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Gupta, Ramanujan, Dyson and Ehrhart: Formulas for Partition Functions, Congruences, Cranks, and Polyhedral Geometry

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GUPTA, RAMANUJAN, DYSON AND EHRHART: FORMULAS FOR PARTITION FUNCTIONS, CONGRUENCES, CRANKS, AND POLYHEDRAL GEOMETRY

A Thesis

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ABSTRACT

Rodriguez, Joselyne, Gupta, Ramanujan, Dyson and Ehrhart: Formulas for Partition Functions, Congruences, Cranks, and Polyhedral Geometry. Master of Science (MS), May, 2021, 30 pp., 4 tables, 7 figures, 9 references.

We will revisit Gupta's result regarding properties of a formula for restricted partitions and generalize this. We will then use this result to prove an infinite family of congruences for a certain restricted partition function. We find and prove combinatorial witnesses, also known as cranks, for the congruences using polyhedral geometry.

DEDICATION

I dedicate this thesis to my family. A very special thank you to my mother and grandmother, Aracely Cantu and Maria Elizondo, for always encouraging me and going with my eccentric ideas, I love you very much. My sister, Yadira, I am beyond lucky to have you in my life; thank you for being my confidant. Sonia and Frank Rios, for taking me into your family and never giving me the idea that I can't reach any of my goals.

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And most importantly, I dedicate this work to my husband, Francisco Aniceto, whose support knows no bounds. I can't thank you enough for believing in me; I will always appreciate it. Thank you for never leaving my side. I love you more than zero to one.

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CHAPTER I

INTEGER PARTITIONS

The function $p(n)$ counts the amount of ways there are to write an integer n as a sum of positive integers, [8]. Partition functions can be altered so that we only focus on a certain number of parts or a specific set of part sizes. The study of partitions has been influenced by many mathematicians such as Gupta, Ramanujan, Dyson and Ehrhart. We begin by looking at partition functions with specific sets of part sizes, their corresponding generating functions, quasipolynomials, and congruences that can be seen from the quasipolynomials. We will then go on to show how those congruences can be proven using polyhedral geometry and Ehrhart theory.

Definition 1. *[1] A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$, such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition.*

Partitions are how we write positive integers as a sum of positive integers. For example, there are five partitions of 4:

Table 1.1: $p(4) = 5$

Partitions of 4
4
3+1
2+2
2+1+1
1+1+1+1

Definition 2. *[1] A generating function is a power series designed to keep track of number sequences. If $A = \{a_n\}_{n=b}^{\infty} = \{a_b, a_{b+1}, \ldots\}$ is a sequence of real numbers, then*

$$f_A(q) = \sum_{n=b}^{\infty} a_n q^n$$

is called the generating function for the sequence A.

We provide a quick bit of information from Calculus II that is useful for generating functions. For $|q| < 1$:

$$\frac{1}{1-q^j} = \sum_{n=0}^{\infty} q^{nj}.$$

We use the standard q-rising factorial notation throughout.

$$(q;q)_d = \prod_{i=1}^{d}(1-q^i)$$

The generating function for the general partition function, $p(n)$, is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} = \frac{1}{(q;q)_\infty}.$$

The generating function for partitions of n into parts of sizes 1 through m, $p(n,m)$, is

$$\sum_{n=0}^{\infty} p(n,m)q^n = \prod_{i=1}^{m} \frac{1}{1-q^i} = \frac{1}{(q;q)_m}.$$

Example 1. *Here is the generating function for the partitions of n into at most four parts, denoted by $p(n,4)$.*

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n = \frac{1}{(q;q)_4} &= p(0,4)+p(1,4)q+p(2,4)q^2+p(3,4)q^3+p(4,4)q^4+p(5,4)q^5+\cdots \\ &= 1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}+\cdots\end{aligned}$$

The coefficients of q correspond to the amount of partitions of the number given by the exponent on

q.

$p(0,4)=1$	$p(12,4)=34$	$p(24,4)=169$
$p(1,4)=1$	$p(13,4)=39$	$p(25,4)=185$
$p(2,4)=2$	$p(14,4)=47$	$p(26,4)=206$
$p(3,4)=3$	$p(15,4)=54$	$p(27,4)=225$
$p(4,4)=5$	$p(16,4)=64$	$p(28,4)=249$
$p(5,4)=6$	$p(17,4)=72$	$p(29,4)=270$
$p(6,4)=9$	$p(18,4)=84$	$p(30,4)=297$
$p(7,4)=11$	$p(19,4)=94$	$p(31,4)=321$
$p(8,4)=15$	$p(20,4)=108$	$p(32,4)=351$
$p(9,4)=18$	$p(21,4)=120$	$p(33,4)=378$
$p(10,4)=23$	$p(22,4)=136$	$p(34,4)=411$
$p(11,4)=27$	$p(23,4)=150$	$p(35,4)=441...$

There are formulas for $p(n,4)$.

$$
\begin{aligned}
p(12k,4) &= 1\binom{k+3}{3}+30\binom{k+2}{3}+39\binom{k+1}{3}+2\binom{k}{3} = && 12k^3+15k^2+6k+1\\
p(12k+1,4) &= 1\binom{k+3}{3}+35\binom{k+2}{3}+35\binom{k+1}{3}+1\binom{k}{3} = && 12k^3+18k^2+8k+1\\
p(12k+2,4) &= 2\binom{k+3}{3}+39\binom{k+2}{3}+30\binom{k+1}{3}+1\binom{k}{3} = && 12k^3+21k^2+12k+2\\
p(12k+3,4) &= 3\binom{k+3}{3}+42\binom{k+2}{3}+27\binom{k+1}{3} = && 12k^3+24k^2+15k+3\\
p(12k+4,4) &= 5\binom{k+3}{3}+44\binom{k+2}{3}+23\binom{k+1}{3} = && 12k^3+27k^2+20k+5\\
p(12k+5,4) &= 6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3} = && 12k^3+30k^2+24k+6\\
p(12k+6,4) &= 9\binom{k+3}{3}+48\binom{k+2}{3}+15\binom{k+1}{3} = && 12k^3+33k^2+30k+9\\
p(12k+7,4) &= 11\binom{k+3}{3}+50\binom{k+2}{3}+11\binom{k+1}{3} = && 12k^3+36k^2+35k+11\\
p(12k+8,4) &= 15\binom{k+3}{3}+48\binom{k+2}{3}+9\binom{k+1}{3} = && 12k^3+39k^2+42k+15\\
p(12k+9,4) &= 18\binom{k+3}{3}+48\binom{k+2}{3}+6\binom{k+1}{3} = && 12k^3+42k^2+48k+18\\
p(12k+10,4) &= 23\binom{k+3}{3}+44\binom{k+2}{3}+5\binom{k+1}{3} = && 12k^3+45k^2+56k+23\\
p(12k+11,4) &= 27\binom{k+3}{3}+42\binom{k+2}{3}+3\binom{k+1}{3} = && 12k^3+48k^2+63k+27 \qquad (1.1)
\end{aligned}
$$

A collection of formulas like this is called a quasipolynomial.

Definition 3. *A function $f(n)$ is a quasipolynomial if there exists polynomials $f_0(n), f_1(n), \ldots, f_{d-1}(n)$, called constituents such that for all $n \in \mathbb{Z}$*

$$
f(n) = \begin{cases} f_0(n) & \text{if } n \equiv 0 \pmod d \\ f_1(n) & \text{if } n \equiv 1 \pmod d \\ \vdots & \\ f_{d-1}(n) & \text{if } n \equiv d-1 \pmod d \end{cases}
$$

The period of the quasipolynomial is the number of constituents.

