v_0 = \frac{nR^{1+1/n}}{n+1}\left[\frac{K}{2\mu}\right]^{1/n}. \tag{20}$$

Bingham Fluid

Here we can have

$$\tau_{rz} = \tau_{zr} = \left[\mu + \frac{\tau_y}{\left|\frac{dv}{dr}\right|}\right]\frac{dv}{dr} \qquad when\ \frac{dv}{dr} \neq 0, \tag{21}$$

and

$$|\tau_{rz}| = |\tau_{zr}| \leq \tau_y \qquad when\ \frac{dv}{dr} = 0, \tag{22}$$

where τ_y represents yield shear stress.

For Newtonian, $\tau_y = 0$. Also,

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{r\theta} = \tau_{\theta r} = 0$$

From the equilibrium equation,

$$|\tau_{zr}| \leq \tau_y \quad when \quad r \leq r_b = \frac{2\tau_y}{K}$$

This implies that inside a cylindrical surface of radius r_b the material flows like solid plug. Now (13) and (21) yield us

$$\frac{dv}{dr} = \frac{\tau_y}{\mu} - \frac{rK}{2\mu} \tag{23}$$

Integrating (23) with respect to r, for $r_b \le r \le R$, we get

$$v(r) = \frac{\tau_y}{\mu} r - \frac{r^2 K}{4\mu} + C_4$$

Here C_4 is a constant of integration. The sticking condition at the cylinder boundary

$$v(R) = 0$$

allows us to evaluate C_4 as

$$C_4 = \frac{R^2 K}{4\mu} - \frac{\tau_y R}{\mu}$$

Thus, finally we can express $v(r)$ as

$$\begin{aligned} v(r) &= \frac{K}{4\mu}\left[R^2 - r^2\right] - \frac{\tau_y}{\mu}\left[R - r\right] \\ &= \frac{KR^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right] - \frac{\tau_y R}{\mu}\left[1 - \frac{r}{R}\right] \end{aligned} \tag{24}$$

When $\tau_y = 0$, we obtain the results same as Newtonian fluid

$$v(r) = \frac{KR^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]. \tag{25}$$

Using the notation v_b for velocity for the solid plug in the region $r \le r_b$, i.e., putting $r = r_b$ in (24), we get

$$\begin{aligned} v_b &= \frac{K}{4\mu}\left[R^2 - r_b^2\right] - \frac{\tau_y}{\mu}\left[R - r_b\right] \\ &= \frac{KR^2}{4\mu}\left[1 - \left(\frac{r_b}{R}\right)^2 - \frac{4\tau_y}{KR}\left(1 - \frac{r_b}{R}\right)\right] \\ &= \frac{KR^2}{4\mu}\left[1 - \left(\frac{r_b}{R}\right)^2 - \frac{2r_b}{R}\left(1 - \frac{r_b}{R}\right)\right] \\ &= \frac{KR^2}{4\mu}\left[1 - \left(\frac{r_b}{R}\right)^2 - \frac{2r_b}{R} + 2\left(\frac{r_b}{R}\right)^2\right] \\ &= \frac{KR^2}{4\mu}\left(1 - \frac{r_b}{R}\right)^2 \\ &= \frac{KR^2}{4\mu}\left(1 - \frac{2\tau_y}{KR}\right)^2 \end{aligned}$$

Thus, for Bingham fluid,

$$v(r) = \frac{KR^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]-\frac{\tau_y R}{\mu}\left[1-\frac{r}{R}\right], \quad r \geq r_b \tag{26}$$

and

$$v_b = \frac{KR^2}{4\mu}\left(1-\frac{2\tau_y}{KR}\right)^2, \quad 0 \leq r \leq r_b \tag{27}$$

