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Using the Differential Fay Identities for ribbon graph enumeration

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USING THE DIFFERENTIAL FAY IDENTITIES
FOR RIBBON GRAPH ENUMERATION

A Thesis

by

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USING THE DIFFERENTIAL FAY IDENTITIES

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ABSTRACT

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In this paper we are exploiting connections between the Differential Fay Identities and the partition function of Gaussian Unitary Random Matrices. The partition function has a asymptotic expression whose terms give the generating function for the enumeration of ribbon graphs partitioned by genus. These ribbon graphs $(k; j)$, of k vertices of degree j have an orientation in genus g where the edges are represented as ribbons. Furthermore, we will develop connections between the Differential Fay Identities and introduce several theorems between the hierarchies of the identities. Lastly, we will briefly explore the first Differential Fay Identity that contains the KP equation.

DEDICATION

I want to first thank God who gave me the strengths and serenity to finish my work with Dr. Pierce. I also want to thank my loving family and wife. Their limitless love and support helped guide me through difficult times. My wife, Roxanna Soto, my mother Patricia Soto, my father Indalecio Soto, my sisters Patricia and Viridiana, my brothers Oziel and Uriel, my nieces Sophia and Ava Soto, my nephews Oziel II and Noah Soto, my pet Mia, I love you guys.

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CHAPTER I

INTRODUCTION

Analytic combinatorics is a branch of mathematics that involves the study of countable discrete structures using methods from complex and asymptotic analysis [2]. In this research, we will explore the connection between the Differential Fay Identities and the partition function of Gaussian Unitary Random Matrices using analytic combinatorics on a generating function that counts *ribbon graphs* by genus. The simple yet intriguing *ribbon graphs* are graphs that consist of vertices, edges, and faces for which the vertices and edges have an orientation on a minimal genus g, and for which the edges around the vertex are also oriented. They can be represented as graphs with vertices as discs and edges as ribbons which do not twist. Our ribbon graphs are considered to be connected and labeled (see Figure 1.1).

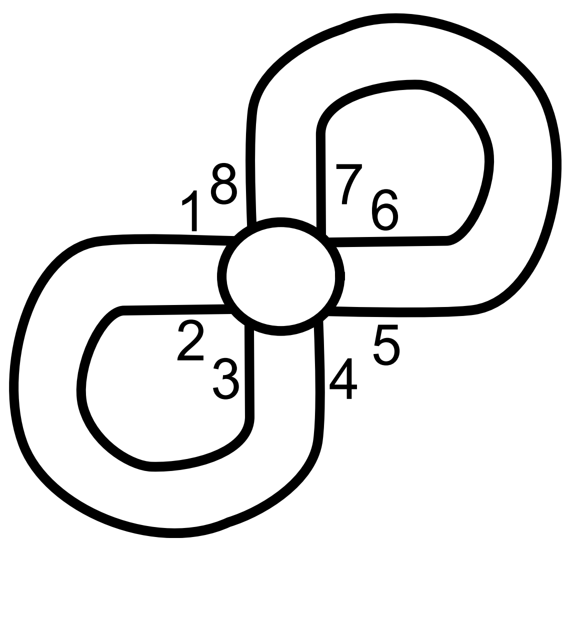

Figure 1.1: Ribbon graph of vertex 1 of degree 4 on a minimal genus 0.

The figure above is an example of a ribbon graph with a vertex of degree 4 with an orientation on a minimal genus 0 surface that we encounter in our research. From the figure above, most of the terminology involving ribbon graphs are self-explanatory except for the minimal genus. The minimal genus of a ribbon graph is the minimal integer g such that the graph can be drawn on without crossing itself on a sphere with g handles, i.e., an oriented surface of genus g (see Figure 1.2 and 1.3).

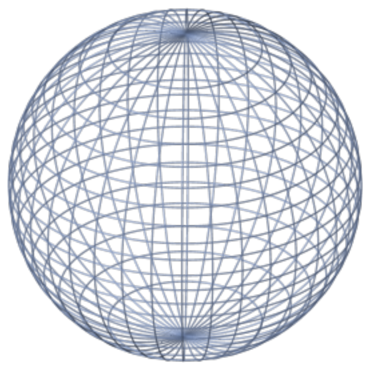

Figure 1.2: Genus 0 is the commonly known, *sphere*.

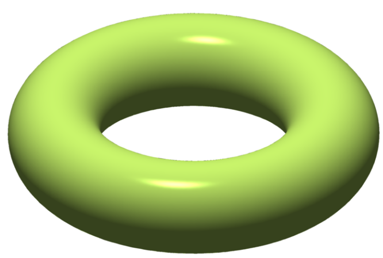

Figure 1.3: Genus 1 is known as *torus*.

We will focus on more complex ribbon graphs that will require the use of *Euler characteristics* to determine their vertices, degrees (also known as edges), and the minimal genus surface they can be drawn on. Furthermore, we will introduce several theorems between the hierarchies derived from the Differential Fay Identities. Finally, we will briefly discuss the KP equation that is hidden within one of the identities.

1.1 Thesis Chapter Outline

Here a basic outline of the thesis chapters is presented.

II:

The objective of the thesis is introduced with some background information. We consider the first identity where we look to make connections between its hierarchies. We then set certain conditions on the identity to uncover new identities within the 1^{st} Differential Fay Identity.

III:

We repeat our asymptotic expansion procedure on the 2^{nd} and 3^{rd} identities with the exception of the conditions implemented on the 1^{st} identity. Then we make several observations and conclude with several propositions involving the hierarchies of the two identities.

CHAPTER II

MATHEMATICAL FORMULATION

The term,

$$Z_N(t_j) = \int_{H_N} \exp\left(-N\mathrm{Tr}\left[V_j(M;t_j)\right]\right) dM \tag{2.1}$$

where

$$V_j(\lambda;t_j) = \frac{1}{2}\lambda^2 + \sum_{j=1}^{2n} t_j \lambda^j \tag{2.2}$$

called the potential function, is integrated over the space of $n \times n$ Hermitian matrices, is a normalization factor and is commonly called the partition function of the corresponding random matrix ensemble [5]. As stated in [5], there have been significant findings in the case where j is even. Our work will be closely associated with the special case where $j = 4$ and where we let $t_4 = t$ for simplicity. Therefore,(2.1) becomes

$$Z_N(t_4) = \int_{H_N} \exp\left(-N\mathrm{Tr}\left[V_4(M;t_4)\right] dM\right) \tag{2.3}$$

or

$$Z_N(t) = \int_{H_N} \exp\left(-N\mathrm{Tr}\left[\frac{1}{2}M^2 + tM^4\right] dM\right), \tag{2.4}$$

which is the partition function of Gaussian Unitary Random Matrices. Furthermore, one can find, in this particular case, there exists an equilibrium measure from which the existence of an asymptotic expansion of the logarithmic partition function is shown in [1, 5]. Therefore, we will have

$$\log \frac{Z_N(t)}{Z_N(0)} = N^2 e_0(t) + e_1(t) + N^{-2} e_2(t) + \cdots \tag{2.5}$$

as $N \to \infty$. Furthermore, the function $e_g(t)$ in (2.5) has an analytic extension to a non-trivial neighborhood of $t = 0$ with a Taylor expansion

$$e_g(t_1, t_2, t_3, \ldots) = \sum_{j_1, j_2, j_3, \ldots} k_g(j_1, j_2, j_3, \ldots) \frac{(-t_1)^{j_1} (-t_2)^{j_2} (-t_3)^{j_3} \cdots}{j_1! j_2! j_3! \cdots} \tag{2.6}$$

where $k_g(j_1, j_2, j_3, \ldots)$ counts the number of connected ribbon graphs with k vertices of degree j on a minimal genus surface g [5]. We will be using the *Euler characteristic*, denoted by χ, letting $\chi = 2 - 2g$ where g is the genus of the ribbon graph. From graph theory, Euler characteristic can be represented as $\chi = k - j + f$ where k is the number of vertices, j is the degrees (or edges) of the vertices, and f is the number of faces. Thus, one can solve the combinatoric problem by understanding the $e_g(t)$ as terms in the asymptotic expansion of $\log\left(\frac{Z_N(t)}{Z_N(0)}\right)$.

2.1 Problem Statement

Before we state our objective, we will briefly cover some useful information for the upcoming sections. Firstly, we will be using the following two differential operators

$$D(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \partial_{t_n} \quad \text{and} \quad D(\mu) = \sum_{n=1}^{\infty} \frac{\mu^{-n}}{n} \partial_{t_n} \tag{2.7}$$

where λ, μ are formal parameters also known as *spectral parameters*. Also, the relationship between the tau function, $\log \tau$, and the logarithmic partition function, $\log\left(\frac{Z_N(t)}{Z_N(0)}\right)$, is

$$\log \tau = \frac{1}{\hbar^2} e_0 + e_1 + \hbar^2 e_2 + \cdots \tag{2.8}$$

where $\tau = \frac{Z_N(t)}{Z_N(0)}$ and $\hbar = \frac{1}{N}$ is also a formal parameter called the *Planck constant*. In the realm of Physics, the Planck constant is the physical constant, that is, it is the quantum of action in quantum mechanics. Also, as $\hbar \to 0$, we have the following

$$
\begin{aligned}
T_n &\to \hbar t_n, && (2.9)\\
D(\lambda) &\to \hbar D(\lambda), && (2.10)\\
\partial_{t_n} &\to \hbar \partial_{t_n}, && (2.11)\\
\log \tau &\to e_0. && (2.12)
\end{aligned}
$$

For example, for $n = 1$, we have $\partial_{t_1} = \partial_1$ and so forth. In this paper, we are going to restrict ourselves to the following time variables: t_1, t_2, and t_3 where $t_1 = x$, $t_2 = y$, and $t_3 = t$. Moreover, the time variable t_0 causes a lot of complications for us. In the context of combinatorics the time variables $t_1, t_2, t_3, \ldots$ count the number of vertices of a ribbon graph, while the t_0 counts the number of faces of a ribbon graph. Also, we will frequently use the following two Taylor expansions:

$$
\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots && (2.13)\\
\exp(-x) &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \cdots . && (2.14)
\end{aligned}
$$

The following set of equations are known as the Differential Fay Identities and will be our main focus throughout the remainder of this paper.

$$\exp\left[\left(e^{\hbar D(\lambda)} - 1\right)\left(e^{\hbar D(\mu)} - 1\right)\log\tau\right] = 1 - \frac{1}{\lambda - \mu}\partial_1\left(e^{\hbar D(\lambda)} - e^{\hbar D(\mu)}\right)\log\tau$$

$$\exp\left[\left(e^{\hbar D(\lambda)} - 1\right)\left(e^{\hbar D(\mu)} - 1\right)\log\tau\right] - \frac{\lambda}{\lambda - \mu}\exp\left[\left(e^{\hbar D(\lambda)} - 1\right)\left(e^{-\hbar\partial_0} - 1\right)\log\tau\right] + \frac{\mu}{\lambda - \mu}\exp\left[\left(e^{\hbar D(\mu)} - 1\right)\left(e^{-\hbar\partial_0} - 1\right)\log\tau\right] = 0$$

$$\exp\left[\left(e^{\hbar D(\lambda)} - 1\right)\left(e^{-\hbar D(\mu)} - 1\right)\log\tau\right] - 1 + \frac{1}{\lambda\mu}\exp\left(\left[\left(e^{\hbar D(\lambda)} - 1\right)\left(e^{\hbar\partial_0} - 1\right) + \left(e^{-\hbar D(\mu)} - 1\right)\left(e^{-\hbar\partial_0} - 1\right) - \left(e^{\hbar\partial_0} - 1\right)\left(e^{-\hbar\partial_0} - 1\right)\right]\log\tau\right) = 0$$

Figure 2.1: The Differential Fay Identities, [6]

The tau function as it is defined by the integral in (2.1) satisfies a family of coupled nonlinear differential equations closely associated with integrable hierarchies such as the Toda lattice. These equations can be succinctly stated in the triple of generating equations (see Figure 2.1) in formal parameters λ and μ [4, 6]. These equations should be interpreted in the following way: one first expands them as a series in λ and μ assuming convergence, and then associates coefficients of like powers of λ and μ.

Due to the lengthy and tedious nature of some equations we will encounter, I will use, at times, color to help the reader identify the sections of the equation we are currently working on. Moreover, since some equations are too long for standard LaTeX format, I will be interchanging between the use standard equation format and figures to display these kinds of equations.