We will show how to obtain these formulas for $p(n,m)$ in Chapter II, where we will continue using generating functions and their corresponding rational functions and quasipolynomials to generalize one of Hansraj Gupta's result from 1975.

CHAPTER II

GENERALIZING GUPTA'S RESULT

We begin by highlighting an interesting result seen in the quasipolynomial for $p(n,4)$ in (1.1).

$$\begin{aligned}
p(12k,4) &= 1 + 30 + 39 + 2 = 72\\
p(12k+1,4) &= 1 + 35 + 35 + 1 = 72\\
p(12k+2,4) &= 2 + 39 + 30 + 1 = 72\\
p(12k+3,4) &= 3 + 42 + 27 = 72\\
p(12k+4,4) &= 5 + 44 + 23 = 72\\
p(12k+5,4) &= 6 + 48 + 18 = 72\\
p(12k+6,4) &= 9 + 48 + 15 = 72\\
p(12k+7,4) &= 11 + 50 + 11 = 72\\
p(12k+8,4) &= 15 + 48 + 9 = 72\\
p(12k+9,4) &= 18 + 48 + 6 = 72\\
p(12k+10,4) &= 23 + 44 + 5 = 72\\
p(12k+11,4) &= 27 + 42 + 3 = 72
\end{aligned} \tag{2.1}$$

The sum of the coefficients of the constituents in this quasipolynomial is 72! In *A Technique in Partitions* [7], Hansraj Gupta provides a formula for finding the sum of the coefficients of the constituents of a quasipolynomial for partition functions. This formula was dependent on there being a part of size one in the set. Our focus will be on redoing Gupta's result so that we may use the most general set of part sizes, without parts of size one. We state Gupta's result:

Theorem 1. *[7] Let $A = \{a_1, a_2, ..., a_m\}$ be a nonempty set of m natural numbers with 1 as an element. The least common multiple of these numbers is given by:* $\operatorname{lcm}(A)$. *For the set A and $0 \leq r < lcm(A)$,*

$$\sum_{k\geq 0} h^*_{\operatorname{lcm}(A)k+r} = \frac{\operatorname{lcm}(A)^{m-1}}{a_1 a_2 ... a_m}$$

The least common multiple of A is the least common multiple of the numbers in the set. For example: The least common multiple of $A = \{1,2,3,4\}$ is $\operatorname{lcm}(A) = 12$.

Theorem 1 is Gupta's result, which requires a part of size one to be in the set. The next theorem is the generalized version of this result, in which we are allowed to choose parts of any size. The proof of this theorem is also an iterative one like that of Gupta's, with some extra steps to make up for the fact that we are not including 1 in our set of part sizes.

Theorem 2. *Let S be a finite collection of natural numbers with repetitions allowed. Let $k \in \mathbb{Z}$ and $0 \leq r < \operatorname{lcm}(S)$. Let h^*_j be a coefficient of a constituent expressed in the binomial basis of the quasipolynomial dependent of the set S. Then,*

$$\sum_{k\geq 0} h^*_{\operatorname{lcm}(S)k+r} = \frac{\operatorname{lcm}(S)^{|S|-1}}{\prod_{d=1}^{|S|-1} \operatorname{lcm}(\gcd(\{s_i\}_{i=1}^{d}), s_{d+1})}. \tag{2.2}$$

Remark 1. *The coefficients h^* in 2.2 will be revisited in Chapter V within the context of polyhedral geometry.*

Definition 4. *$p(n,S)$ is the function that counts the number of partitions of n into parts that come from the finite set S.*

Because a certain polynomial is so useful, we state it as a definition below.

Definition 5. *Let S be any set of positive integers. The polynomial $E_S(q)$ is defined to be*

$$E_S(q) = \prod_{i=1}^{|S|} \left(\sum_{n=0}^{\frac{\operatorname{lcm}(S)}{i}-1} q^{i\cdot n} \right) = \sum_{x=0}^{|S|\cdot\operatorname{lcm}(S)-\sum_{i=1}^{|S|} s_i} h^*_x q^x. \tag{2.3}$$

Equivalently, $E_S(q)$ is the polynomial such that,

$$E_S(q) \times (q;q)_S = (1-q^{\mathrm{lcm}(S)})^{|S|}. \tag{2.4}$$

Example 2. *Let $S = \{2,4,4,5\}$, then $\mathrm{lcm}(S) = 20$. The generating function is*

$$\sum_{n\geq 0} p(n|S)q^n = \frac{1}{(1-q^2)(1-q^4)^2(1-q^5)} = \frac{E_S(q)}{E_S(q)(1-q^2)(1-q^4)^2(1-q^5)}.$$

$E_S(q) = (1+q^5+q^{10}+q^{15})(1+q^4+q^8+q^{12}+q^{16})^2(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}+q^{14}+q^{16}+q^{18})$.

$$\sum_{n\geq 0} p(n|S)q^n = \frac{1}{(1-q^2)(1-q^4)^2(1-q^5)} = \frac{E_S(q)}{E_S(q)(1-q^2)(1-q^4)^2(1-q^5)}$$

$$= \frac{E_S(q)}{(1-q^{20})^4} = E_S(q) \times \sum_{k\geq 0} \binom{k+3}{3} q^{20k}$$

We have rewritten the generating function in terms of binomial coefficients. Multiplying and collecting like terms, we get:

$$p(n|S) = \begin{cases}
p(20k|S) = \quad 1\binom{k+3}{3} + 24\binom{k+2}{3} + 24\binom{k+1}{3} + 1\binom{k}{3} & p(20k+10|S) = 7\binom{k+3}{3} + 36\binom{k+2}{3} + 7\binom{k+1}{3} \\
p(20k+1|S) = \quad 0\binom{k+3}{3} + 18\binom{k+2}{3} + 29\binom{k+1}{3} + 3\binom{k}{3} & p(20k+11|S) = 3\binom{k+3}{3} + 34\binom{k+2}{3} + 13\binom{k+1}{3} \\
p(20k+2|S) = \quad 1\binom{k+3}{3} + 28\binom{k+2}{3} + 21\binom{k+1}{3} & p(20k+12|S) = 11\binom{k+3}{3} + 33\binom{k+2}{3} + 6\binom{k+1}{3} \\
p(20k+3|S) = \quad 0\binom{k+3}{3} + 21\binom{k+2}{3} + 28\binom{k+1}{3} + 1\binom{k}{3} & p(20k+13|S) = 6\binom{k+3}{3} + 33\binom{k+2}{3} + 11\binom{k+1}{3} \\
p(20k+4|S) = \quad 3\binom{k+3}{3} + 29\binom{k+2}{3} + 18\binom{k+1}{3} & p(20k+14|S) = 13\binom{k+3}{3} + 34\binom{k+2}{3} + 3\binom{k+1}{3} \\
p(20k+5|S) = \quad 1\binom{k+3}{3} + 24\binom{k+2}{3} + 24\binom{k+1}{3} + 1\binom{k}{3} & p(20k+15|S) = 7\binom{k+3}{3} + 36\binom{k+2}{3} + 7\binom{k+1}{3} \\
p(20k+6|S) = \quad 3\binom{k+3}{3} + 34\binom{k+2}{3} + 13\binom{k+1}{3} & p(20k+16|S) = 18\binom{k+3}{3} + 29\binom{k+2}{3} + 3\binom{k+1}{3} \\
p(20k+7|S) = \quad 1\binom{k+3}{3} + 28\binom{k+2}{3} + 21\binom{k+1}{3} & p(20k+17|S) = 11\binom{k+3}{3} + 33\binom{k+2}{3} + 6\binom{k+1}{3} \\
p(20k+8|S) = \quad 6\binom{k+3}{3} + 33\binom{k+2}{3} + 11\binom{k+1}{3} & p(20k+18|S) = 21\binom{k+3}{3} + 28\binom{k+2}{3} + 1\binom{k+1}{3} \\
p(20k+9|S) = \quad 3\binom{k+3}{3} + 29\binom{k+2}{3} + 18\binom{k+1}{3} & p(20k+19|S) = 13\binom{k+3}{3} + 34\binom{k+2}{3} + 3\binom{k+1}{3}
\end{cases}$$

Observe that the sum of the coefficients of the constituents is 50, in accordance with Gupta's result.