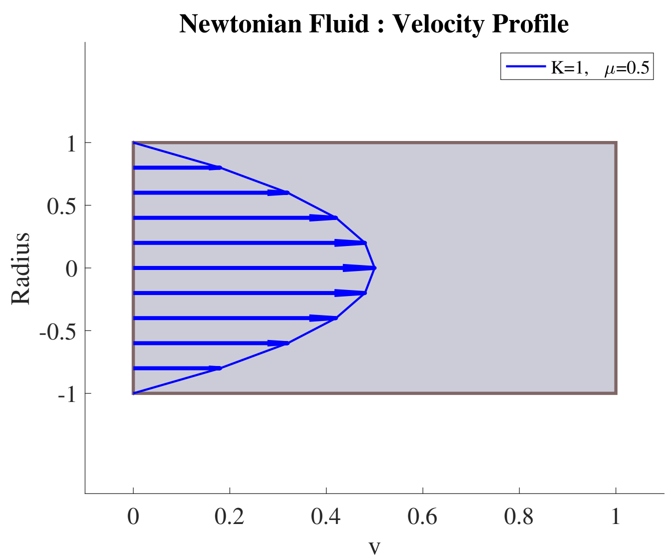
CHAPTER IV

RESULTS AND DISCUSSION

Newtonian Fluid

Velocity profile is displayed in Figure 13.

Figure 13: Velocity profile for Newtonian Fluid



Results for the velocity component for various K is shown in Figure 14.

Figure 14: Velocity for Newtonian Fluid for various K

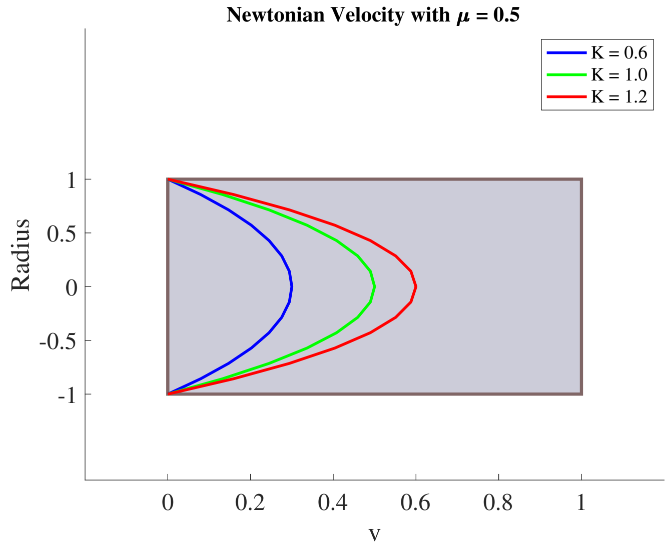
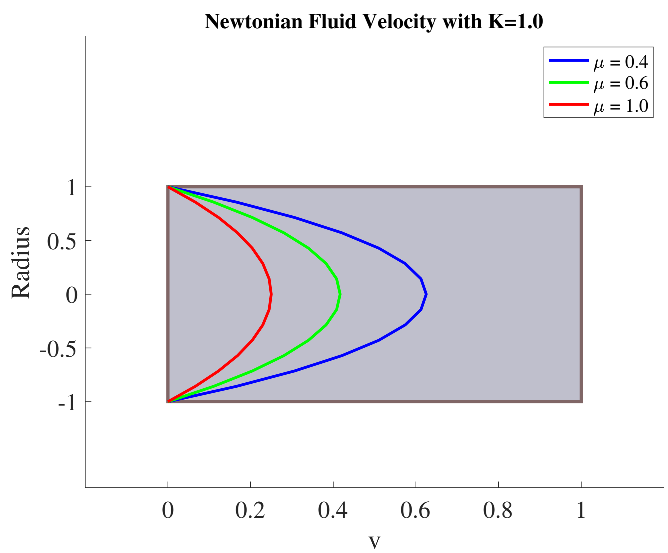


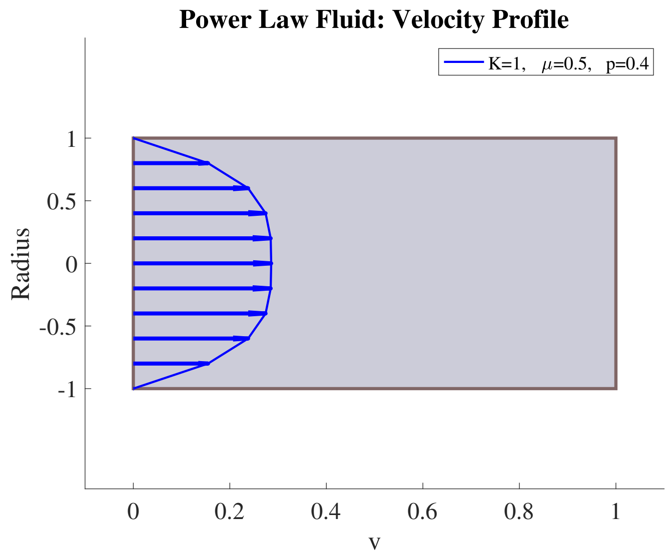
Figure 15 is used to display results for the velocity component for various μ.