2.2 1^{st} Differential Fay Identity

In these sections, we will first focus on first identity from Figure 2.1. In order to simplify things, we will first asymptotically expand the Left-Hand-Side (LHS) of the identity, then we will similarly expand the Right-Hand-Side (RHS) of the identity. First, only consider the exponent term of the LHS $\left(e^{\hbar D(\lambda)}-1\right)\left(e^{\hbar D(\mu)}-1\right)\log\tau$. Using the Taylor expansion of $\exp(x)$, we expand the exponent part while collecting the powers of $\hbar$ as $\hbar \to 0$. We obtain the following

$$\begin{aligned}
&\hbar^0\left(D(\lambda)D(\mu)e_0\right)+\hbar^1\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right) \\
&\quad+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right. \\
&\quad\left.+D(\lambda)D(\mu)e_1\right) \\
&\quad+\hbar^3\left(\left[\frac{1}{4!}D(\lambda)D(\mu)^4+\frac{1}{2!3!}D(\lambda)^2D(\mu)^3+\frac{1}{2!3!}D(\lambda)^3D(\mu)^2\right.\right. \\
&\quad\left.\left.+\frac{1}{4!}D(\lambda)^4D(\mu)\right]e_0+\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_1\right) \\
&\quad+\mathscr{O}\left(\hbar^4\right).
\end{aligned} \tag{2.15}$$

Inserting this expansion version of the exponent into the LHS, we now acquire the following term

$$\begin{aligned}
\exp&\left[\hbar^0\left(D(\lambda)D(\mu)e_0\right)+\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)\right. \\
&\quad+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right. \\
&\quad\left.\left.+D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\right].
\end{aligned} \tag{2.16}$$

After the previous steps, we then factor out the leading order $\hbar^0$ and get

$$e^{D(\lambda)D(\mu)e_0}\exp\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right) \tag{2.17}$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.$$
$$\left.+D(\lambda)D(\mu)e_1\right)$$
$$+\mathscr{O}\left(\hbar^3\right)\Bigg].$$

If we expand this term using the Taylor expansion, we finally obtain the desire result of

$$e^{D(\lambda)D(\mu)e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right) \tag{2.18}$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.$$
$$\left.+D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]$$
$$+\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.$$
$$\left.+D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]^2$$
$$+\mathscr{O}\left(\hbar^3\right)\Bigg].$$

and therefore, we are done for the LHS of the equation. Now focusing on the RHS of the equation. Using the Taylor expansion of $\exp(x)$ and distributing the term e_0 and $\frac{1}{\lambda-\mu}$ while collecting the powers of $\hbar$ on $1-\frac{1}{\lambda-\mu}\partial_1\left(e^{\hbar D(\lambda)}-e^{\hbar D(\mu)}\right)\log\tau$, the RHS becomes

$$\hbar^0\left[1-\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)-D(\mu)\right)e_0\right]-\hbar^1\left[\frac{1}{2!}\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)^2-D(\mu)^2\right)e_0\right] \tag{2.19}$$
$$-\hbar^2\frac{1}{\lambda-\mu}\left[\frac{1}{3!}\partial_1\left(D(\lambda)^3-D(\mu)^3\right)e_0\right.$$
$$\left.+\partial_1\left(D(\lambda)-D(\mu)\right)e_1\right]$$
$$-\hbar^3\frac{1}{\lambda-\mu}\left[\frac{1}{4!}\partial_1\left(D(\lambda)^4-D(\mu)^4\right)e_0\right.$$
$$\left.+\frac{1}{2!}\partial_1\left(D(\lambda)^2-D(\mu)^2\right)e_1\right]$$
$$-\mathscr{O}\left(\hbar^4\right)$$

Therefore, we obtain our desire result for the RHS of the identity. Then combining the results of the LHS and the RHS, the 1^{st} Differential Fay Identity becomes

$$e^{D(\lambda)D(\mu)e_0}+\left[\hbar e^{D(\lambda)D(\mu)e_0}\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)\right. \tag{2.20}$$
$$+\hbar^2e^{D(\lambda)D(\mu)e_0}\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.$$
$$\left.\left.+D(\lambda)D(\mu)e_1\right)+e^{D(\lambda)D(\mu)e_0}\mathscr{O}\left(\hbar^3\right)\right]$$
$$+\frac{1}{2!}e^{D(\lambda)D(\mu)e_0}\left[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)\right.$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.$$
$$\left.\left.+D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\right]^2+e^{D(\lambda)D(\mu)e_0}\mathscr{O}\left(\hbar^3\right)\Big]$$
$$=\hbar^0\left[1-\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)-D(\mu)\right)e_0\right]$$
$$-\hbar^1\left[\frac{1}{2!}\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)^2-D(\mu)^2\right)e_0\right]$$
$$-\hbar^2\frac{1}{\lambda-\mu}\left[\frac{1}{3!}\partial_1\left(D(\lambda)^3-D(\mu)^3\right)e_0+\partial_1\left(D(\lambda)-D(\mu)\right)e_1\right]$$
$$-\hbar^3\frac{1}{\lambda-\mu}\left[\frac{1}{4!}\partial_1\left(D(\lambda)^4-D(\mu)^4\right)e_0+\frac{1}{2!}\partial_1\left(D(\lambda)^2-D(\mu)^2\right)e_1\right]$$
$$-\mathscr{O}\left(\hbar^4\right)$$

Now, collecting the powers of $\hbar$ on either side of the identity, we obtain the following equation

$$\begin{aligned}
&\hbar^0 e^{D(\lambda)D(\mu)e_0} + \hbar^1 e^{D(\lambda)D(\mu)e_0}\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2 D(\mu)\right]e_0\right) \\
&+\hbar^2 e^{D(\lambda)D(\mu)e_0}\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3 + \frac{1}{2!2!}D(\lambda)^2D(\mu)^2 + \frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right. \\
&\left.+\frac{1}{2!}\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2D(\mu)\right]e_0\right)^2 + D(\lambda)D(\mu)e_1\right) + \mathscr{O}\left(\hbar^3\right) \\
&= \hbar^0\left[1 - \frac{1}{\lambda-\mu}\partial_1\left(D(\lambda) - D(\mu)\right)e_0\right] - \hbar^1\left[\frac{1}{2!}\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)^2 - D(\mu)^2\right)e_0\right] \\
&-\hbar^2\frac{1}{\lambda-\mu}\left[\frac{1}{3!}\partial_1\left(D(\lambda)^3 - D(\mu)^3\right)e_0 + \partial_1\left(D(\lambda) - D(\mu)\right)e_1\right] - \mathscr{O}\left(\hbar^3\right).
\end{aligned} \tag{2.21}$$

Comparing coefficients of $\hbar^0, \hbar^1$, and $\hbar^2$, we obtain the following hierarchies. For the case h^0, we have

$$e^{D(\lambda)D(\mu)e_0} = 1 - \frac{1}{\lambda-\mu}\partial_1\left(D(\lambda) - D(\mu)\right)e_0 \tag{2.22}$$

For the case $\hbar^1$, we have

$$e^{D(\lambda)D(\mu)e_0}\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2D(\mu)\right]e_0\right) = \frac{1}{2!}\frac{-1}{\lambda-\mu}\partial_1\left(D(\lambda)^2 - D(\mu)^2\right)e_0 \tag{2.23}$$

And finally, for the case $\hbar^2$, we have

$$\begin{aligned}
e^{D(\lambda)D(\mu)e_0}&\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3 + \frac{1}{2!2!}D(\lambda)^2D(\mu)^2 + \frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right. \\
&\left.+\frac{1}{2!}\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2D(\mu)\right]e_0\right)^2 + D(\lambda)D(\mu)e_1\right) \\
&= \frac{-1}{\lambda-\mu}\left[\frac{1}{3!}\partial_1\left(D(\lambda)^3 - D(\mu)^3\right)e_0 + \partial_1\left(D(\lambda) - D(\mu)\right)e_1\right]
\end{aligned} \tag{2.24}$$

Proposition 1. *If e_0 satisfies equation (2.22) then equation (2.23) follows.*

Proof. Before we try to prove the proposition, we will need the Clairaut's Theorem stated in [3] that states *if* $f(x,y)$ *and its partial derivatives* f_x, f_y, f_{xy}, *and* f_{yx} *are defined throughout an open region containing a point* (a,b) *and are all continuous at* (a,b), *then*

$$f_{xy}(a,b) = f_{yx}(a,b). \tag{2.25}$$

Since the function $e_g(t)$ has an analytic extension to a non-trivial neighborhood of $t = 0$ then we can assume that the conditions in the Clairaut's theorem holds. Therefore, we can assume that the differential operators $D(\lambda)$ and $D(\mu)$ are commutative, i.e., $D(\lambda)D(\mu) = D(\mu)D(\lambda)$. Therefore, consider equation (2.22)

$$e^{D(\lambda)D(\mu)e_0} = 1 - \frac{1}{\lambda - \mu}\partial_1 \left(D(\lambda) - D(\mu)\right) e_0 \tag{2.26}$$

Multiplying both sides of the equation by the differential operator $D(\lambda) + D(\mu)$, we have

$$[D(\lambda) + D(\mu)]e^{D(\lambda)D(\mu)e_0} = [D(\lambda) + D(\mu)] \left(1 - \frac{1}{\lambda - \mu}\partial_1 \left(D(\lambda) - D(\mu)\right) e_0\right). \tag{2.27}$$

Applying the differential operator to the LHS, we obtain

$$[D(\lambda) + D(\mu)]e^{D(\lambda)D(\mu)e_0} = e^{D(\lambda)D(\mu)e_0}[D(\lambda) + D(\mu)]D(\lambda)D(\mu)e_0 \tag{2.28}$$

$$= e^{D(\lambda)D(\mu)e_0}\left[D(\lambda)^2 D(\mu) + D(\lambda)D(\mu)^2\right] e_0. \tag{2.29}$$

Now, applying the differential operator to the RHS we obtain

$$\underbrace{[D(\lambda) + D(\mu)] \cdot 1}_{=0} - [D(\lambda) + D(\mu)] \left(\frac{1}{\lambda - \mu}\partial_1 \left(D(\lambda) - D(\mu)\right) e_0\right) \tag{2.30}$$

The first term equals to zero because any differential operator that differentiates a constant equals to zero. In the second term, we can interchange the terms due to the commutative property we obtain from Clairaut's Theorem. Hence, the RHS becomes

$$-\left[D(\lambda)+D(\mu)\right]\left(\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)-D(\mu)\right)e_0\right) \Rightarrow \frac{-1}{\lambda-\mu}\partial_1\left(D(\lambda)^2-D(\mu)^2\right)e_0. \tag{2.31}$$

Therefore,

$$e^{D(\lambda)D(\mu)e_0} = 1-\frac{1}{\lambda-\mu}\partial_1\left(D(\lambda)-D(\mu)\right)e_0 \tag{2.32}$$

$$\Rightarrow$$

$$e^{D(\lambda)D(\mu)e_0}\left[D(\lambda)^2D(\mu)+D(\lambda)D(\mu)^2\right]e_0 = \frac{-1}{\lambda-\mu}\partial_1\left(D(\lambda)^2-D(\mu)^2\right)e_0. \tag{2.33}$$

Hence, Proposition 1 is proved. □

Conjecture 1. *We claim that if e_g satisfies every hierarchy associated with an even power of $\hbar$ in the 1^{st} Differential Fay identity then the hierarchy associated with the next odd power of $\hbar$ follows.*

2.3 First Case: $\mu \to \lambda$

In this section, we will expand the 1^{st} Differential Fay Identity under the case where $\mu \to \lambda$. Just like in the previous section, we will solely focus on the LHS of the identity. First, using the Taylor expansion of $\exp(x)$, we can expand the LHS of the identity as $\mu \to \lambda$ to

$$\exp\left(\left(e^{D(\lambda)} - 1\right)^2 \log\tau\right) = 1 - \frac{1}{\lambda-\mu}\partial_1\left(e^{D(\lambda)} - e^{D(\mu)}\right)\log\tau \tag{2.34}$$

$$\exp\left(\left(D(\lambda) + \frac{1}{2!}D^2(\lambda) + \cdots\right)^2 \log\tau\right) = 1 - \frac{1}{\lambda-\mu}\partial_1\left(e^{D(\lambda)} - e^{D(\mu)}\right)\log\tau. \tag{2.35}$$