To prove this result, we will examine the Ehrhart numerator $E_S(q)$.

We set q to be 20^{th} root of unity, to organize the exponents modulo 20. We begin expanding E_S by selecting any two of the factors $\sum_{n=0}^{\frac{\mathrm{lcm}(S)}{i}-1} q^{i\cdot n}$ we wish.

$$\sum_{n=0}^{\frac{20}{2}-1} q^{2n} \times \sum_{n=0}^{\frac{20}{4}-1} q^{4n} = (1+q^2+q^4+q^6+q^8+q^{10}+q^{12}+q^{14}+q^{16}+q^{18}) \times (1+q^4+q^8+q^{12}+q^{16})$$

$$= 5(1+q^2+q^4+...+q^{18}) = \frac{20}{\mathrm{lcm}(2,4)} \sum_{n=0}^{\frac{20}{2}-1} q^{\gcd(2,4)n} = 5\sum_{n=0}^{9} q^{2n} \quad (2.5)$$

Notice how setting q to be a 20^{th} root of unity organizes the exponents so that there are no exponents higher than 19. This is an iterative process and we continue by multiplying by another factor from $E_S(q)$.

$$5 \times \sum_{n=0}^{9} q^{2n} \times \sum_{n=0}^{\frac{20}{5}-1} q^{5n} = 5 \times \frac{20}{\mathrm{lcm}(\gcd(2,4),5)} \sum_{n=0}^{\frac{20}{\gcd(\gcd(2,4),5)}-1} q^{\gcd(\gcd(2,4),5)n} = 5 \times 2 \times \sum_{n=0}^{19} q^n. \quad (2.6)$$

Our final factor from $E_S(q)$ is $\sum_{n=0}^{\frac{20}{4}-1} q^{4n}$, and we consider the product of it with the right-hand side of (2.6).

$$10 \times \sum_{n=0}^{19} q^n \times \sum_{n=0}^{\frac{20}{4}-1} q^{4n} = 10 \times \frac{20}{\mathrm{lcm}(\gcd(\gcd(2,4),5),4)} \sum_{n=0}^{\frac{20}{\gcd(\gcd(\gcd(2,4),5),4)}-1} q^{\gcd(\gcd(\gcd(2,4),5),4)n}$$

$$= 10 \times 5 \times \sum_{n=0}^{19} q^n = 50 \times \sum_{n=0}^{19} q^n \quad (2.7)$$

In the end, we see that 50 *is the coefficient of this polynomial. Theorem 2 says:*

$$\sum_{k\geq 0} h^*_{20k+r} = \frac{20}{\mathrm{lcm}(2,4)} \times \frac{20}{\mathrm{lcm}(\gcd(2,4),5)} \times \frac{20}{\mathrm{lcm}(\gcd(\gcd(2,4),5),4)} = 5 \times 2 \times 5 = 50.$$

The proof of Theorem 2 will follow this same process.

Proof of Theorem 2. We set q to be an $\mathrm{lcm}(S)^{th}$ root of unity, so that the exponents on q are automatically organized modulo $\mathrm{lcm}(S)$.

$$\sum_{n=0}^{\frac{\mathrm{lcm}(S)}{s_i}-1} q^{n\cdot s_i} \times \sum_{n=0}^{\frac{\mathrm{lcm}(S)}{s_j}-1} q^{n\cdot s_j} = \frac{\mathrm{lcm}(S)}{\mathrm{lcm}(s_i,s_j)} \times \sum_{n=0}^{\frac{\mathrm{lcm}(S)}{\gcd(s_i,s_j)}-1} q^{n\cdot \gcd(s_i,s_j)}. \tag{2.8}$$

To have all the terms $q^{a\cdot s_i} \times q^{b\cdot s_j}$, for $a \in \left\{0,\ldots,\frac{\mathrm{lcm}(S)}{s_i}-1\right\}$ and $b \in \left\{0,\ldots,\frac{\mathrm{lcm}(S)}{s_j}-1\right\}$, be considered, the exponent on q in (2.8) must be $n \cdot \gcd(s_i,s_j)$. This exponent accounts for all possible linear combinations of $a \cdot s_i$ and $b \cdot s_j$, hence $\gcd(s_i,s_j)$. Therefore we have a polynomial with exactly $\frac{\mathrm{lcm}(S)}{\gcd(s_i,s_j)}$ terms. Because q is an $lcm(S)^{th}$ root of unity, the largest exponent possible on q is $\mathrm{lcm}(S)-1$, and the top index of the summation in the product must be $\frac{\mathrm{lcm}(S)}{\gcd(s_i,s_j)}-1$.

$$\sum_{n=0}^{\frac{\mathrm{lcm}(S)}{s_i}-1} q^{n\cdot s_i} \times \sum_{n=0}^{\frac{\mathrm{lcm}(S)}{s_j}-1} q^{n\cdot s_j}$$

Since there are a total of $\frac{\mathrm{lcm}(S)}{s_i} \times \frac{\mathrm{lcm}(S)}{s_j}$ terms, including multiplicity, each term has a multiplicity of exactly $\frac{\mathrm{lcm}(S)}{\mathrm{lcm}(s_i,s_j)}$. This is because $\frac{\mathrm{lcm}(S)}{\gcd(s_i,s_j)} \times \frac{\mathrm{lcm}(S)}{\mathrm{lcm}(s_i,s_j)} = \frac{\mathrm{lcm}(S)}{s_i} \times \frac{\mathrm{lcm}(S)}{s_j}$. Since S is a finite set, iterating this procedure will give the desired result from Theorem 2.

$$\sum_{k\geq 0} h_{\mathrm{lcm}(S)k+r} = \frac{\mathrm{lcm}(S)^{|S|-1}}{\prod_{d=1}^{|S|-1} \mathrm{lcm}(\gcd(\{s_i\}_{i=1}^{d}), s_{d+1})}.$$

□

This generalization will be used in Chapter III to prove a new set of partition formulas.

CHAPTER III

MAKING USE OF GUPTA'S RESULT

We will use Theorem 2 as a basis for a new formula. This new theorem will focus on divisibility patterns for the coefficients of the constituents in the binomial basis, but for a new set of part sizes. These congruences are similar to those proven by Srinivasa Ramanujan in 1919 [5]:

$$p(5k+4) \equiv 0 \pmod{5}$$

$$p(7k+5) \equiv 0 \pmod{7}$$

$$p(11k+6) \equiv 0 \pmod{11}.$$

The proof of this new theorem will use definitions presented in the beginning as well as the result from Theorem 2. It is important to note that the set that will be used in Theorem 3, though very specific, was made as general as possible, so that it can still follow the style of Theorem 2. We begin with a definition of a partition function.