Figure 15: Velocity for Newtonian Fluid for various μ



Velocity profile is displayed in Figure 16.

Figure 16: Velocity profile for Power Law Fluid



Results for the velocity component for various K is shown in Figure 17.

Figure 17: Velocity for Power Law Fluid for various K

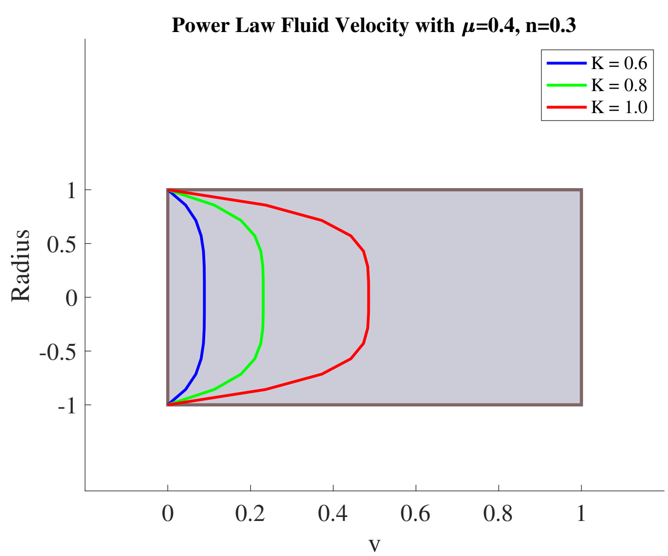
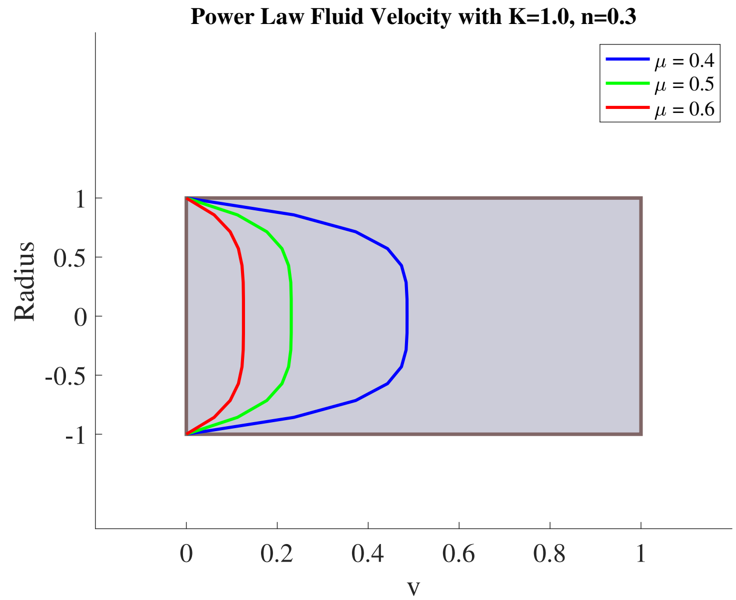


Figure 18 is used to display results for the velocity component for various μ.

Figure 18: Velocity for Power Law Fluid for various μ



Results for the velocity component for various n is shown in Figure 19 and Figure 20.