Then, expanding the identity using the Taylor series of $\exp(x)$ on the LHS of the equation, we have

$$1 + \left(D(\lambda) + \frac{1}{2!}D^2(\lambda) + \frac{1}{3!}D^3(\lambda) + \cdots\right)^2 \log\tau + \cdots = 1 - \frac{1}{\lambda-\mu}\partial_1\left(e^{D(\lambda)} - e^{D(\mu)}\right)\log\tau. \tag{2.36}$$

Now expanding both sides of equation (2.36) at the same time, we obtain that equation (2.36) expands to

$$
\begin{aligned}
1+\frac{1}{\lambda^2}\left(\partial_1^2\log\tau\right)+\frac{1}{\lambda^3}\left(\partial_1^3\log\tau+\partial_1\partial_2\log\tau\right) \\
&+\frac{1}{\lambda^4}\left(\frac{1}{2}\left(\partial_1^2\log\tau\right)^2+\frac{7}{12}\partial_1^4\log\tau+\frac{3}{2}\partial_1^2\partial_2\log\tau+\frac{1}{4}\partial_2^2\log\tau\right. \\
&\left.+\frac{2}{3}\partial_1\partial_3\log\tau\right) \\
&+\frac{1}{\lambda^5}\left(\left(\partial_1^2\log\tau\right)\left(\partial_1^3\log\tau\right)+\frac{1}{6}\left(\partial_1^5\log\tau\right)+\frac{7}{6}\left(\partial_1^3\partial_2\log\tau\right)\right. \\
&+\left(\partial_1\partial_2\log\tau\right)\left(\partial_2^2\log\tau\right)+\frac{3}{4}\left(\partial_1\partial_2^2\log\tau\right)+\partial_1^2\partial_3\log\tau \\
&\left.+\frac{1}{3}\left(\partial_2\partial_3\log\tau\right)\right) \\
&+\cdots \\
&=1-\left(-\frac{1}{\lambda^2}\left(\partial_1^2\log\tau\right)-\frac{1}{\lambda^3}\left(\partial_1^3\log\tau+\partial_1\partial_2\log\tau\right)\right. \\
&-\frac{1}{\lambda^4}\left(\frac{1}{2}\partial_1^4\log\tau+\frac{3}{2}\partial_1^2\partial_2\log\tau+\partial_1\partial_3\log\tau\right) \\
&-\frac{1}{\lambda^5}\left(\frac{1}{6}\partial_1^5\log\tau+\partial_1^3\partial_2\log\tau+\frac{1}{2}\partial_1\partial_2^2\log\tau+\frac{4}{3}\partial_1^2\partial_3\log\tau\right. \\
&\left.+\partial_1\partial_4\log\tau\right) \\
&\left.-\cdots\right),
\end{aligned}
\tag{2.37}
$$

which quickly becomes

$$
\begin{aligned}
1+\frac{1}{\lambda^2}\left(\partial_1^2\log\tau\right)+\frac{1}{\lambda^3}\left(\partial_1^3\log\tau+\partial_1\partial_2\log\tau\right) \\
+\frac{1}{\lambda^4}\Bigg(\frac{1}{2}\left(\partial_1^2\log\tau\right)^2+\frac{7}{12}\partial_1^4\log\tau+\frac{3}{2}\partial_1^2\partial_2\log\tau \\
+\frac{1}{4}\partial_2^2\log\tau+\frac{2}{3}\partial_1\partial_3\log\tau\Bigg) \\
+\frac{1}{\lambda^5}\left(\left(\partial_1^2\log\tau\right)\left(\partial_1^3\log\tau\right)+\frac{1}{6}\left(\partial_1^5\log\tau\right)+\frac{7}{6}\left(\partial_1^3\partial_2\log\tau\right)\right. \\
+\left(\partial_1\partial_2\log\tau\right)\left(\partial_2^2\log\tau\right)+\frac{3}{4}\left(\partial_1\partial_2^2\log\tau\right)+\partial_1^2\partial_3\log\tau \\
\left.+\frac{1}{3}\left(\partial_2\partial_3\log\tau\right)\right)+\cdots \\
=1+\frac{1}{\lambda^2}\left(\partial_1^2\log\tau\right)+\frac{1}{\lambda^3}\left(\partial_1^3\log\tau+\partial_1\partial_2\log\tau\right) \\
+\frac{1}{\lambda^4}\left(\frac{1}{2}\partial_1^4\log\tau+\frac{3}{2}\partial_1^2\partial_2\log\tau+\partial_1\partial_3\log\tau\right) \\
+\frac{1}{\lambda^5}\left(\frac{1}{6}\partial_1^5\log\tau+\partial_1^3\partial_2\log\tau+\frac{1}{2}\partial_1\partial_2^2\log\tau+\frac{4}{3}\partial_1^2\partial_3\log\tau+\partial_1\partial_4\log\tau\right) \\
+\cdots
\end{aligned}
\tag{2.38}
$$

after collecting the powers of $\frac{1}{\lambda}$. Now, comparing coefficients of $\frac{1}{\lambda^2}$, $\frac{1}{\lambda^3}$, and $\frac{1}{\lambda^4}$, we obtain the following hierarchy of equations. For the case $\frac{1}{\lambda^2}$, we obtain the trivial equation that is

$$
\partial_1^2\log\tau=\partial_1^2\log\tau \tag{2.39}
$$

For the case $\frac{1}{\lambda^3}$, we obtain another trivial equation that is

$$
\partial_1^3\log\tau+\partial_1\partial_2\log\tau=\partial_1^3\log\tau+\partial_1\partial_2\log\tau, \tag{2.40}
$$

For the case $\frac{1}{\lambda^4}$, we obtain the first non-trivial equation that is

$$
\begin{aligned}
\frac{1}{2}\left(\partial_1^2\log\tau\right)^2+\frac{7}{12}\partial_1^4\log\tau+\frac{3}{2}\partial_1^2\partial_2\log\tau+\frac{1}{4}\partial_2^2\log\tau+\frac{2}{3}\partial_1\partial_3\log\tau \\
=\frac{1}{2}\partial_1^4\log\tau+\frac{3}{2}\partial_1^2\partial_2\log\tau+\partial_1\partial_3\log\tau.
\end{aligned}
\tag{2.41}
$$

Now, if we substitute the asymptotic expansion of the tau function from (2.8) into (2.41), simultaneously asymptotically expanding the equation while collecting the powers of $\hbar$, and assuming $\hbar \to 0$, we obtain

$$\frac{1}{2}\left(\hbar^2\partial_1^2(\log\tau)\right)^2+\frac{1}{12}\hbar^4\partial_1^4\log\tau+\frac{1}{4}\hbar^2\partial_2^2\log\tau-\frac{1}{3}h^2\partial_1\partial_3\log\tau=0 \tag{2.42}$$

$$\begin{aligned}&\frac{1}{2}\left(\hbar^2\partial_1^2\left(\frac{1}{\hbar^2}e_0+e_1+\hbar^2e_2+\cdots\right)\right)^2+\frac{1}{12}\hbar^4\partial_1^4\left(\frac{1}{\hbar^2}e_0+e_1+\hbar^2e_2+\cdots\right)\\&+\frac{1}{4}\hbar^2\partial_2^2\left(\frac{1}{\hbar^2}e_0+e_1+\hbar^2e_2+\cdots\right)-\frac{1}{3}\hbar^2\partial_1\partial_3\left(\frac{1}{\hbar^2}e_0+e_1+\hbar^2e_2+\cdots\right)=0\end{aligned} \tag{2.43}$$

$$\begin{aligned}&h^0\left(\frac{1}{4}\partial_2^2e_0+\frac{1}{2}\left(\partial_1^2e_0\right)^2-\frac{1}{3}\partial_1\partial_3e_0\right)\\&+\hbar^2\left(\left(\partial_1^2e_0\right)\left(\partial_1^2e_1\right)+\frac{1}{4}\partial_2^2e_1+\frac{1}{12}\partial_1^4e_0-\frac{1}{3}\partial_1\partial_3e_1\right)\\&+\hbar^4\left(\left(\partial_1^2e_0\right)\left(\partial_1^2e_2\right)+\frac{1}{4}\partial_2^2e_2+\frac{1}{2}\left(\partial_1^2e_1\right)^2+\frac{1}{12}\partial_1^4e_1-\frac{1}{3}\partial_1\partial_3e_2\right)\\&+\hbar^6\left(\left(\partial_1^2e_0\right)\left(\partial_1^2e_3\right)+\left(\partial_1^2e_1\right)\left(\partial_1^2e_2\right)+\frac{1}{4}\partial_2^2e_3+\frac{1}{12}\partial_1^4e_2-\frac{1}{3}\partial_1\partial_3e_3\right)\\&+\cdots=0.\end{aligned} \tag{2.44}$$

Now, comparing coefficients of $\hbar^0, \hbar^2, \hbar^4, \ldots$ we obtain the following hierarchy of equations. For the case h^0, we have

$$\frac{1}{4}\partial_2^2e_0+\frac{1}{2}\left(\partial_1^2e_0\right)^2-\frac{1}{3}\partial_1\partial_3e_0=0. \tag{2.45}$$

For the case $\hbar^2$, we have

$$\left(\partial_1^2e_0\right)\left(\partial_1^2e_1\right)+\frac{1}{4}\partial_2^2e_1+\frac{1}{12}\partial_1^4e_0-\frac{1}{3}\partial_1\partial_3e_1=0. \tag{2.46}$$

For the case $\hbar^4$, we have

$$\left(\partial_1^2e_0\right)\left(\partial_1^2e_2\right)+\frac{1}{4}\partial_2^2e_2+\frac{1}{2}\left(\partial_1^2e_1\right)^2+\frac{1}{12}\partial_1^4e_1-\frac{1}{3}\partial_1\partial_3e_2=0. \tag{2.47}$$

For the case $\hbar^6$, we have

$$\left(\partial_1^2 e_0\right)\left(\partial_1^2 e_3\right)+\left(\partial_1^2 e_1\right)\left(\partial_1^2 e_2\right)+\frac{1}{4}\partial_2^2 e_3+\frac{1}{12}\partial_1^4 e_2-\frac{1}{3}\partial_1\partial_3 e_3=0. \tag{2.48}$$

These are the equations we were striving for because they are counting the very exclusive ribbon graphs. To see this, remember that from (2.6), we have that g is the genus of the ribbon graph. Then, for the partial derivatives

$$\partial_j^k e_g\left(t_1,t_2,t_3,\dots\right), \tag{2.49}$$

we can interpret this symbolically as how many ways can we draw ribbon graphs of k vertices of degree j on the minimal genus surface g where $t_n=0$ for all $n=1,2,3,\dots$. Consider the equation (2.45), we will use the symbolic interpretation of the partial derivatives to determine if the equation holds true. First consider the first term, $\partial_2^2 e_0$, which equals to two. Why? The reason the first term equal to two is because there's only two ways to draw a ribbon graph with 2 vertices of degree 2 on a minimal genus surface 0 (see Figures 2.2 and 2.3).

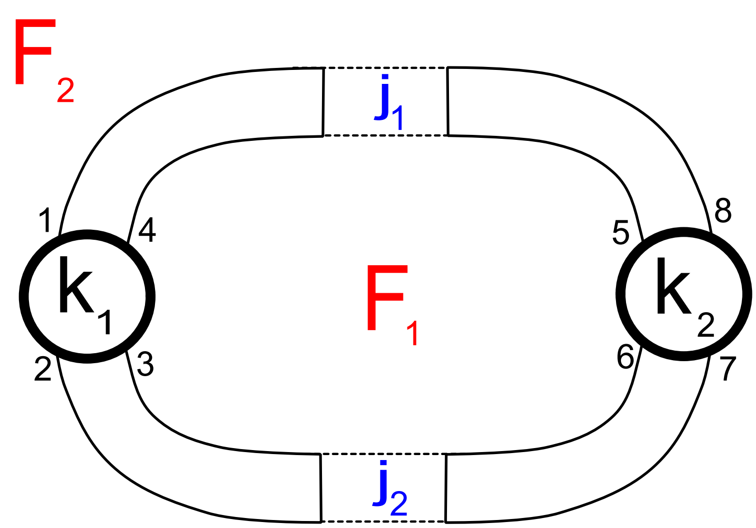

Figure 2.2: 1^{st} ribbon graph with 2 vertices of degree 2 on a minimal genus 0

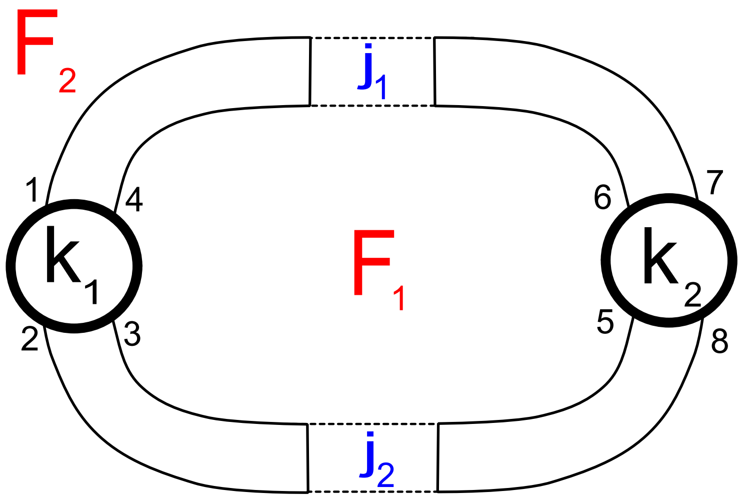

Figure 2.3: 2^{nd} ribbon graph with 2 vertices of degree 2 on a minimal genus 0.