Theorem 3. *Let $S = \{a,b,c\}$ be a set of three relatively prime numbers, with one of them being an even integer. For $j \in \mathbb{N}$, we define the set $S_j = \{ja, jb, jc\}$. Then,*

$$p\left(jabck + \frac{2jabc - ja - jb - jc}{2}, S_j\right) \equiv 0 \left(\text{mod } \frac{abc}{2}\right). \tag{3.1}$$

Proof. We will show the following:

$$p\left(jabck + \frac{2jabc - ja - jb - jc}{2}, S_j\right) = \frac{abc}{2}\binom{k+2}{2} + \frac{abc}{2}\binom{k+1}{2}. \tag{3.2}$$

Using the definition of $E_S(q)$, we know that $E_{S_j}(q)$ is a reciprocal polynomial of degree $3jabc -$

$ja - jb - jc$.

We will now show that

$$p\left(jabck + \frac{2jabc - ja - jb - jc}{2}, S_j\right) = h^*_{\frac{2jabc-ja-jb-jc}{2}}\binom{k+2}{2} + h^*_{jabc+\frac{2jabc-ja-jb-jc}{2}}\binom{k+1}{2}$$

where

$$h^*_{\frac{2jabc-ja-jb-jc}{2}} = h^*_{jabc+\frac{2jabc-ja-jb-jc}{2}} \tag{3.3}$$

Because $E_{S_j}(q)$ is a reciprocal polynomial, we compute:

$$\begin{aligned} h^*_{\frac{2jabc-ja-jb-jc}{2}} &= h^*_{3jabc-ja-jb-jc-\frac{2jabc-ja-jb-jc}{2}} \\ &= h^*_{\frac{6jabc-2ja-2jb-2jc-2jabc+ja+jb+jc}{2}} \\ &= h^*_{\frac{4jabc-ja-jb-jc}{2}} \\ &= h^*_{\frac{2jabc+2jabc-ja-jb-jc}{2}} \\ &= h^*_{jabc+\frac{2jabc-ja-jb-jc}{2}} \end{aligned}$$

$$h^*_{\frac{2jabc-ja-jb-jc}{2}} = h^*_{jabc+\frac{2jabc-ja-jb-jc}{2}}$$

are the only two coefficients for this constituent because $2jabc + \frac{2jabc-ja-jb-jc}{2} > 3jabc - ja - jb - jc$, which is the degree of $E_{S_j}(q)$. In other words, $h^*_{2jabc+\frac{2jabc-ja-jb-jc}{2}} = 0$.

Now, using the result from Theorem 2, we show that the sum of the coefficients from any constituent is $jabc$.

$$\begin{aligned} \sum_{k\geq 0} h^*_{\operatorname{lcm}(S_j)k+r} = \frac{\operatorname{lcm}(S_j)^{|S_j|-1}}{\prod_{d=1}^{|S_j|-1}\operatorname{lcm}(\gcd(\{s_i\}_{i=1}^{d}, s_{d+1}))} &= \frac{(jabc)^2}{\prod_{d=1}^{2}\operatorname{lcm}(\gcd(\{s_i\}_{i=1}^{d}, s_{d+1}))} \\ &= \frac{(jabc)^2}{\operatorname{lcm}(ja, jb)\cdot\operatorname{lcm}(\gcd(ja, jb), jc)} = \frac{(jabc)^2}{jab\cdot jc} = jabc. \end{aligned} \tag{3.4}$$

Because $h^*_{\frac{2jabc-ja-jb-jc}{2}} + h^*_{jabc+\frac{2jabc-ja-jb-jc}{2}} = jabc$, it follows from (3.3) that

$$h^*_{\frac{2jabc-ja-jb-jc}{2}} = h^*_{jabc+\frac{2jabc-ja-jb-jc}{2}} = \frac{jabc}{2}.$$

Therefore, we have shown (3.2);

$$p\left(jabck + \frac{2jabc-ja-jb-jc}{2}, S_j\right) = \frac{jabc}{s}\binom{k+2}{2} + \frac{jabc}{s}\binom{k+1}{2}.$$

Thus,

$$p\left(jabck + \frac{2jabc-ja-jb-jc}{2}, S_j\right) \equiv 0 \left(\bmod \frac{jabc}{2}\right).$$

□

In the next Chapter, we will examine a special case of Theorem 3 where $S = \{1,2,\ell\}$ for any odd number $\ell \geq 3$.

Special Case 1. *Let $S = \{1,2,\ell\}$. Then for $k \geq 0$ and any odd number $\ell \geq 3$,*

$$p\left(2\ell k + \frac{3\ell-3}{2}, \{1,2,\ell\}\right) = \ell\binom{k+2}{2} + \ell\binom{k+1}{2} \equiv 0 \pmod{\ell}. \tag{3.5}$$

CHAPTER IV

RANKS AND CRANKS

In *Some Guesses in the Theory of Partitions*[6], Freeman Dyson asks for proofs of Ramanujan's identities in which it is clear to see the way the divisions are made. He began by inventing the *rank* of a partition.

Definition 6. *The rank of a partition is the largest part minus the number of parts.*

Dyson wanted to use the rank to show that that the congruences Ramanujan found create equally-populated sub-classes of the partitions of those values of n. For example, in Table 4.1 we apply the rank to the seven partitions of 5. If we take those values modulo 7, we get seven sub-classes, each with one partition.

Example 3. $p(5) = 7$. *Table 4.1 displays the seven partitions of 5, the respective rank, and the rank modulo 7.*

Table 4.1: $p(5) = 7$

$\lambda \vdash 5$	rank(λ)	rank(λ) (mod 7)
5	4	4
4+1	2	2
3+2	1	1
3+1+1	0	0
2+2+1	-1	6
2+1+1+1	-2	5
1+1+1+1+1	-4	3

This provides a complete set of residues for modulo 7.

A similar result is achieved for rank values of $p(5k+4)$ modulo 5, but not for the rank values of $p(11k+6)$ modulo 11. Thus, the rank failed to be the proof he was looking for. He then proposed the *crank*, a new statistic that would also witness the divisibility for modulo 11.

Definition 7. *A crank is a statistic on partitions that is not the rank, and witnesses any Ramanujan-like partition congruence.*

In 1988, George Andrews and Frank Garvan [2] found a crank that showed the divisibility for $p(11k+6)$ modulo 11. It is defined by:

Definition 8. *[2] For a partition π, let $l(\pi)$ denote the largest part of pi, $\omega(\pi)$ denote the number of ones in π, and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The crank is given by*

$$c(\pi) = \begin{cases} l(\pi) & if \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & if \omega(\pi) > 0. \end{cases}$$

We will apply this crank to the partitions of 6 in the table below.

Table 4.2: $p(6) = 11$

$\lambda \vdash 5$	$l(\pi)$	$\omega(\pi)$	$\mu(\pi)$	$c(\pi)$	$c(\pi) \pmod{11}$
6	6	0	1	6	6
5+1	5	1	1	0	0
4+2	4	0	2	4	4
4+1+1	4	2	1	-1	10
3+3	3	0	2	3	3
3+2+1	3	1	2	1	1
3+1+1+1	3	3	0	-3	8
2+2+2	2	0	3	2	2
2+2+1+1	2	2	0	-2	9
2+1+1+1+1	2	4	0	-4	7
1+1+1+1+1+1	1	6	0	-6	5

Theorem 4. *For $\ell \equiv 1 \pmod 4$, $0 \le r < \ell$ and $k \ge 0$ in Special Case 1, the crank $4\lambda_2 - 3\lambda_3$, witnesses the divisibility. For $\ell \equiv 3 \pmod 4$ in Special Case 1, the crank $2\lambda_1 - 2\lambda_2 + \lambda_3$, witnesses the divisibility.*

Example 4.