Figure 19: Velocity for Power Law Fluid for various $n < 1$

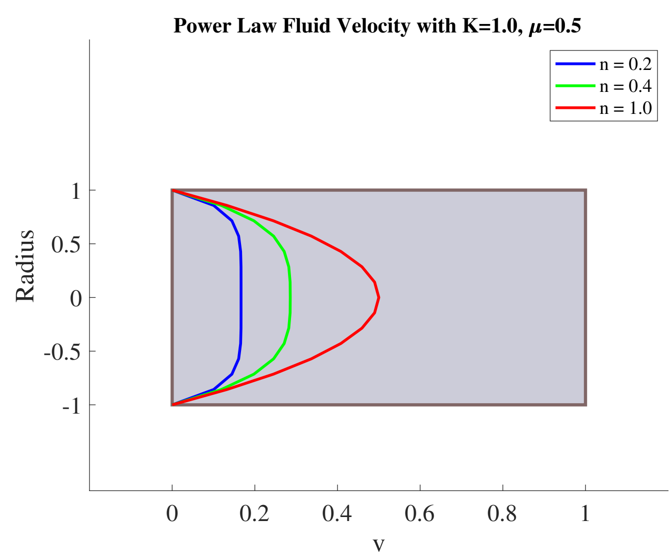
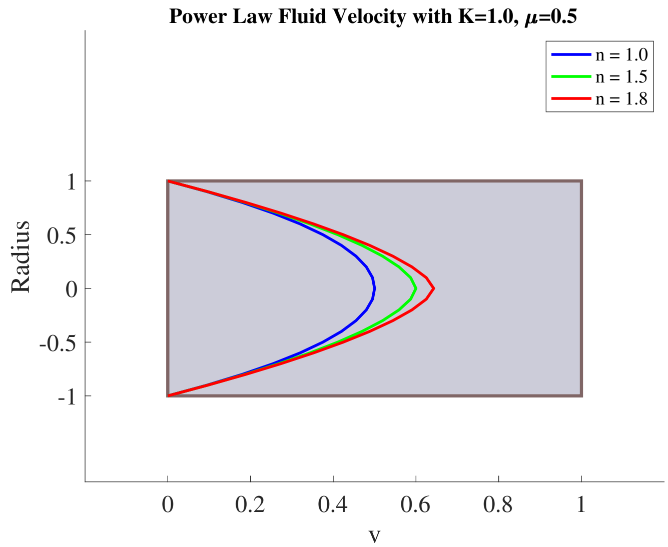


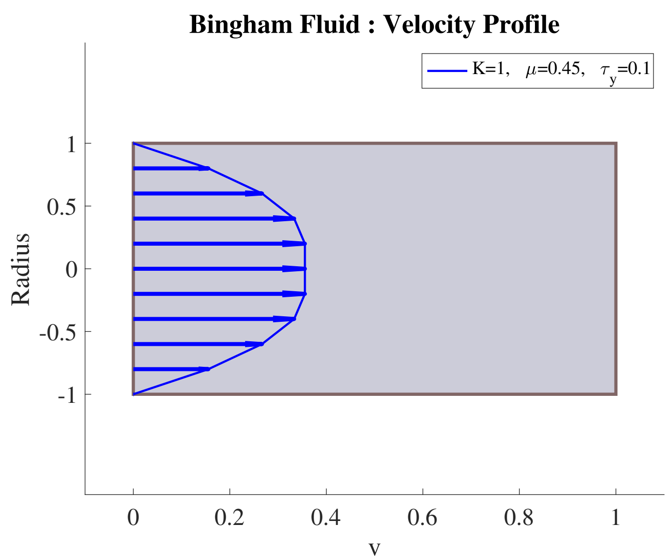
Figure 20: Velocity for Power Law Fluid for various $n > 1$



Non-Newtonian Bingham Fluid

Velocity profile is displayed in Figure 21.

Figure 21: Velocity profile for Bingham Fluid



Results for the velocity component for various K is shown in Figure 22.