How do we know that these two ribbon graphs can be drawn on the minimal genus surface 0? Well, using Euler characteristic, we can determine the genus of any graph. That is, for both ribbon graphs we have that

$$\begin{aligned} 2-2g &= 2-2+2 \\ 2-2g &= 2 \\ -2g &= 0 \Rightarrow g = 0. \end{aligned}$$

Therefore, the genus of the ribbon graphs in Figures 2.2 and 2.3 is 0. Similarly, we apply this to the next two terms. The second term equals to one because there's only one way to draw a ribbon graph with 2 vertices of degree 1 on a minimal genus 0 (see Figure 2.4)

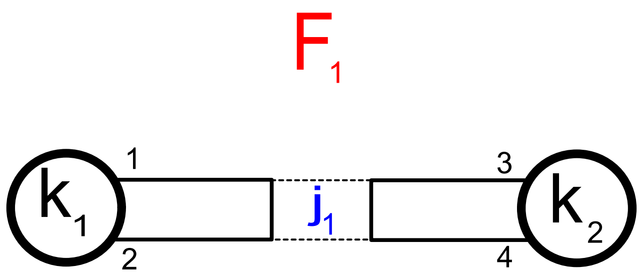

Figure 2.4: The only ribbon graph with 2 vertices of degree 1 on a minimal genus 0.

The last term of equation (2.45) equals to three because there's only three ways to draw a ribbon graph that involves 1 vertex of degree 1 and 1 vertex of degree 3 on a minimal genus surface 0 (see Figures 2.5, 2.6, and 2.7)

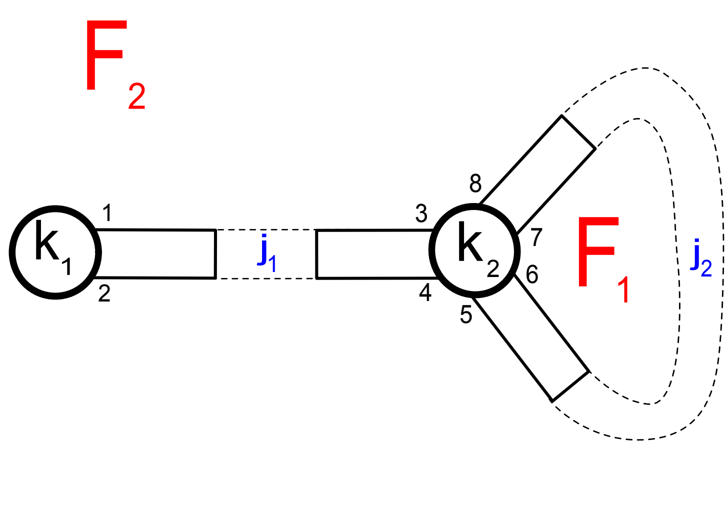

Figure 2.5: 1^{st} ribbon graph with 1 vertex of degree 1 and 1 vertex with degree 3 on genus 0.

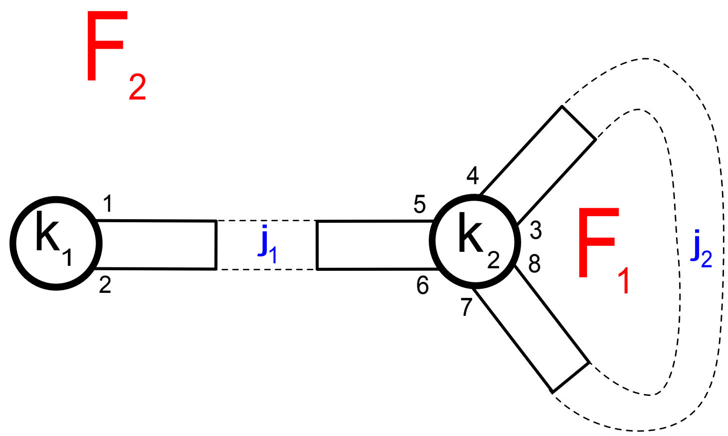

Figure 2.6: 2^{nd} ribbon graph with 1 vertex of degree 1 and 1 vertex with degree 3 on genus 0.

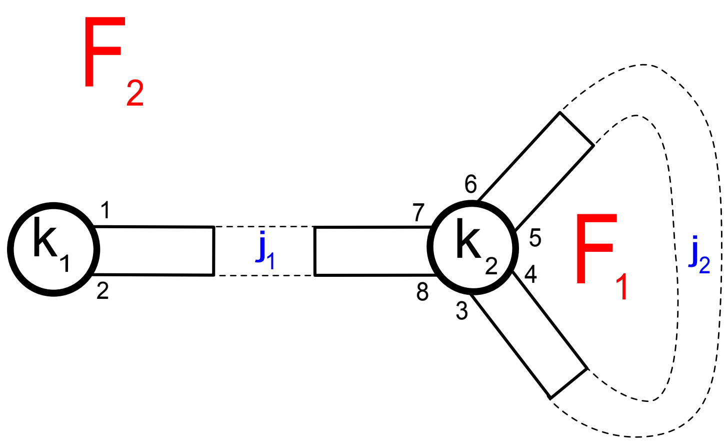

Figure 2.7: 3^{rd} ribbon graph with 1 vertex of degree 1 and 1 vertex with degree 3 on genus 0.

Inputting the values into equation (2.45), we obtain

$$
\begin{aligned}
\frac{1}{4}(2)+\frac{1}{2}(1)^2-\frac{1}{3}(3)&=0\\
\frac{1}{2}+\frac{1}{2}-1&=0.
\end{aligned}
$$

Therefore, equation (2.45) holds true. Unfortunately, this application does not hold for the remaining equations due to the trivial outcomes we encounter while applying the procedure.

2.4 Second Case: $\mu \to \infty$

In this section, we will now expand the 1^{st} Differential Fay Identity under the case where $\mu \to \infty$. First, using the Taylor expansion of $\exp(x)$ where $x = D(\mu)$, we can expand the LHS of the identity as $\mu \to \infty$ to

$$\exp\left[\left(e^{D(\lambda)}-1\right)\left(e^{D(\mu)}-1\right)\log\tau\right] = 1-\frac{1}{\lambda-\mu}\partial_1\left(e^{D(\lambda)}-e^{D(\mu)}\right)\log\tau \tag{2.50}$$

$$\exp\left[\left(e^{D(\lambda)}-1\right)\left(D(\mu)+\cdots\right)\log\tau\right] = 1-\frac{1}{\lambda-\mu}\partial_1\left(e^{D(\lambda)}-\left(1+D(\mu)\cdots\right)\right)\log\tau. \tag{2.51}$$

Using the Taylor expansion of the differential operator $D(\mu)$ from (2.7), we can rewrite the LHS of the identity as

$$\exp\left[\left(e^{D(\lambda)}-1\right)\left(\left[\frac{1}{\mu}\partial_1+\cdots\right]\right)\log\tau\right] = 1+\frac{1}{\mu}\left(\frac{1}{1-\frac{\lambda}{\mu}}\right)\partial_1\left(e^{D(\lambda)}-\left(1+\cdots\right)\right)\log\tau \tag{2.52}$$

which will make the process slightly easier to work with. Since the formal parameters $\lambda,\mu \to \infty$ then using geometric series, we have

$$\frac{1}{1-\frac{\lambda}{\mu}} = \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n = 1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}+\frac{\lambda^3}{\mu^3}+\cdots. \tag{2.53}$$

Thus, the equation (2.52) becomes

$$\exp\left(\left(e^{D(\lambda)}-1\right)\left(\left[\frac{1}{\mu}\partial_1+\cdots\right]\cdots\right)\log\tau\right) = 1+\frac{1}{\mu}\left(1+\frac{\lambda}{\mu}\cdots\right)\partial_1\left(\left(e^{(\lambda)}-1\right)\cdots\right)\log\tau \tag{2.54}$$

while using the Taylor expansion of $D(\mu)$. Using the Taylor series for $\exp(x)$ where

$$x = \left(e^{D(\lambda)}-1\right)\left(\left[\frac{1}{\mu}\partial_1+\cdots\right]+\frac{1}{2!}\left[\frac{1}{\mu}\partial_1+\cdots\right]^2+\cdots\right)\log\tau, \tag{2.55}$$

we get that the LHS of equation (2.54) expands to

$$1+\left(e^{D(\lambda)}-1\right)\left(\left[\frac{1}{\mu}\partial_1+\cdots\right]+\frac{1}{2!}\left[\frac{1}{\mu}\partial_1+\cdots\right]^2+\frac{1}{3!}\left[\frac{1}{\mu}\partial_1+\cdots\right]^3+\cdots\right)\log\tau \tag{2.56}$$
$$+\frac{1}{2!}\left[\left(e^{D(\lambda)}-1\right)\left(\left[\frac{1}{\mu}\partial_1+\cdots\right]+\frac{1}{2!}\left[\frac{1}{\mu}\partial_1+\cdots\right]^2+\frac{1}{3!}\left[\frac{1}{\mu}\partial_1+\cdots\right]^3+\cdots\right)\log\tau\right]^2$$
$$+\cdots$$

while the RHS is left alone for now. Now, collecting the powers of $\frac{1}{\mu}, \frac{1}{\mu^2}, \frac{1}{\mu^3}, \ldots$ after expanding the LHS, we obtain

$$1+\frac{1}{\mu}\left(e^{D(\lambda)}-1\right)\partial_1\log\tau \tag{2.57}$$
$$+\frac{1}{\mu^2}\left(\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1(\log\tau)\right]^2+\frac{1}{2}\left(e^{D(\lambda)}-1\right)\partial_1^2\log\tau+\frac{1}{2}\left(e^{D(\lambda)}-1\right)\partial_2\log\tau\right)$$
$$+\frac{1}{\mu^3}\left(\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_1^2\log\tau\right]\right.$$
$$+\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_2\log\tau\right]+\frac{1}{6}\left[\left(e^{D(\lambda)}-1\right)\partial_1^3\log\tau\right]$$
$$\left.+\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\partial_2\log\tau\right]+\frac{1}{3}\left[\left(e^{D(\lambda)}-1\right)\partial_3\log\tau\right]\right)$$
$$+\frac{1}{\mu^4}\left(\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\left(\frac{\partial_1^2}{2}+\frac{\partial_2}{2}\right)\log\tau\right]^2+\frac{1}{6}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_1^3\log\tau\right]\right.$$
$$+\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_1\partial_2\log\tau\right]$$
$$+\frac{1}{3}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_3\log\tau\right]+\frac{1}{4}\left[\left(e^{D(\lambda)}-1\right)\partial_1^2\partial_2\log\tau\right]$$
$$+\frac{1}{8}\left[\left(e^{D(\lambda)}-1\right)\partial_2^2\log\tau\right]+\frac{1}{3}\left[\left(e^{D(\lambda)}-1\right)\partial_1\partial_3\log\tau\right]$$
$$\left.+\frac{1}{4}\left[\left(e^{D(\lambda)}-1\right)\partial_4\log\tau\right]\right)$$
$$+\cdots$$