Table 4.3: $p(16, \{1,2,5\}) = 20$
$16 = \lambda_1 + \lambda_2 + \lambda_3^3$.
Crank: $4\lambda_2 - 3\lambda_3$.

$\lambda \vdash 16$	$c(\lambda)$	$c(\lambda) \pmod 5$
16	0	0
15+1	4	4
14+2	8	3
13+3	12	2
12+4	16	1
$12+1+1^3$	1	1
11+5	20	0
$11+2+1^3$	5	0
10+6	24	4
$10+3+1^3$	9	4
9+7	28	3
$9+4+1^3$	13	3
8+8	32	2
$8+5+1^3$	17	2
$8+2+2^3$	2	2
$7+6+1^3$	21	1
$7+3+2^3$	6	1
$6+4+2^3$	10	0
$5+5+2^3$	14	4
$4+3+3^3$	3	3

Here we see how the crank $4\lambda_2 - 3\lambda_3$ witnesses the congruence $p\big(16, \{1,2,5\}\big) \equiv 0 \pmod 5$.

We now define a new notation for partition functions that show the crank value r, modulo ℓ, the number partitioned n, and the set of part sizes S.

Definition 9. *Let $\mathscr{M}(r, \ell, n, S)$ denote the set of partitions λ of n into at most part sizes from the set S, such that the crank value $c(\lambda)$ is congruent to $r \pmod \ell$, and $M(r, \ell, n, S)$ enumerate the amount of those partitions.*

From Table 4.3 we have the following:

$$
\begin{aligned}
\mathscr{M}\big(0,5,16,\{1,2,5\}\big) &= \{16, 11+5, 11+2+1^3, 6+4+2^3\} \quad \text{so} \quad M\big(0,5,16,\{1,2,5\}\big)=4,\\
\mathscr{M}\big(1,5,16,\{1,2,5\}\big) &= \{12+4, 12+1+1^3, 7+6+1^3, 6+4+2^3\} \quad \text{so} \quad M\big(1,5,16,\{1,2,5\}\big)=4,\\
\mathscr{M}\big(2,5,16,\{1,2,5\}\big) &= \{13+3, 8+8, 8+5+1^3, 8+2+2^3\} \quad \text{so} \quad M\big(2,5,16,\{1,2,5\}\big)=4\\
\mathscr{M}\big(3,5,16,\{1,2,5\}\big) &= \{14+2, 9+7, 9+4+1^3, 4+3+3^3\} \quad \text{so} \quad M\big(3,5,16,\{1,2,5\}\big)=4,\\
\text{and} \quad \mathscr{M}\big(4,5,16,\{1,2,5\}\big) &= \{15+1, 10+6, 10+3+1^3, 5+5+2^3\} \quad \text{so} \quad M\big(4,5,16,\{1,2,5\}\big)=4.
\end{aligned}
\tag{4.1}
$$

After taking the crank values modulo 5, we can see that there are 5 sub-classes in (4.1), each with four partitions. In order to prove Theorem 4 with generality, we turn to polyhedral geometry and Ehrhart Theory.

CHAPTER V

POLYHEDRAL GEOMETRY AND EHRHART THEORY

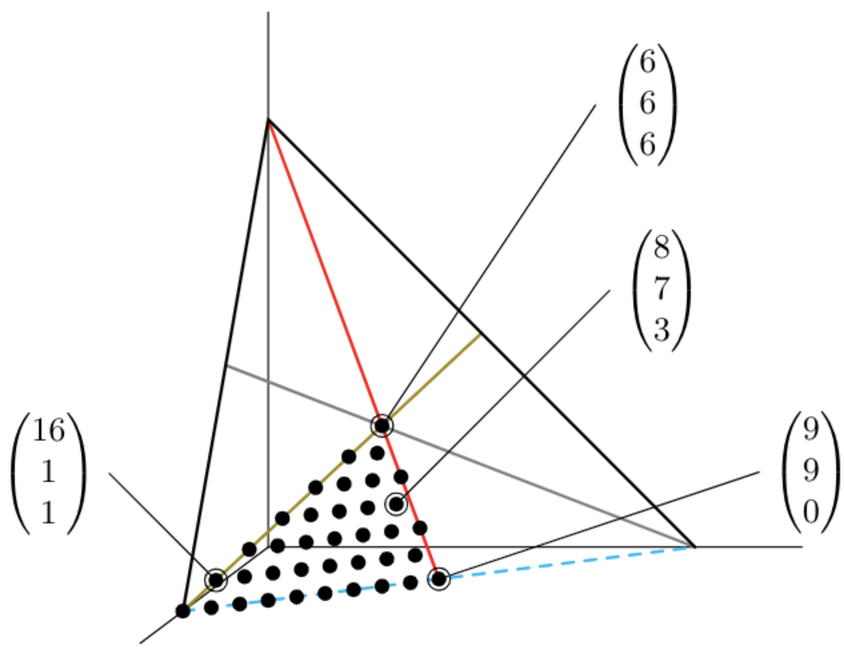


Figure 5.1: 37 integer lattice points in the set $P(18,\{1,2,3\})$.

The material in this chapter is a synthesis of the underlying combinatorics of Gupta's arithmetic in Chapter III with polyhedral geometry. We begin with the partition function $p(n,\{1,2,\ell\})$ that enumerates the partitions of positive integers into parts of sizes $1,2$, and ℓ. These partitions can be represented as integer vectors or lattice points. In other words, partitions will now be referred to as vectors at height $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3^{\ell-2}$. The set of integer lattice points, denoted by

$$P(n,\{1,2,\ell\}) = \left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3^{\ell-2} \end{pmatrix} \in \mathbb{Z}^\ell \middle| \lambda_1 + \lambda_2 + \lambda_3^{\ell-2} = n \text{ and } \lambda_1 \geq \lambda_2 \geq \lambda_3^{\ell-2} \geq 0 \right\} \tag{5.1}$$

fit inside an ℓ-dimensional partition triangle denoted by $\mathscr{P}(n,\{1,2,\ell\})$, see Figure 5.1.

This partition triangle $\mathscr{P}(n,\{1,2,\ell\})$ lives in a polyhedral cone, called the partition cone C_ℓ, which is the set of all linear combinations with non-negative real coefficients of a finite set of generators $v_1, v_2, \ldots, v_\ell \in \mathbb{Q}^\ell$ and is defined by

$$C_\ell = \{x \in \mathbb{R}^\ell | x_1 \geq x_2 \geq \cdots \geq x_\ell \geq 0\}. \tag{5.2}$$

When these generators are linearly independent, the cone is called *simplicial* [3, 4].