Figure 22: Velocity for Bingham Fluid for various K

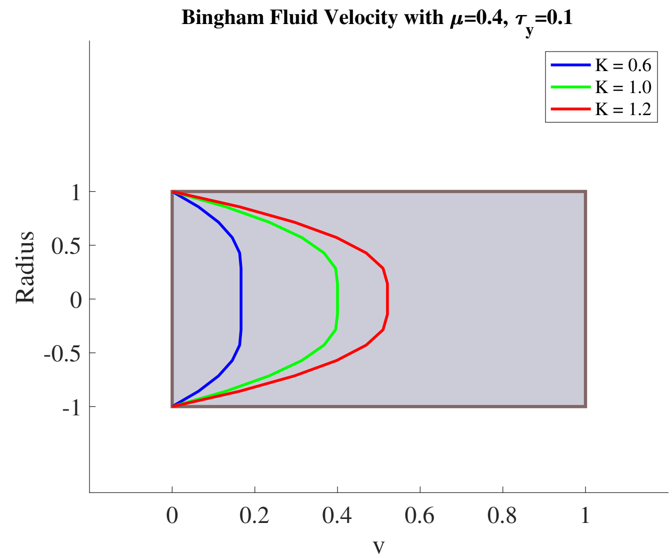
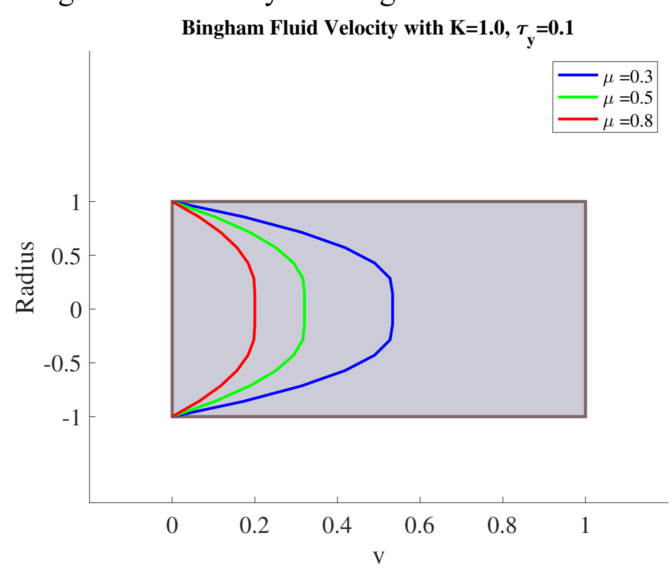


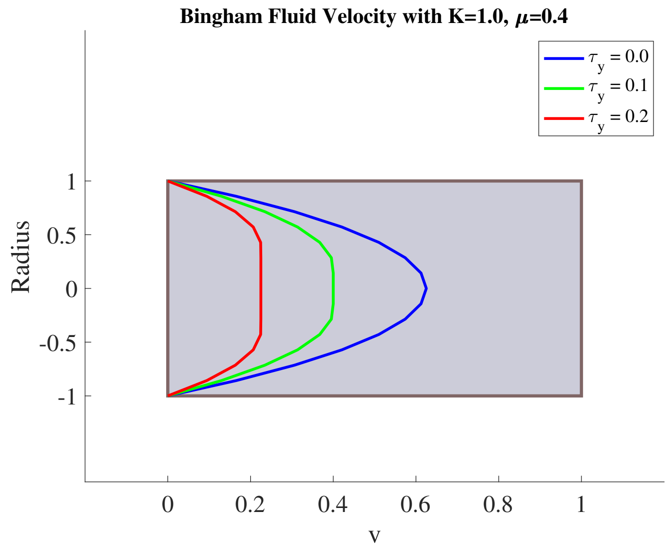
Figure 23 is used to display results for the velocity component for various μ.

Figure 23: Velocity for Bingham Fluid for various μ



Results for the velocity component for various n is shown in Figure 24.

Figure 24: Velocity for Bingham Fluid for various τ_y



CHAPTER V

CONCLUSION

We considered Newtonian and non-Newtonian fluid flow through a cylindrical pipe. We assume that fluid is incompressible and viscous. We obtain velocity profile for both Newtonian fluid and non-Newtonian fluids, specifically shear-thinning, shear- thickening and Bingham plastic fluids. The flow is governed by the equation of continuity (conservation of mass) and the momentum equation. After presenting the governing system in the cylindrical coordinate system and assuming that the flow is due to the pressure drop and wall shear stress, we derived the expressions for the velocity component in the axial direction for these cases. Some computational results of the velocity profiles for various cases were presented. It was observed that velocity profile is parabolic for Newtonian fluid whereas it is flatter for a shear- thinning and sharper for a shear-thickening fluid. For a Bingham fluid, the velocity reaches a constant value known as the plug velocity in the central plug flow region and it decreases gradually to zero at the pipe wall. Furthermore, regardless of the type of fluid, we observed that the higher the pressure gradient, the higher the velocity profile, whereas the higher the viscosity (or apparent viscosity for Non-Newtonian fluids), the lesser the velocity profile.

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BIOGRAPHICAL SKETCH

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