Similarly, collecting the powers of $\frac{1}{\mu}, \frac{1}{\mu^2}, \frac{1}{\mu^3}, \ldots$ after expanding the RHS, we get

$$
\begin{aligned}
&1+\frac{1}{\mu}\left(e^{D(\lambda)}-1\right)\partial_1\log\tau \\
&+\frac{1}{\mu^2}\left(\lambda\left(e^{D(\lambda)}-1\right)\partial_1\log\tau-\partial_1^2\log\tau\right) \\
&+\frac{1}{\mu^3}\left(\lambda^2\left(e^{D(\lambda)}-1\right)\partial_1\log\tau-\frac{1}{2}\partial_1^3\log\tau-\frac{1}{2}\partial_1\partial_2\log\tau-\lambda\partial_1^2\log\tau\right) \\
&+\frac{1}{\mu^4}\left(\lambda^3\left(e^{D(\lambda)}-1\right)\partial_1\log\tau-\frac{1}{6}\partial_1^4\log\tau-\frac{1}{2}\partial_1^2\partial_2\log\tau-\frac{1}{3}\partial_1\partial_3\log\tau\right. \\
&\left.-\frac{1}{2}\lambda\partial_1^3\log\tau-\frac{1}{2}\lambda\partial_1\partial_2\log\tau-\lambda^2\partial_1^2\log\tau\right) \\
&+\cdots
\end{aligned}
\tag{2.58}
$$

Finally, comparing the coefficients of the powers of $\frac{1}{\mu}$, $\frac{1}{\mu^2}$, $\frac{1}{\mu^3}, \ldots$ we get the following hierarchy of equations. For the case $\frac{1}{\mu}$, we the first trivial equation under the case when $\mu \to \infty$ that is

$$
\left(e^{D(\lambda)}-1\right)\partial_1\log\tau = \left(e^{D(\lambda)}-1\right)\partial_1\log\tau. \tag{2.59}
$$

For the case $\frac{1}{\mu^2}$, we obtain the first non-trival equation under the case when $\mu \to \infty$ that is

$$
\begin{aligned}
\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1(\log\tau)\right]^2+\frac{1}{2}\left(e^{D(\lambda)}-1\right)\partial_1^2\log\tau+\frac{1}{2}\left(e^{D(\lambda)}-1\right)\partial_2\log\tau \\
=\lambda\left(e^{D(\lambda)}-1\right)\partial_1\log\tau-\partial_1^2\log\tau.
\end{aligned}
\tag{2.60}
$$

For the case $\frac{1}{\mu^3}$, we obtain the second non-trival equation under the case when $\mu \to \infty$ that is

$$
\begin{aligned}
&\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_1^2\log\tau\right] \\
&\quad+\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_2\log\tau\right] \\
&\quad+\frac{1}{6}\left[\left(e^{D(\lambda)}-1\right)\partial_1^3\log\tau\right] \\
&\quad+\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\partial_2\log\tau\right] \\
&\quad+\frac{1}{3}\left[\left(e^{D(\lambda)}-1\right)\partial_3\log\tau\right] \\
&\quad=\lambda^2\left(e^{D(\lambda)}-1\right)\partial_1\log\tau-\frac{1}{2}\partial_1^3\log\tau \\
&\quad-\frac{1}{2}\partial_1\partial_2\log\tau-\lambda\partial_1^2\log\tau.
\end{aligned}
\tag{2.61}
$$

For the case $\frac{1}{\mu^4}$, we obtain the third non-trival equation under the case when $\mu\to\infty$ that is

$$
\begin{aligned}
&\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\left(\frac{\partial_1^2}{2}+\frac{\partial_2}{2}\right)\log\tau\right]^2+\frac{1}{6}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_1^3\log\tau\right] \\
&\quad+\frac{1}{2}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_1\partial_2\log\tau\right] \\
&\quad+\frac{1}{3}\left[\left(e^{D(\lambda)}-1\right)\partial_1\log\tau\right]\left[\left(e^{D(\lambda)}-1\right)\partial_3\log\tau\right] \\
&\quad+\frac{1}{4}\left[\left(e^{D(\lambda)}-1\right)\partial_1^2\partial_2\log\tau\right]+\frac{1}{8}\left[\left(e^{D(\lambda)}-1\right)\partial_2^2\log\tau\right] \\
&\quad+\frac{1}{3}\left[\left(e^{D(\lambda)}-1\right)\partial_1\partial_3\log\tau\right]+\frac{1}{4}\left[\left(e^{D(\lambda)}-1\right)\partial_4\log\tau\right] \\
&\quad=\lambda^3\left(e^{D(\lambda)}-1\right)\partial_1\log\tau-\frac{1}{6}\partial_1^4\log\tau-\frac{1}{2}\partial_1^2\partial_2\log\tau \\
&\quad-\frac{1}{3}\partial_1\partial_3\log\tau-\frac{1}{2}\lambda\partial_1^3\log\tau-\frac{1}{2}\lambda\partial_1\partial_2\log\tau-\lambda^2\partial_1^2\log\tau.
\end{aligned}
\tag{2.62}
$$

Then we will solely focus on the equation associated with the case $\frac{1}{\mu^2}$ where we will now perform an asymptotic expansion on equation (2.60) with respect to λ. Therefore, using the Taylor expansion of the differential operator $D(\lambda)$, equation (2.60) becomes

$$
\begin{aligned}
&\frac{1}{2}\left[\left(D(\lambda)+\cdots\right)\partial_1\left(\log\tau\right)\right]^2+\frac{1}{2}\left(D(\lambda)+\cdots\right)\partial_1^2\log\tau+\frac{1}{2}\left(D(\lambda)+\cdots\right)\partial_2\log\tau \\
&\qquad=\lambda\left(D(\lambda)+\cdots\right)\partial_1\log\tau-\partial_1^2\log\tau,
\end{aligned}
\tag{2.63}
$$

which expands to

$$
\begin{aligned}
&\frac{1}{\lambda}\left(\frac{1}{2}\partial_1^3\log\tau+\frac{1}{2}\partial_1\partial_2\log\tau\right)+\frac{1}{\lambda^2}\left(\frac{1}{2}\left[\partial_1^2\log\tau\right]^2+\frac{1}{4}\left(\partial_1^4\log\tau+\partial_1^2\partial_2\log\tau\right)\right.\\
&\left.+\frac{1}{4}\left(\partial_1^2\partial_2\log\tau+\partial_2^2\log\tau\right)\right)\\
&+\frac{1}{\lambda^3}\left(\frac{1}{2}\left(\left[\partial_1^2\log\tau\right]\left[\partial_1^3\log\tau\right]+\left[\partial_1^2\log\tau\right]\left[\partial_1\partial_2\log\tau\right]\right)\right.\\
&+\frac{1}{2}\left(\frac{1}{6}\partial_1^5\log\tau+\frac{1}{2}\partial_1^3\partial_2\log\tau+\frac{1}{3}\partial_1^2\partial_3\log\tau\right)\\
&\left.+\frac{1}{2}\left(\frac{1}{6}\partial_1^3\partial_2\log\tau+\frac{1}{2}\partial_1\partial_2^2\log\tau+\frac{1}{3}\partial_2\partial_3\log\tau\right)\right)\\
&+\cdots\\
&=\frac{1}{\lambda}\left(\frac{1}{2}\partial_1^3\log\tau+\frac{1}{2}\partial_1\partial_2\log\tau\right)\\
&+\frac{1}{\lambda^2}\left(\frac{1}{6}\partial_1^4\log\tau+\frac{1}{2}\partial_1^2\partial_2\log\tau+\frac{1}{3}\partial_1\partial_3\log\tau\right)\\
&+\frac{1}{\lambda^3}\left(\frac{1}{24}\partial_1^5\log\tau+\frac{1}{4}\partial_1^3\partial_2\log\tau+\frac{1}{8}\partial_1\partial_2^2\log\tau\right.\\
&\left.+\frac{1}{3}\partial_1^2\partial_3\log\tau+\frac{1}{4}\partial_1\partial_4\log\tau\right)\\
&+\cdots.
\end{aligned}
\tag{2.64}
$$

After all that work, we once more compare the coefficients of the λ to get the following hierarchy of equations. For the case $\frac{1}{\lambda}$, we obtain another trivial equation

$$
\frac{1}{2}\partial_1^3\log\tau+\frac{1}{2}\partial_1\partial_2\log\tau=\frac{1}{2}\partial_1^3\log\tau+\frac{1}{2}\partial_1\partial_2\log\tau. \tag{2.65}
$$

For the case $\frac{1}{\lambda^2}$, we get our first non-trivial equation that is

$$
\begin{aligned}
\frac{1}{2}\left[\partial_1^2\log\tau\right]^2+\frac{1}{4}\left(\partial_1^4\log\tau+\partial_1^2\partial_2\log\tau\right)+\frac{1}{4}\left(\partial_1^2\partial_2\log\tau+\partial_2^2\log\tau\right)\\
=\frac{1}{6}\partial_1^4\log\tau+\frac{1}{2}\partial_1^2\partial_2\log\tau+\frac{1}{3}\partial_1\partial_3\log\tau.
\end{aligned}
\tag{2.66}
$$

For the case $\frac{1}{\lambda^3}$, we get our second non-trivial equation that is

$$\frac{1}{2}\Bigg(\left[\partial_1^2 \log\tau\right]\left[\partial_1^3 \log\tau\right] + \left[\partial_1^2 \log\tau\right]\left[\partial_1\partial_2 \log\tau\right]\Bigg) \tag{2.67}$$
$$+\frac{1}{2}\left(\frac{1}{6}\partial_1^5 \log\tau + \frac{1}{2}\partial_1^3\partial_2 \log\tau + \frac{1}{3}\partial_1^2\partial_3 \log\tau\right)$$
$$+\frac{1}{2}\left(\frac{1}{6}\partial_1^3\partial_2 \log\tau + \frac{1}{2}\partial_1\partial_2^2 \log\tau + \frac{1}{3}\partial_2\partial_3 \log\tau\right)$$
$$= \frac{1}{24}\partial_1^5 \log\tau + \frac{1}{4}\partial_1^3\partial_2 \log\tau + \frac{1}{8}\partial_1\partial_2^2 \log\tau$$
$$+\frac{1}{3}\partial_1^2\partial_3 \log\tau + \frac{1}{4}\partial_1\partial_4 \log\tau.$$

Now, turning our focus on the equation that's associated with the case $\frac{1}{\lambda^2}$ where we moved all the terms to one side. We then have the following equation the terms for $\frac{1}{\lambda^2}$,

$$6\left[\partial_1^2 \log\tau\right]^2 + \partial_1^4 \log\tau + 3\partial_2^2 \log\tau - 4\partial_1\partial_3 \log\tau = 0. \tag{2.68}$$

Now, let $u = 2\partial_1^2 \log\tau$. Then substituting this term into equation (2.68), we obtain

$$6\left[\frac{1}{2}u\right]^2 + \frac{1}{2}\partial_1^2 u + 3\partial_2^2 \log\tau - 4\partial_1\partial_3 \log\tau = 0. \tag{2.69}$$

Now, differentiating the entire equation (2.69) by ∂_1^2, we obtain

$$\frac{3}{2}\partial_1^2 u^2 + \frac{1}{2}\partial_1^4 u + 3\partial_2^2\partial_1^2 \log\tau - 4\partial_1\partial_3\partial 1^2 \log\tau = 0 \tag{2.70}$$

which becomes

$$\frac{3}{2}u_{xx}^2 + \frac{1}{2}u_{xxxx} + \frac{3}{2}u_{yy} - 2u_{tx} = 0. \tag{2.71}$$

We can rewrite equation (2.71) to

$$3u_{xx}^2 + u_{xxxx} + 3u_{yy} - 4u_{tx} = 0 \tag{2.72}$$

or

$$\partial_x \left[3u_x^2 + u_{xxx} - 4u_t\right] + 3u_{yy} = 0 \tag{2.73}$$

which is called the KP equation. The KP equation is partial differential equation that describes nonlinear wave motion in two spatial dimensions, x and y. This is one of the infinitely many differential equations one can expect to find in the 1^{st} Differential Fay Identity after setting certain conditions to the identity.

CHAPTER III

2^{nd} AND 3^{rd} DIFFERENTIAL FAY IDENTITIES

In this chapter, we will take a similar approach that we utilized in chapter II. We will use the same process on 2^{nd} and 3^{rd} Differential Fay identity from Figure 2.1. Due to time constraints, the cases in the Chapter I are beyond our scope for the time being.