Definition 10. *[4, 9] Given a simplicial cone, we define a fundamental parallelepiped, $\mathscr{F}_\ell$ as*

$$\mathscr{F}_\ell = V_\ell([0,1)^\ell) = \Pi(v_1,\ldots,v_\ell) = \left\{ \sum_{i=0}^{\ell} \lambda_i v_i \middle| 0 \leq \lambda_i < 1 \right\}. \tag{5.3}$$

For some $m \geq \ell$, the set of lattice points in the fundamental parallelepiped is defined as

$$F_\ell = \Pi_{\mathbb{Z}^\ell} = \mathbb{Z}^m \cap cone_{\mathbb{R}}(v_1,\ldots,v_\ell).$$

The partition cone is tessellated by translations of the fundamental parallelepiped by integer vectors. We now define $\mathscr{H}_i = \mathscr{F}_\ell \cap \{\lambda \in \mathbb{R} | |\lambda| = i\}$ to be the hyper-plane slices of the fundamental parallelepiped at height i. $H_i = \mathbb{Z}^\ell \cap \mathscr{H}_i$ represents the set of lattice points, while $h_i^* = \# H_i$ is the amount of points in that set. $T_k = \{\tau \in \mathbb{Z}^\ell_{\leq 0}\}$ is the set of all non-negative integer vectors whose coordinates sum to k. Using notation from Ehrhart theory [3], we have,

$$p\left(2\ell k + \frac{3\ell-3}{2}, \{1,2,\ell\}\right) = h_r^* \binom{k+2}{2} + h_{2\ell+r}^* \binom{k+1}{2}, \tag{5.4}$$

$$P\left(2\ell k+\frac{3\ell-3}{2},\{1,2,\ell\}\right)=(H_r+V_\ell T_k)\cup(H_{2\ell+r}+V_\ell T_{k-1}), \tag{5.5}$$

$$\mathscr{P}\left(2\ell k+\frac{3\ell-3}{2},\{1,2,\ell\}\right)=(\mathscr{H}_r+V_\ell T_k)\cup(\mathscr{H}_{2\ell+r}+V_\ell T_{k-1}), \tag{5.6}$$

for $0\le r<\ell$ and $k\ge 0$. These slices of the fundamental parallelepiped are translated using the following method: $V_\ell\times\tau+\mu=\lambda$, which represent:

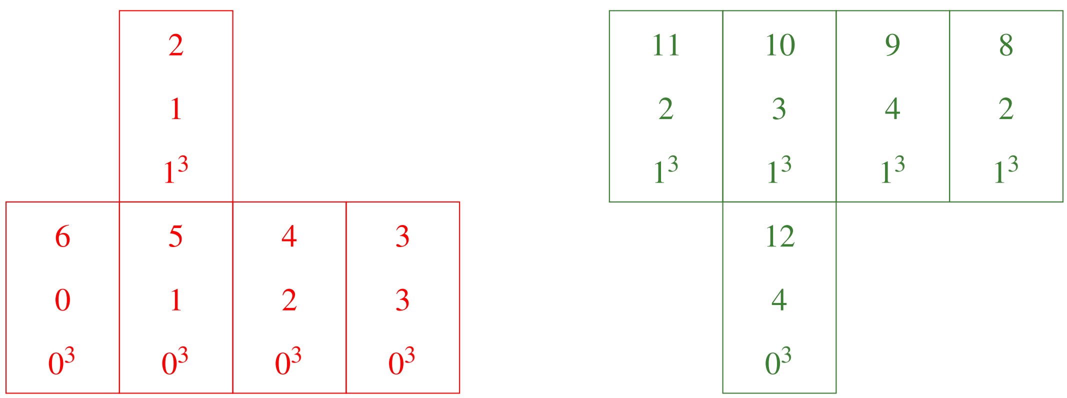
$$\begin{pmatrix} 2\ell & \ell & 2 \\ 0 & \ell & 2 \\ 0^{\ell-2} & 0^{\ell-2} & 2^{\ell-2} \end{pmatrix}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\end{pmatrix}+\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3^{\ell-2}\end{pmatrix}=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3^{\ell-2}\end{pmatrix}. \tag{5.7}$$

Note that the superscripts are not exponents on the third coordinates of the vectors in (5.7), nor on the slices that will follow below, but actually denote that the final $\ell-2$ parts of the partitions are identical and are written as one $\lambda_3^{\ell-2}$. The columns of V_ℓ represent the generators for the partition cone C_ℓ. The translation vector τ, has coordinates that sum to k. The partition μ is found in the fundamental parallelepiped F_ℓ. The vector λ gives the coordinates of the new partition after the translation is applied. This method is known as the box decomposition for a partition λ [4]. The box decomposition shows the geometric and combinatorial implications behind the coefficients of the constituents in a binomial basis.

Theorem 2 tell us that whenever $n=2\ell k+\frac{3\ell-3}{2}\Big|_{k=0,1}$ in Special Case 1, the $\mathscr{H}_i$ slices of the fundamental parallelepiped have exactly ℓ- integer lattice points in them. We will show that each slice has a complete set of residues modulo ℓ, so that each slice, and their translations, represent an ℓ-cycle of partitions. By ℓ-cycle, we mean a collection G of ℓ maps such that for $g\in G$, $g(\lambda)=\lambda'$ where $c(\lambda') \pmod{\ell}=c(\lambda)+x \pmod{\ell}$ for some fixed integer x co-prime to ℓ. Moreover, λ' and λ are contained in the same translated slice, $V_\ell\tau+H_n$ for specified values of n. Hence, every slice of the partition triangle at height $n=2k\ell+\frac{3\ell-3}{2}\Big|_{k\ge 0}$ consists of exactly $\binom{k+2}{2}+\binom{k+1}{2}=(k+1)^2$ ℓ-cycles. To prove that the crank witnesses the divisibility shown in Special Case 1, we will show that the slices of the fundamental parallelepiped at $n=2k\ell+\frac{3\ell-3}{2}\Big|_{k=0,1}$ have a complete set of residues modulo ℓ after applying the crank to them.

To illustrate, we begin with an example. Let $\ell = 5$ so that we may consider partitions whose parts come from the set $S = \{1,2,5\}$. Below we show the slices of the fundamental parallelepiped for $n = 2k\ell + \frac{3\ell-3}{2}\Big|_{k\geq 0}$, then translate them using integer vectors.

Example 5. *Let $\ell = 5$ in Special Case 1, then for $p(10k+6|S) \equiv 0 \pmod 5$, we have the following slices of the fundamental parallelepiped for $k = 0$ and $k = 1$.*



Using (5.7)*, with $k = 3$, we now translate these two slices of the fundamental parallelepiped* $\binom{3+2}{2} + \binom{3+1}{2} = 16$ *times, to create the slice of the partition cone at height* $2\ell k + \frac{3\ell-3}{2} = 2 \times 5 \times 3 + \frac{3\times 5-3}{2} = 36$.

Some examples of using equation (5.7) *to translate the partitions are shown below:*

$$V_\ell \times \tau + \mu = \lambda$$

$$\begin{pmatrix} 10 & 5 & 2 \\ 0 & 5 & 2 \\ 0^3 & 0^3 & 2^3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0^3 \end{pmatrix} = \begin{pmatrix} 33 \\ 3 \\ 0^3 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 5 & 2 \\ 0 & 5 & 2 \\ 0^3 & 0^3 & 2^3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 9 \\ 4 \\ 1^3 \end{pmatrix} = \begin{pmatrix} 24 \\ 9 \\ 1^3 \end{pmatrix}.$$

These coordinates are in bold font within the slice of the partition cone at height 36*. Now we show*

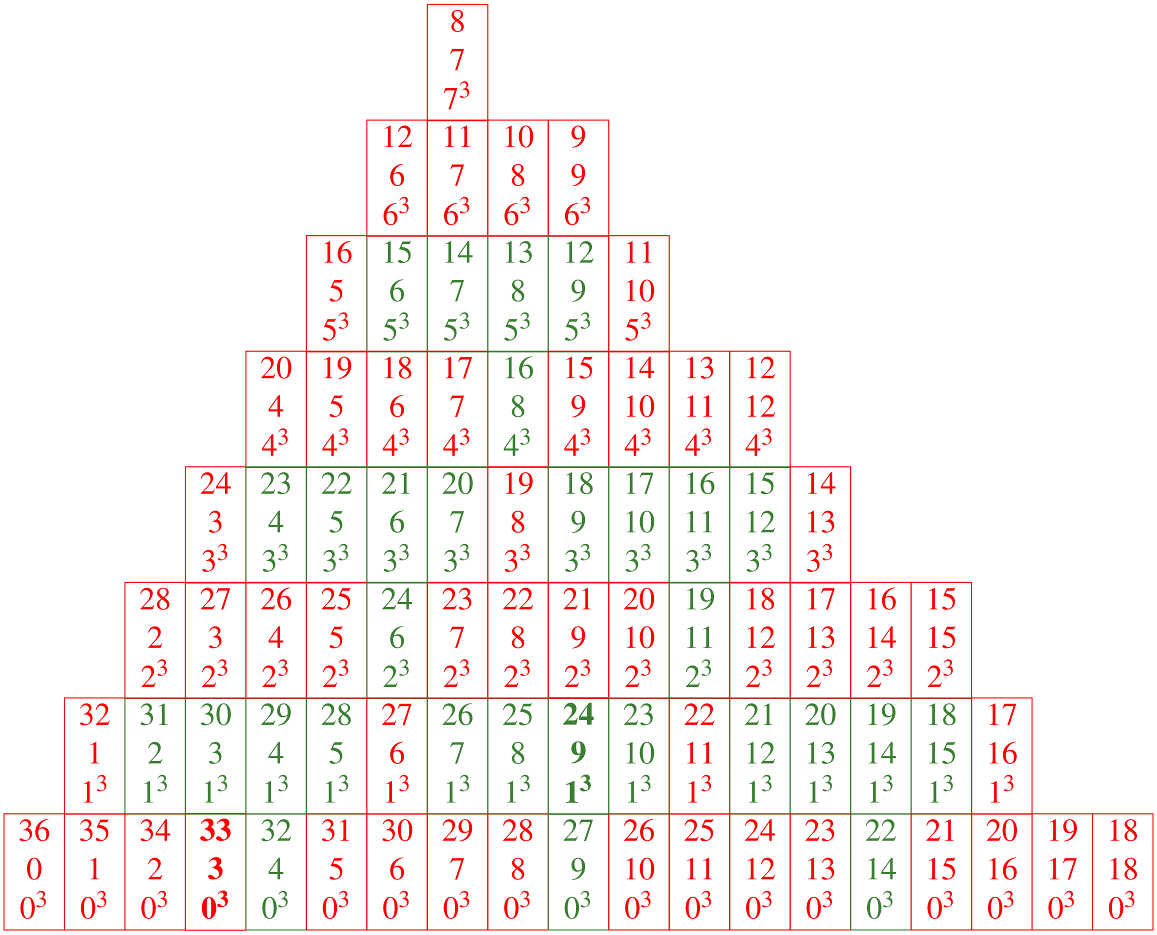


Figure 5.2: Slice of the partition cone C_ℓ at height 36.

that applying the crank to each of these coordinates gives us complete sets of residues within each of the 5*-cycles (*ℓ*-cycles).*

We will now prove Theorem 4 for the case when $\ell \equiv 1 \pmod 4$.

Theorem 4. *For* $\ell \equiv 1 \pmod 4$, $0 \le r < \ell$ *and* $k \ge 0$ *in Special Case 1, the crank* $4\lambda_2 - 3\lambda_3$*, witnesses the divisibility. For* $\ell \equiv 3 \pmod 4$ *in Special Case 1, the crank* $2\lambda_1 - 2\lambda_2 + \lambda_3$*, witnesses the divisibility.*

Proof of Theorem 4. Our proof will detail $\ell \equiv 1 \pmod 4$. For Special Case 1, we examine the slices of the fundamental parallelepiped at $n = 2k\ell + \frac{3\ell-3}{2}\Big|_{k=0,1}$, which is given to us by Gupta's result in Theorem 1, are shown below in Figures 5.4-5.7. It is important to note that the partitions in these slices represent the conjugates of the partitions of n into part sizes $1, 2, \ell$.

							2											
						1	0	4	3									
					0	4	3	2	1	0								
				4	3	2	1	0	4	3	2	1						
			3	2	1	0	4	3	2	1	0	4	3					
		2	1	0	4	3	2	1	0	4	3	2	1	0	4			
	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2	1		
0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2

Figure 5.3: Crank values modulo 5 of the slice of the partition cone at height 36.

	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-1}{2}-\frac{\ell-5}{4}$ $1+\frac{\ell-5}{4}$ $1^{\ell-2}$	
$\frac{3\ell-3}{2}$ 0 $0^{\ell-2}$	$\frac{3\ell-3}{2}-1$ 1 $0^{\ell-2}$	$\cdots$	$\cdots$	$\frac{3\ell-3}{4}$ $\frac{3\ell-3}{4}$ $0^{\ell-2}$

Figure 5.4: $\ell \equiv 1 \pmod 4$: The slice of the fundamental parallelepiped generated by $\begin{pmatrix} 2\ell & \ell & 2 \\ 0 & \ell & 2 \\ 0^{\ell-2} & 0^{\ell-2} & 2^{\ell-2} \end{pmatrix}$ at height $\frac{3\ell-3}{2}$, $(k=0)$.

$\frac{5\ell-1}{2}-\frac{\ell-5}{4}-1$ $\frac{\ell+5}{4}$ $1^{\ell-2}$	$\cdots$	$\cdots$	$\cdots$	$\frac{3\ell+1}{2}$ ℓ $1^{\ell-2}$
	$\frac{11\ell-3}{4}-1$ $\frac{3\ell+1}{4}$ $0^{\ell-2}$	$\cdots$	$\frac{5\ell-1}{2}$ $\ell-1$ $0^{\ell-2}$	

Figure 5.5: $\ell \equiv 1 \pmod 4$: The slice of the fundamental parallelepiped generated by $\begin{pmatrix} 2\ell & \ell & 2 \\ 0 & \ell & 2 \\ 0^{\ell-2} & 0^{\ell-2} & 2^{\ell-2} \end{pmatrix}$ at height $2\ell + \frac{3\ell-3}{2}$, $(k=1)$.

	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-3}{4}+1$ $\frac{\ell-3}{4}+1$ $1^{\ell-2}$	
$\frac{3\ell-3}{2}$ 0 $0^{\ell-2}$	$\cdots$	$\cdots$	$\cdots$	$\frac{3\ell-1}{4}$ $\frac{3\ell-5}{4}$ $0^{\ell-2}$

Figure 5.6: $\ell \equiv 3 \pmod 4$: The slice of the fundamental parallelepiped generated by $\begin{pmatrix} 2\ell & \ell & 2 \\ 0 & \ell & 2 \\ 0^{\ell-2} & 0^{\ell-2} & 2^{\ell-2} \end{pmatrix}$ at height $\frac{3\ell-3}{2}$, $(k=0)$.

$\frac{9\ell-3}{4}-\frac{\ell-5}{4}-1$ $\frac{\ell+5}{4}$ $1^{\ell-2}$	$\cdots$	$\cdots$	$\cdots$	$\frac{3\ell+1}{2}$ ℓ $1^{\ell-2}$
	$\frac{11\ell-5}{4}-1$ $\frac{3\ell+1}{4}$ $0^{\ell-2}$	$\cdots$	$\frac{5\ell-1}{2}$ $\ell-1$ $0^{\ell-2}$	

Figure 5.7: $\ell \equiv 3 \pmod 4$: The slice of the fundamental parallelepiped generated by $\begin{pmatrix} 2\ell & \ell & 2 \\ 0 & \ell & 2 \\ 0^{\ell-2} & 0^{\ell-2} & 2^{\ell-2} \end{pmatrix}$ at height $2\ell+\frac{3\ell-3}{2}$, $(k=1)$.