3.1 2^{nd} Differential Fay Identity

Due to the long nature of the 2^{nd} Differential Fay identity, we will split the LHS of the identity into three parts denoted by color. The first section we will focus on is the first first term: $\exp\left[\left(e^{\hbar D(\lambda)}-1\right)\left(e^{\hbar D(\mu)}-1\right)\log\tau\right]$. Luckily for us, we have already done the hard work to expand this term in Chapter II. Therefore, the first term of the LHS of the 2^{nd} Differential Fay identity expands to

$$
\begin{aligned}
e^{D(\lambda)D(\mu)e_0}\Bigg[1+\Bigg[&\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)\\
&+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.\\
&\left.+D(\lambda)D(\mu)e_1\right)+\mathcal{O}\left(\hbar^3\right)\Bigg]\\
&+\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)\\
&+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.\\
&\left.+D(\lambda)D(\mu)e_1\right)+\mathcal{O}\left(\hbar^3\right)\Bigg]^2+\mathcal{O}\left(\hbar^3\right)\Bigg].
\end{aligned}
\tag{3.1}
$$

Now, we will expand the second term of the LHS of the 2^{nd} Differential Fay identity:

$\exp\left[\left(e^{\hbar D(\lambda)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau\right]$. Notice that this term is similar to the first term we expanded in Chapter II where the only difference is the term $\left(e^{-\hbar\partial_0}-1\right)$. Therefore, using the Taylor expansion of $\exp(-x)$ from (2.14) to expand the exponent part of $\exp\left[\left(e^{\hbar D(\lambda)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau\right]$ while collecting the powers of $\hbar$, we obtain the following

$$\begin{aligned}\hbar^0(-D(\lambda)\partial_0 e_0)+&\hbar^1\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right)\\ &-\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right)\\ &+\hbar^3\left(\left[\frac{1}{4!}D(\lambda)\partial_0^4-\frac{1}{2!3!}D(\lambda)^2\partial_0^3+\frac{1}{2!3!}D(\lambda)^3\partial_0^2-\frac{1}{4!}D(\lambda)^4\partial_0\right]e_0\right.\\ &\left.+\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_1\right)+\mathscr{O}\left(\hbar^4\right).\end{aligned}\tag{3.2}$$

Inserting this expansion version of the exponent into the term $\exp\left[\left(e^{\hbar D(\lambda)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau\right]$, we acquire the following term

$$\begin{aligned}\exp\Bigg[\hbar^0(-D(\lambda)\partial_0 e_0)+&\hbar\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right)\\ &-\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right)\\ &+\mathscr{O}\left(\hbar^3\right)\Bigg].\end{aligned}\tag{3.3}$$

After the previous step, we then factor out the leading order $\hbar^0$ just like we did in Chapter II and obtain

$$\begin{aligned}e^{-D(\lambda)\partial_0 e_0}\exp\Bigg[&\hbar\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right)\\ &-\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right)\\ &+\mathscr{O}\left(\hbar^3\right)\Bigg].\end{aligned}\tag{3.4}$$

If we expand this term using the Taylor expansion of $\exp(x)$, we finally obtain the desire result of

$$e^{-D(\lambda)\partial_0 e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right) \\ +\mathscr{O}\left(\hbar^3\right)\Bigg] \\ +\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right) \\ +\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg]. \tag{3.5}$$

Finally, we focus on the third and final term of the LHS of the 2^{nd} Differential Fay identity: $\exp\left[\left(e^{\hbar D(\mu)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau\right]$. Once again, this term is similar to the previous term and the one following that one where the only slight differences are the exponent terms. Therefore, we can handle the term $\exp\left[\left(e^{\hbar D(\mu)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau\right]$ similarly. Hence, it expands to

$$\hbar^0\left(-D(\mu)\partial_0 e_0\right)+\hbar^1\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\ +\hbar^3\Bigg(\left[\frac{1}{4!}D(\mu)\partial_0^4-\frac{1}{2!3!}D(\mu)^2\partial_0^3+\frac{1}{2!3!}D(\mu)^3\partial_0^2-\frac{1}{4!}D(\mu)^4\partial_0\right]e_0 \\ +\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_1\Bigg)+\mathscr{O}\left(\hbar^4\right). \tag{3.6}$$

Therefore, it becomes

$$\exp\Bigg[\hbar^0\left(-D(\mu)\partial_0 e_0\right)+\hbar\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\ +\mathscr{O}\left(\hbar^3\right)\Bigg]. \tag{3.7}$$

Then, using Taylor expansion, we get

$$e^{-D(\mu)\partial_0 e_0}\exp\Bigg[\hbar\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\ +\mathscr{O}\left(\hbar^3\right)\Bigg], \tag{3.8}$$

which finally reduces to the desire form to

$$e^{-D(\mu)\partial_0 e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\ +\mathscr{O}\left(\hbar^3\right)\Bigg] \\ +\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\ -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\ +\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg]. \tag{3.9}$$

After all our hard work of expanding this tedious equations, we can rewrite the 2^{nd} Differential Fay identity to

$$
\begin{aligned}
& e^{D(\lambda)D(\mu)e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right) \\
&\quad +\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0 \right.\\
&\quad \left. +D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad +\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right) \\
&\quad +\hbar^2\left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0 \right.\\
&\quad \left. +D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
& -\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right) \\
&\quad -\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right) \\
&\quad +\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad +\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right) \\
&\quad -\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right) \\
&\quad +\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
& +\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\
&\quad -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\
&\quad +\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad +\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right) \\
&\quad -\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right) \\
&\quad +\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad =0.
\end{aligned}
\tag{3.10}
$$

Now, collecting the powers of $\hbar$ of the identity, we obtain the following

$$
\begin{aligned}
&\hbar^0 \left[e^{D(\lambda)D(\mu)e_0} - \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} + \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0}\right] + \hbar \Bigg[e^{D(\lambda)D(\mu)e_0} \left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2 D(\mu)\right] e_0\right) - \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} \left(\frac{1}{2!}\left[D(\lambda)\partial_0^2 - D(\lambda)^2\partial_0\right] e_0\right) \\
&\quad + \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0} \left(\frac{1}{2!}\left[D(\mu)\partial_0^2 - D(\mu)^2\partial_0\right] e_0\right)\Bigg] \\
&\quad + \hbar^2 \Bigg[e^{D(\lambda)D(\mu)e_0} \left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3 + \frac{1}{2!2!}D(\lambda)^2 D(\mu)^2 + \frac{1}{3!}D(\lambda)^3 D(\mu)\right] e_0 + D(\lambda)D(\mu)e_1\right) \\
&\quad + \frac{1}{2!} e^{D(\lambda)D(\mu)e_0} \left[\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2 D(\mu)\right] e_0\right)\right]^2 \\
&\quad + \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} \left(\left[\frac{1}{3!}D(\lambda)\partial_0^3 - \frac{1}{2!2!}D(\lambda)^2\partial_0^2 + \frac{1}{3!}D(\lambda)^3\partial_0\right] e_0 + D(\lambda)\partial_0 e_1\right) \\
&\quad - \frac{1}{2!} e^{-D(\lambda)\partial_0 e_0} \left[\frac{\lambda}{\lambda-\mu}\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2 - D(\lambda)^2\partial_0\right] e_0\right)\right]^2 \\
&\quad - \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0} \left(\left[\frac{1}{3!}D(\mu)\partial_0^3 - \frac{1}{2!2!}D(\mu)^2\partial_0^2 + \frac{1}{3!}D(\mu)^3\partial_0\right] e_0 + D(\mu)\partial_0 e_1\right) \\
&\quad + \frac{1}{2!} e^{-D(\mu)\partial_0 e_0} \left[\frac{\mu}{\lambda-\mu}\left(\frac{1}{2!}\left[D(\mu)\partial_0^2 - D(\mu)^2\partial_0\right] e_0\right)\right]^2 \Bigg] \\
&\quad + \mathcal{O}\left(\hbar^3\right) \\
&\quad = 0
\end{aligned}
$$

Figure 3.1: Equation 3.10.1

Comparing the coefficients of $\hbar^0, \hbar^1$, and $\hbar^2$, we obtain the following hierarchies. For the case h^0, we have

$$
e^{D(\lambda)D(\mu)e_0} - \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} + \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0} = 0. \tag{3.11}
$$

For the case $\hbar^1$, we have

$$
\begin{aligned}
e^{D(\lambda)D(\mu)e_0} & \left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 + D(\lambda)^2(\mu)\right] e_0\right) \\
& - \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} \left(\frac{1}{2!}\left[D(\lambda)\partial_0^2 - D(\lambda)^2\partial_0\right] e_0\right) \\
& + \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0} \left(\frac{1}{2!}\left[D(\mu)\partial_0^2 - D(\mu)^2\partial_0\right] e_0\right) \\
& = 0.
\end{aligned} \tag{3.12}
$$

For the case $\hbar^2$, we have

$$
\begin{aligned}
&e^{D(\lambda)D(\mu)e_0}\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0 \\
&\quad+\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}\left[\frac{1}{3!}D(\lambda)\partial_0^3-\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0 \\
&\quad-\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}\left[\frac{1}{3!}D(\mu)\partial_0^3-\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0 \\
&\quad+\frac{1}{2!}e^{D(\lambda)D(\mu)e_0}\left[\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2+D(\lambda)^2D(\mu)\right]e_0\right)\right]^2 \\
&\quad-\frac{1}{2!}e^{-D(\lambda)\partial_0 e_0}\left[\frac{\lambda}{\lambda-\mu}\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0\right)\right]^2 \\
&\quad+\frac{1}{2!}e^{-D(\mu)\partial_0 e_0}\left[\frac{\mu}{\lambda-\mu}\left(\frac{1}{2!}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0\right)\right]^2 \\
&\quad=\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}D(\mu)\partial_0 e_1-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}D(\lambda)\partial_0 e_1 \\
&\quad-e^{D(\lambda)D(\mu)e_0}D(\lambda)D(\mu)e_1
\end{aligned}
\tag{3.13}
$$

Proposition 2. *If e_0 satisfies equation (3.11) then equation (3.12) follows.*

Proof. First, consider the equation (3.11).

$$
e^{D(\lambda)D(\mu)e_0}-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}+\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}=0. \tag{3.14}
$$

We must find a differential operator that once applied to (3.11), it will differentiate into (3.12) just like in Proposition 1 in Chapter II. Therefore, consider the differential $[D(\lambda)+D(\mu)-\partial_0]$. Applying this differential operator to both sides of equation (3.11)

$$
\begin{aligned}
&[D(\lambda)+D(\mu)-\partial_0]\left(e^{D(\lambda)D(\mu)e_0}-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}+\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}\right) \\
&\quad=[D(\lambda)+D(\mu)-\partial_0]\cdot 0
\end{aligned}
\tag{3.15}
$$

yields

$$
\begin{aligned}
e^{D(\lambda)D(\mu)e_0}&[D(\lambda)+D(\mu)-\partial_0]D(\lambda)D(\mu)e_0 \\
&-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}[D(\lambda)+D(\mu)-\partial_0](-D(\lambda)\partial_0 e_0) \\
&+\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}[D(\lambda)+D(\mu)-\partial_0](-D(\mu)\partial_0 e_0) \\
&=0
\end{aligned} \tag{3.16}
$$

after applying the differential operator. Notice that the RHS of the equation becomes 0 because any sort of differential operation that differentiates a constant, i.e., 0, will be 0. Then distributing the terms $D(\lambda)D(\mu)e_0$, $-D(\lambda)\partial_0 e_0$, and $-D(\mu)\partial_0 e_0$ without the e_0 results in

$$
\begin{aligned}
e^{D(\lambda)D(\mu)e_0}&\left[D(\lambda)^2D(\mu)+D(\lambda)D(\mu)^2-D(\lambda)D(\mu)\partial_0\right]e_0 \\
&-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0-D(\lambda)D(\mu)\partial_0\right]e_0 \\
&+\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0-D(\lambda)D(\mu)\partial_0\right]e_0 \\
&=0.
\end{aligned} \tag{3.17}
$$