We now show that applying the crank to the ℓ coordinates gives us a complete set of residues modulo ℓ in each of these slices. We examine the slice of the fundamental parallelepiped shown in Figure 5.4, which contains the conjugates of the partitions of $\frac{3\ell-3}{2}$ into parts of sizes $\{1,2,\ell\}$.

The leftmost vector in the $\lambda_3 = 0^{\ell-2}$ row has $\lambda_2 = 0$, giving a crank value of 0.

$$c(\lambda) = c\left(\frac{3\ell-3}{2},0,0\right) = 4(0)-3(0) = 0 \pmod{\ell}.$$

There are $\frac{3\ell+1}{4}$ vectors in the $\lambda_3 = 0^{\ell-2}$ row, each consecutive in the λ_2 coordinate. This is because the partition of $\frac{3\ell-3}{2}$ with the largest λ_2 possible is $\begin{pmatrix} \frac{3\ell-3}{4} \\ \frac{3\ell-3}{4} \\ 0^{\ell-2} \end{pmatrix}$. The vector to the right of this would be $\begin{pmatrix} \frac{3\ell-7}{4} \\ \frac{3\ell+1}{4} \\ 0^{\ell-2} \end{pmatrix}$, which is no longer is not a valid partition. As we move from left to right, the crank value $c(\lambda)$ of the vectors increases by $4 \pmod{\ell}$. The extra vector would have had crank value of $c(\lambda) = 1$. However, since it is not in this slice, the last crank value in this row is

$c(\lambda) = 4\lambda_2 - 3\lambda_3 \equiv -3 \pmod{\ell}$.

$$c(\lambda) = c\left(\frac{3\ell-3}{4}, \frac{3\ell-3}{4}, 0\right) = 4\left(\frac{3\ell-3}{4}\right) - 3(0) = -3 \pmod{\ell}.$$

Thus, we never get a crank value of 1 in this row. Because ℓ is an odd number, clearly $\gcd(4,\ell) = 1$ and therefore the crank value $c(\lambda)$ of each vector must be distinct.

Theorem 2 tells us that there are ℓ vectors in total in this slice. Because there are $\frac{3\ell+1}{4}$ in the $\lambda_3 = 0^{\ell-2}$ row, there must be $\ell - \frac{3\ell+1}{4} = \frac{\ell-1}{4}$ vectors in the $\lambda_3 = 1^{\ell-2}$ row. The partition of $\frac{3\ell-3}{2}$ with the smallest λ_2 value possible in this row is $\begin{pmatrix} \frac{\ell-1}{4} \\ 1 \\ 1^{\ell-2} \end{pmatrix}$, which has a crank value of $c(\lambda) = 1$.

$$c\left(\frac{\ell-1}{4}, 1, 1\right) = 4(1) - 3(1) = 1 \pmod{\ell}.$$

Since the crank value increases by 4, the remaining $\frac{\ell-5}{4}$ vectors in this row have a crank value congruent to $1 \pmod{4}$. Together, the crank values of the $\frac{3\ell+1}{4}$ vectors in the $\lambda_3 = 0^{\ell-2}$ row and the crank values of the $\frac{\ell-1}{4}$ vectors in the $\lambda_3 = 1^{\ell-2}$ row make a complete set of residues modulo ℓ.

A similar argument will show that the slice displayed in Figure 5.5, which contains the conjugates of the partitions of $2\ell + \frac{3\ell-3}{2}$ into parts of sizes $\{1,2,\ell\}$, likewise carries a complete set of residues modulo ℓ.

We now show that the sets of residues in these slices of the fundamental parallelepiped are preserved under translations using the $V_\ell \times \tau + \mu = \lambda$ method. We begin with a partition that appears in the fundamental parallelepiped, $\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3^{\ell-2} \end{pmatrix}$, and we apply the translation vectors.

$$\begin{pmatrix} 2\ell & \ell & 2 \\ 0 & \ell & 2 \\ 0^{\ell-2} & 0^{\ell-2} & 2^{\ell-2} \end{pmatrix} \times \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3^{\ell-2} \end{pmatrix} = \begin{pmatrix} 2\ell\tau_1 + \ell\tau_2 + 2\tau_3 + \mu_1 \\ \ell\tau_2 + 2\tau_3 + \mu_2 \\ 2^{\ell-2}\tau_3 + \mu_3^{\ell-2} \end{pmatrix}. \tag{5.8}$$

When we reduce the right side of (5.8) modulo ℓ, we get $\begin{pmatrix} 2\tau_3 + \mu_1 \\ 2\tau_3 + \mu_2 \\ 2^{\ell-2}\tau_3 + \mu_3^{\ell-2} \end{pmatrix}$. Applying the crank to the partition corresponding to this vector gives: $4(2\tau_3 + \mu_2) - 3(2\tau_3 + \mu_3) \pmod{\ell}$, which is $2\tau_3 + (4\mu_2 - 3\mu_3) \pmod{\ell}$. Given $\mu \in F_\ell$, the crank value $c(\lambda)$ of $\lambda = V_\ell \times \tau + \mu$, i.e., any integer translation of μ, is increased by $2\tau_3 \pmod{\ell}$. Therefore, any integer vector translation of the partitions in the fundamental parallelepiped simply permutes the set of residues.

Using the symmetry of the fundamental parallelepiped, the slice at $n = 2k\ell + \frac{3\ell-3}{2}\Big|_{k=1}$, as well as its translations, likewise have complete sets of residues modulo ℓ. Thus, for $\ell \equiv 1 \pmod 4$, the crank $4\lambda_2 - 3\lambda_3$ witnesses the congruence found in Special Case 1. A similar argument will establish the case for $\ell \equiv 3 \pmod 4$ where the crank $2\lambda_1 - 2\lambda_2 + \lambda_3$ witnesses the divisibility. $\square$

CHAPTER VI

CONCLUSION

Our future work will be to obtain a similar result for Theorem 3, in which we have:

Theorem 3. *Let $S = \{a, b, c\}$ be a set of three relatively prime numbers, with one of them being an even integer. For $j \in \mathbb{N}$, we define the set $S_j = \{ja, jb, jc\}$. Then,*

$$p\left(jabck + \frac{2jabc - ja - jb - jc}{2}, S_j\right) \equiv 0 \left(\text{mod}\, \frac{abc}{2}\right).$$

Currently, we know that the cranks $4\lambda_2 - 3\lambda_3$ for $\ell \equiv 1 \pmod 4$ and $2\lambda_1 - 2\lambda_2 + \lambda_3$ for $\ell \equiv \pmod 4$ can be applied to the partitions from the set of part sizes $\{j, 2j, j\ell\}$. We will also work towards extending Theorem 3 to include more than three part sizes in the set.

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BIOGRAPHICAL SKETCH

While completing her coursework at Mission Collegiate High School, Joselyne Rodriguez received an Associate of Art in Interdisciplinary Studies from South Texas College in 2016. That fall, she enrolled at the University of Texas at San Antonio and received a Bachelor of Science with a major in Mathematics. She was employed as a team teacher for Mission High School for the Spring 2019 semester. In August 2019, she entered the mathematics graduate program at the University of Texas Rio Grande Valley. Joselyne Rodriguez earned a Master of Science in Mathematics at the University of Texas Rio Grande Valley in May 2021; she can be contacted at joselyne.rodriguez01@utrgv.edu.