Now, collecting the common terms of $\left[D(\lambda)^2D(\mu)+D(\lambda)D(\mu)^2\right]e_0$, $\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0$, $\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0$, and $D(\lambda)D(\mu)\partial_0 e_0$ yields

$$
\begin{aligned}
e^{D(\lambda)D(\mu)e_0}&\left[D(\lambda)^2D(\mu)+D(\lambda)D(\mu)^2\right]e_0 \\
&-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}\left[D(\lambda)\partial_0^2-D(\lambda)^2\partial_0\right]e_0 \\
&+\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}\left[D(\mu)\partial_0^2-D(\mu)^2\partial_0\right]e_0 \\
&-\left[e^{D(\lambda)D(\mu)e_0}-\frac{\lambda}{\lambda-\mu}e^{-D(\lambda)\partial_0 e_0}+\frac{\mu}{\lambda-\mu}e^{-D(\mu)\partial_0 e_0}\right]D(\lambda)D(\mu)\partial_0 e_0 \\
&=0.
\end{aligned} \tag{3.18}
$$

Notice that from (3.18) that we have a known term, that is,

$$e^{D(\lambda)D(\mu)e_0} - \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} + \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0} \tag{3.19}$$

which equals to 0 by our assumption. Therefore, (3.18) is reduced too

$$\begin{aligned} & e^{D(\lambda)D(\mu)e_0} \left[D(\lambda)^2 D(\mu) + D(\lambda)D(\mu)^2\right] e_0 \\ & \quad - \frac{\lambda}{\lambda-\mu} e^{-D(\lambda)\partial_0 e_0} \left[D(\lambda)\partial_0^2 - D(\lambda)^2 \partial_0\right] e_0 \\ & \quad + \frac{\mu}{\lambda-\mu} e^{-D(\mu)\partial_0 e_0} \left[D(\mu)\partial_0^2 - D(\mu)^2 \partial_0\right] e_0 \\ & \quad = 0. \end{aligned} \tag{3.20}$$

Hence, Proposition 2 is proved. □

Conjecture 2. *We claim that if e_g satisfies every hierarchy associated with an even power of $\hbar$ in the 2^{nd} Differential Fay identity then the hierarchy associated with the next odd power of $\hbar$ follows.*

3.2 3^{rd} Differential Fay Identity

In order final section of Chapter III, we will be focusing on our final Differential Fay Identity. Due to its long structure, we will split the LHS of the identity into two. The first section we will focus on is the first first term: $\exp\left[\left(e^{\hbar D(\lambda)}-1\right)\left(e^{-\hbar D(\mu)}-1\right)\log\tau\right]$. This term is similar to the first terms encounter in Chapter II and the first section of this Chapter. The only small difference between term that expands to equation (3.1) is the negative sign. Therefore, the exponent part of the first term of the LHS of the 3^{rd} Differential Fay identity expands to

$$\begin{aligned}
-\hbar^0\left(D(\lambda)D(\mu)e_0\right)+\hbar^1\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2-D(\lambda)^2D(\mu)\right]e_0\right)& \\
+\hbar^2\left(-\left[\frac{1}{3!}D(\lambda)D(\mu)^3-\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.& \\
\left.-D(\lambda)D(\mu)e_1\right)& \\
+\hbar^3\left(\left[\frac{1}{4!}D(\lambda)D(\mu)^4-\frac{1}{2!3!}D(\lambda)^2D(\mu)^3+\frac{1}{2!3!}D(\lambda)^3D(\mu)^2\right.\right.& \\
\left.\left.-\frac{1}{4!}D(\lambda)^4D(\mu)\right]e_0+\frac{1}{2!}\left[D(\lambda)D(\mu)^2-D(\lambda)^2D(\mu)\right]e_1\right)& \\
+\mathscr{O}\left(\hbar^4\right).&
\end{aligned} \tag{3.21}$$

Therefore, factoring out the leading term and applying the Taylor coefficients yields

$$\begin{aligned}
e^{-D(\lambda)D(\mu)e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2-D(\lambda)^2D(\mu)\right]e_0\right)& \\
+\hbar^2\left(-\left[\frac{1}{3!}D(\lambda)D(\mu)^3-\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.& \\
\left.-D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]& \\
+\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2-D(\lambda)^2D(\mu)\right]e_0\right)& \\
+\hbar^2\left(-\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0\right.& \\
\left.-D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg].&
\end{aligned} \tag{3.22}$$

Now we will expand the final term of the 3^{rd} Differential Fay identity:

$$\exp\left(\left[\left(e^{\hbar D(\lambda)}-1\right)\left(e^{\hbar\partial_0}-1\right)+\left(e^{-\hbar D(\mu)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\right.\right.$$
$$\left.\left.-\left(e^{\hbar\partial_0}-1\right)\left(e^{-\hbar\partial_0}-1\right)\right]\log\tau\right). \tag{3.23}$$

Since this term is too long to expand all together, we will split the exponent part it into three sections denoted by color. First, let us expand the first term of the exponent of (3.23): $\left(e^{\hbar D(\lambda)}-1\right)\left(e^{\hbar\partial_0}-1\right)\log\tau$. Fortunately for us, this term is very similar to the term that expands equation (3.2) in Section II. Therefore, this expands to

$$\hbar^0\left(D(\lambda)\partial_0 e_0\right)+\hbar^1\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right]e_0\right) \tag{3.24}$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right)$$
$$+\hbar^3\left(\left[\frac{1}{4!}D(\lambda)\partial_0^4+\frac{1}{2!3!}D(\lambda)^2\partial_0^3+\frac{1}{2!3!}D(\lambda)^3\partial_0^2+\frac{1}{4!}D(\lambda)^4\partial_0\right]e_0\right.$$
$$\left.+\frac{1}{2!}\left[D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right]e_1\right)+\mathscr{O}\left(\hbar^4\right).$$

Similarly, we can expand the next term in the exponent of (3.23): $\left(e^{-\hbar D(\mu)}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau$. This yields the following expansion

$$\hbar^0\left(D(\mu)\partial_0 e_0\right)-\hbar^1\left(\frac{1}{2!}\left[D(\mu)\partial_0^2+D(\mu)^2\partial_0\right]e_0\right) \tag{3.25}$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right)$$
$$-\hbar^3\left(\left[\frac{1}{4!}D(\mu)\partial_0^4+\frac{1}{2!3!}D(\mu)^2\partial_0^3+\frac{1}{2!3!}D(\mu)^3\partial_0^2+\frac{1}{4!}D(\mu)^4\partial_0\right]e_0\right.$$
$$\left.+\frac{1}{2!}\left[D(\mu)\partial_0^2+D(\mu)^2\partial_0\right]e_1\right)+\mathscr{O}\left(\hbar^4\right).$$

Finally, we can expand the last term in the exponent of (3.23): $\left(e^{\hbar\partial_0}-1\right)\left(e^{-\hbar\partial_0}-1\right)\log\tau$. This yields the following expansion

$$-\hbar^0\left(\partial_0^2 e_0\right)+\hbar^1\left(\frac{1}{2!}\left[\partial_0^3-\partial_0^3\right]e_0\right) \tag{3.26}$$
$$-\hbar^2\left(\left[\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right]e_0+\partial_0^2 e_1\right)$$
$$+\hbar^3\left(\left[\frac{1}{4!}\partial_0^5-\frac{1}{2!3!}\partial_0^5+\frac{1}{2!3!}\partial_0^5-\frac{1}{4!}\partial_0^5\right]e_0+\frac{1}{2!}\left[\partial_0^3-\partial_0^3\right]e_1\right)$$
$$-\hbar^4\left(\left[\frac{1}{5!}\partial_0^6-\frac{1}{2!4!}\partial_0^6+\frac{1}{3!3!}\partial_0^6-\frac{1}{2!4!}\partial_0^6+\frac{1}{5!}\partial_0^6\right]e_0\right.$$
$$\left.+\left[\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right]e_1+\partial_0^2 e_2\right)+\mathscr{O}\left(\hbar^5\right).$$

Notice that the odd powers of $\hbar$ in (3.26) are all zero. Consequently, equation (3.23) becomes

$$\exp\left(\hbar^0\left(D(\lambda)\partial_0 e_0\right)+\hbar^1\left(\frac{1}{2!}\left[D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right]e_0\right)\right. \tag{3.27}$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right]e_0+D(\lambda)\partial_0 e_1\right)$$
$$+\hbar^3\left(\left[\frac{1}{4!}D(\lambda)\partial_0^4+\frac{1}{2!3!}D(\lambda)^2\partial_0^3+\frac{1}{2!3!}D(\lambda)^3\partial_0^2+\frac{1}{4!}D(\lambda)^4\partial_0\right]e_0\right.$$
$$\left.+\frac{1}{2!}\left[D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right]e_1\right)+\mathscr{O}\left(\hbar^4\right)$$
$$+\hbar^0\left(D(\mu)\partial_0 e_0\right)-\hbar^1\left(\frac{1}{2!}\left[D(\mu)\partial_0^2+D(\mu)^2\partial_0\right]e_0\right)$$
$$+\hbar^2\left(\left[\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right]e_0+D(\mu)\partial_0 e_1\right)$$
$$-\hbar^3\left(\left[\frac{1}{4!}D(\mu)\partial_0^4+\frac{1}{2!3!}D(\mu)^2\partial_0^3+\frac{1}{2!3!}D(\mu)^3\partial_0^2+\frac{1}{4!}D(\mu)^4\partial_0\right]e_0\right.$$
$$\left.+\frac{1}{2!}\left[D(\mu)\partial_0^2+D(\mu)^2\partial_0\right]e_1\right)+\mathscr{O}\left(\hbar^4\right)$$
$$+\hbar^0\left(\partial_0^2 e_0\right)+\hbar^2\left(\left[\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right]e_0+\partial_0^2 e_1\right)$$
$$+\hbar^4\left(\left[\frac{1}{5!}\partial_0^6-\frac{1}{2!4!}\partial_0^6+\frac{1}{3!3!}\partial_0^6-\frac{1}{2!4!}\partial_0^6+\frac{1}{5!}\partial_0^6\right]e_0\right.$$
$$\left.\left.+\left[\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right]e_1+\partial_0^2 e_2\right)+\mathscr{O}\left(\hbar^6\right)\right).$$

Now, collecting the powers of $\hbar$ in the exponent of equation (3.27), equation (3.27) becomes

$$\exp\Bigg[\hbar^0\left(\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0\right)+\hbar\left(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right)\right. \tag{3.28}$$

$$-\left(D(\mu)\partial_0^2+D(\mu)^2\partial_0\right)\Big]e_0$$

$$+\hbar^2\left(\left[\left(\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right)\right.\right.$$

$$+\left(\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right)$$

$$+\left(\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right)\Bigg]e_0$$

$$\left.+\left(D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg].$$

First, we factor our the leading term from equation (3.28) that yields

$$e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}\exp\Bigg[\hbar\left(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right)\right. \tag{3.29}$$

$$\left.-\left(D(\mu)\partial_0^2+D(\mu)^2\partial_0\right)\Big]e_0\right)$$

$$\hbar^2\left(\left[\left(\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right)\right.\right.$$

$$+\left(\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right)$$

$$+\left(\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right)\Bigg]e_0$$

$$\left.+\left(D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg].$$

Then using the Taylor expansion of $\exp(x)$ on (3.29) yields

$$
\begin{aligned}
e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}\Bigg[1+\Bigg[&\hbar\left(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right)\right. \\
&\left.-\left(D(\mu)\partial_0^2+D(\mu)^2\partial_0\right)\Big]e_0\right) \\
&\hbar^2\left(\Bigg[\left(\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right)\right. \\
&+\left(\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right) \\
&+\left(\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right)\Bigg]e_0 \\
&\left.+\left(D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&+\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right)\right. \\
&\left.-\left(D(\mu)\partial_0^2+D(\mu)^2\partial_0\right)\Big]e_0\right) \\
&\hbar^2\left(\Bigg[\left(\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right)\right. \\
&+\left(\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right) \\
&+\left(\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right)\Bigg]e_0 \\
&\left.+\left(D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]^2 \\
&+\mathscr{O}\left(\hbar^3\right)\Bigg].
\end{aligned}
\tag{3.30}
$$

Finally, inputting equations (3.22) and (3.30) into 3^{rd} Differential Fay identity, we get this enormous equation

$$
\begin{aligned}
& e^{-D(\lambda)D(\mu)e_0}\Bigg[1+\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2-D(\lambda)^2D(\mu)\right]e_0\right) \\
&\quad +\hbar^2\left(-\left[\frac{1}{3!}D(\lambda)D(\mu)^3-\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0 \right.\\
&\quad \left. -D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad +\frac{1}{2!}\Bigg[\hbar\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2-D(\lambda)^2D(\mu)\right]e_0\right) \\
&\quad +\hbar^2\left(-\left[\frac{1}{3!}D(\lambda)D(\mu)^3+\frac{1}{2!2!}D(\lambda)^2D(\mu)^2+\frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0 \right.\\
&\quad \left. -D(\lambda)D(\mu)e_1\right)+\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad -1 \\
& +\frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}\Bigg[1+\Bigg[\hbar\Bigg(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right) \\
&\quad -\left(D(\mu)\partial_0^2+D(\mu)^2\partial_0\right)\Big]e_0\Bigg) \\
&\quad \hbar^2\Bigg(\Bigg[\left(\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right) \\
&\quad +\left(\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right) \\
&\quad +\left(\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right)\Bigg]e_0 \\
&\quad +\left(D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right)e_1\Bigg)+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad +\frac{1}{2!}\Bigg[\hbar\Bigg(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2+D(\lambda)^2\partial_0\right) \\
&\quad -\left(D(\mu)\partial_0^2+D(\mu)^2\partial_0\right)\Big]e_0\Bigg) \\
&\quad \hbar^2\Bigg(\Bigg[\left(\frac{1}{3!}D(\lambda)\partial_0^3+\frac{1}{2!2!}D(\lambda)^2\partial_0^2+\frac{1}{3!}D(\lambda)^3\partial_0\right) \\
&\quad +\left(\frac{1}{3!}D(\mu)\partial_0^3+\frac{1}{2!2!}D(\mu)^2\partial_0^2+\frac{1}{3!}D(\mu)^3\partial_0\right) \\
&\quad +\left(\frac{1}{3!}\partial_0^4-\frac{1}{2!2!}\partial_0^4+\frac{1}{3!}\partial_0^4\right)\Bigg]e_0 \\
&\quad +\left(D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right)e_1\Bigg)+\mathscr{O}\left(\hbar^3\right)\Bigg]^2+\mathscr{O}\left(\hbar^3\right)\Bigg] \\
&\quad =0.
\end{aligned} \tag{3.31}
$$

After our extremely tedious hard work, collecting the terms of $\hbar^0, \hbar^1, \hbar^2$ in the equation (3.31) yields in

$$
\begin{aligned}
\hbar^0 \left[e^{-D(\lambda)D(\mu)e_0} - 1 + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0}\right] + \hbar \Bigg[& e^{-D(\lambda)D(\mu)e_0} \left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 - D(\lambda)^2 D(\mu)\right] e_0\right) \\
& + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} \left(\frac{1}{2!}\left[\left(D(\lambda)\partial_0^2 + D(\lambda)^2\partial_0\right) - \left(D(\mu)\partial_0^2 + D(\mu)^2\partial_0\right)\right] e_0\right)\Bigg] \\
& + \hbar^2 \Bigg[-e^{-D(\lambda)D(\mu)e_0} \left(\left[\frac{1}{3!}D(\lambda)D(\mu)^3 - \frac{1}{2!2!}D(\lambda)^2 D(\mu)^2 + \frac{1}{3!}D(\lambda)^3 D(\mu)\right] e_0 + D(\lambda)D(\mu)e_1\right) \\
& + \frac{1}{2!} e^{-D(\lambda)D(\mu)e_0} \left[\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 - D(\lambda)^2 D(\mu)\right] e_0\right)\right]^2 \\
& + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} \Bigg(\Bigg[\left(\frac{1}{3!}D(\lambda)\partial_0^3 + \frac{1}{2!2!}D(\lambda)^2\partial_0^2 + \frac{1}{3!}D(\lambda)^3\partial_0\right) \\
& + \left(\frac{1}{3!}D(\mu)\partial_0^3 + \frac{1}{2!2!}D(\mu)^2\partial_0^2 + \frac{1}{3!}D(\mu)^3\partial_0\right) + \left(\frac{1}{3!}\partial_0^4 - \frac{1}{2!2!}\partial_0^4 + \frac{1}{3!}\partial_0^4\right)\Bigg] e_0 \\
& + \left(D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right) e_1\Bigg) \\
& + \frac{1}{2!}\frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} \left[\left(\frac{1}{2!}\left[\left(D(\lambda)\partial_0^2 + D(\lambda)^2\partial_0\right) - \left(D(\mu)\partial_0^2 + D(\mu)^2\partial_0\right)\right] e_0\right)\right]^2\Bigg] \\
& + \mathcal{O}\left(\hbar^3\right) \\
& = 0
\end{aligned}
$$

Figure 3.2: Equation 3.31.1

Now, comparing the coefficients of $\hbar^0, \hbar^1$, and $\hbar^2$ we obtain the following hierarchies. For the case h^0, we have

$$
e^{-D(\lambda)D(\mu)e_0} + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} - 1 \quad = \quad 0. \tag{3.32}
$$

For the case h^1, we have

$$
\begin{aligned}
e^{-D(\lambda)D(\mu)e_0} & \left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 - D(\lambda)^2 D(\mu)\right] e_0\right) \\
& + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} \Bigg(\frac{1}{2!}\Big[\left(D(\lambda)\partial_0^2 + D(\lambda)^2\partial_0\right) \\
& - \left(D(\mu)\partial_0^2 + D(\mu)^2\partial_0\right)\Big] e_0\Bigg) \\
& = 0.
\end{aligned} \tag{3.33}
$$

For the case h^2, we have

$$
\begin{aligned}
\frac{1}{2!}e^{-D(\lambda)D(\mu)e_0}\left[\left(\frac{1}{2!}\left[D(\lambda)D(\mu)^2 - D(\lambda)^2D(\mu)\right]e_0\right)\right]^2 &- e^{-D(\lambda)D(\mu)e_0}\left[\frac{1}{3!}D(\lambda)D(\mu)^3 - \frac{1}{2!2!}D(\lambda)^2D(\mu)^2 + \frac{1}{3!}D(\lambda)^3D(\mu)\right]e_0 \\
&+ \frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}\left[\left(\frac{1}{3!}D(\lambda)\partial_0^3 + \frac{1}{2!2!}D(\lambda)^2\partial_0^2 + \frac{1}{3!}D(\lambda)^3\partial_0\right)\right. \\
&\left. + \left(\frac{1}{3!}D(\mu)\partial_0^3 + \frac{1}{2!2!}D(\mu)^2\partial_0^2 + \frac{1}{3!}D(\mu)^3\partial_0\right) + \left(\frac{1}{3!}\partial_0^4 - \frac{1}{2!2!}\partial_0^4 + \frac{1}{3!}\partial_0^4\right)\right]e_0 \\
&+ \frac{1}{2!}\frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}\left[\left(\frac{1}{2!}\left[\left(D(\lambda)\partial_0^2 + D(\lambda)^2\partial_0\right) - \left(D(\mu)\partial_0^2 + D(\mu)^2\partial_0\right)\right]e_0\right)\right]^2 \\
&= e^{-D(\lambda)D(\mu)e_0}D(\lambda)D(\mu)e_1 - \frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}\left(D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right)e_1
\end{aligned}
$$

Figure 3.3: Equation 3.33.1

Proposition 3. *If e_0 satisfies equation (3.32) then equation (3.33) follows.*

Proof. First, consider the equation (3.32).

$$
e^{-D(\lambda)D(\mu)e_0} + \frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0} - 1 = 0. \tag{3.34}
$$

Again, we must find a differential operator that once applied to (3.34), it will differentiate into (3.33). Therefore, consider the differential $[D(\lambda) - D(\mu)]$. Applying this differential operator to both sides of equation (3.34)

$$
\begin{aligned}
[D(\lambda) - D(\mu)]\left(e^{-D(\lambda)D(\mu)e_0} + \frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0} - 1\right) \\
= [D(\lambda) - D(\mu)] \cdot 0
\end{aligned} \tag{3.35}
$$

yields

$$
\begin{aligned}
e^{-D(\lambda)D(\mu)e_0}&[D(\lambda) - D(\mu)]\left(-D(\lambda)D(\mu)e_0\right) \\
&+ \frac{1}{\lambda\mu}e^{\left[D(\lambda)\partial_0+D(\mu)\partial_0+\partial_0^2\right]e_0}[D(\lambda) - D(\mu)]\left(\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0\right) \\
&= 0
\end{aligned} \tag{3.36}
$$

Then distributing the terms $-D(\lambda)D(\mu)e_0$ and $\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right] e_0$ without the e_0 results in

$$
\begin{aligned}
e^{-D(\lambda)D(\mu)e_0} & \left[D(\lambda)D(\mu)^2 - D(\lambda)^2 D(\mu)\right] e_0 \\
& + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} \Big[D(\lambda)^2\partial_0 + D(\lambda)D(\mu)\partial_0 \\
& + D(\lambda)\partial_0^2 - D(\mu)D(\lambda)\partial_0 - D(\mu)^2\partial_0 - D(\mu)\partial_0^2 \Big] e_0 \\
& = 0.
\end{aligned} \tag{3.37}
$$

Now, using the same assumption we made in Proposition 1 in Chapter II, equation (3.37) becomes

$$
\begin{aligned}
e^{-D(\lambda)D(\mu)e_0} & \left(\left[D(\lambda)D(\mu)^2 - D(\lambda)^2 D(\mu)\right] e_0\right) \\
& + \frac{1}{\lambda\mu} e^{\left[D(\lambda)\partial_0 + D(\mu)\partial_0 + \partial_0^2\right]e_0} \Big(\Big[\left(D(\lambda)\partial_0^2 + D(\lambda)^2\partial_0\right) \\
& - \left(D(\mu)\partial_0^2 + D(\mu)^2\partial_0\right) \Big] e_0 \Big) \\
& = 0.
\end{aligned} \tag{3.38}
$$

Hence, Proposition 3 is proved. □

Conjecture 3. *We claim that if e_g satisfies every hierarchy associated with an even power of $\hbar$ in the 3^{rd} Differential Fay identity then the hierarchy associated with the next odd power of $\hbar$ follows.*

CHAPTER IV

RESULTS AND DISCUSSION

4.1 Conclusion

We have derived differential equations which govern the behavior of e_0 in all time variables. These were previously unknown as previous studies have focused on single time parameters, or on the full expression of $\log\tau$ rather than individual terms of the expansion [5]. These equations thus enable us to understand the terms in the Taylor expansions for e_0 coming from multiplying time parameters such as t_{4^j}, t_{6^k}, giving the number of genus 0 ribbon graphs with j vertices of degree 4 and k vertices of degree 6.

In the process, we have also explained a number of the symmetries in the Differential Fay identities. Of particular importance is proposition 1, 2 and 3 in which we showed that an asymptotic expansion of $\log\tau$ in even powers of the Planck constant, $\hbar$, is supported by the identities.

Furthermore, it remains to be seen if it is possible to derive from the equations we have derived a closed form expression for e_0 or its Taylor coefficients. If such solution could be found, it would constitute a solution of the original combinatoric problem we started from.

4.2 Future Work

Here are some of the future goals we would like to achiece in the near future. The main goal we would strive to work on is proving the three conjectures that were mention in Chapter II and Chapter III. Following this, we would like to formulate a general formula that will generate the hierarchy of equations that we encountered in the Differential Fay Identities under the general case $\hbar \to 0$, as well in the case of $\mu \to \lambda$ and $\mu \to \infty$. Finally, we would like to explore what happens to the 2^{nd} and 3^{rd} Differential Fay Identities when we apply the two conditions we set on the 1^{st}

Differential Fay Identity in Chapter II.

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BIOGRAPHICAL SKETCH

Indalecio Soto Jr., born June 10, 1990, grew up in Mission, Tx, attending Mission High school graduating top 10% in 2008. He attended the University of Texas-Pan American where he graduated in 2012 with a Bachelors of Science in Applied Mathematics. He went on to earn his Masters of Science in Applied Mathematics from the University of Texas-Pan American in 2014. Professionally he served as a tutor at the university, and later became a graduate teaching assistant during the GAANN Fellowship. While there he was an active member in several organization related to the Math department which includes the Society for Industrial and Applied Mathematics (Treasurer), the Society for the Advancement of Chicanos and Native Americans in Science, Wiener's Society of Mathematicians, and the Society for Optics and Photonics. His permanent mailing address is 4405 Fudge Street, Edinburg, TX, 78542.