8-2021

Integrable Equation with No Solitary Traveling Waves

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INTEGRABLE EQUATION WITH NO SOLITARY TRAVELING WAVES

A Thesis

by

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Submitted to the Graduate College of
The University of Texas Rio Grande Valley
In partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2021

Major Subject: Mathematics

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August 2021

ABSTRACT

Rodriguez, Miguel, Integrable Equation With No Solitary Traveling Waves. Master of Science (MS), August, 2021, 35 pp., 4 figures, 12 references.

We consider the negative order KdV (NKdV) hierarchy which generates nonlinear integrable equations. Selecting different seed functions produces different evolution equations. We apply the traveling wave setting to study one of these equations. Assuming a particular type of solution leads us to solve a cubic equation. New solutions are found, but none of these are classical solitary traveling wave solutions.

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CHAPTER I

INTRODUCTION

The study of water waves is an important area of research in mathematical physics. Several integrable systems in the form of nonlinear partial differential equations are used to describe their motion. There has been an effort to discover new solutions to these systems, particularly soliton solutions.

1.1 History

Solitary waves were first described by John Scott Russell in his 1838 report (Russell [1838]). While riding on horseback by the Union Canal in Scotland, he witnessed "...a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed". He followed this "Wave of Translation" for approximately two miles, eventually losing it in the windings of the channel. Later, in laboratory experiments Russell replicated this phenomenon. He confirmed the existence of solitary waves, described as long, shallow, water waves of permanent form. This was controversial since the current mathematical theories did not support their existence (Ablowitz and Clarkson [1991]).

The issue was solved when Korteweg and de Vries derived a new equation (Korteweg and De Vries [1895]). The Korteweg-de Vries (KdV) equation is given by

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

where u is the dependent variable, t is the temporal variable, x is the spatial variable, and subscripts denote partial derivatives. This nonlinear evolution equation governs the one-dimensional motion of small amplitude, surface gravity waves propagating in shallow water. It was known to have the solitary wave solution

$$u(x,t) = 2k^2 \operatorname{sech}^2\big(k(x - 4k^2 t - x_0)\big) \tag{1.2}$$

where k, and x_0 are constants (Ablowitz and Clarkson [1991]). Zabusky and Kruskal discovered that these solutions interact elastically with each other, thereby calling them solitons (Zabusky and Kruskal [1965]). These solitons retain their shape and velocity after passing through each other, differing only by a phase shift.

A new development came when Miura gave the transformation

$$u = -(v^2 + v_x) \tag{1.3}$$

which produces a solution to the KdV equation from a solution to the modified KdV (mKdV) equation (Miura [1968]). Miura's transformation helped in proving that the KdV equation has an infinite number of conservation laws. It also motivated the development of the Inverse Scattering Transform by Gardner, Greene, Kruskal, and Miura, which could solve the inital-value problem of the KdV equation (Gardner et al. [1967]). The Inverse Scattering Method would linearize the KdV equation by relating it to the time-independent Schrodinger scattering problem

$$L\psi = \psi_{xx} + u\psi = \lambda\psi \tag{1.4}$$

which is well known from quantum mechanics (Ablowitz and Clarkson [1991]). Here, the eigenvalues and the eigenfunctions produce the scattering data. Then, the inverse scattering problem reconstructs the potential u, which is also the solution to the KdV equation.

Lax later gave a general method to associate nonlinear evolution equations with linear operators, where the eigenvalues of the linear operator correspond to the integrals of the evolution equation (Lax [1968]). Consider the linear system

$$L\psi = \lambda\psi \tag{1.5}$$

$$\psi_t = M\psi \tag{1.6}$$

where L is the spectral problem operator, and M is the time evolution operator. The compatibility condition for this system is given by Lax's equation

$$L_t + [L,M] = 0 \tag{1.7}$$

where $[L,M]$ is the commutator of L and M. Taking

$$L = \partial^2 + u \tag{1.8}$$

$$M = -4\partial^3 - 3(\partial u + u\partial) \tag{1.9}$$

equation (1.7) becomes the KdV equation, where L and M are called its Lax pair. In fact, many evolution equations can arise from the compatibility condition (1.7) by guessing the correct operators.

Following this discovery, Lax used a recursion procedure to introduce the KdV hierarchy (Lax [1975]). The positive order KdV hierarchy includes as its member the known KdV equation (Qiao and Fan [2012]). The negative order KdV (NKdV) hierarchy includes some equations that are gauge-equivalent to the popular Camassa-Holm equation

$$m_t + m_x u + 2mu_x = 0 \tag{1.10}$$

$$m = u - u_{xx} \tag{1.11}$$

derived by Roberto Camassa and Darryl Holm (Camassa and Holm [1993]). This equation has gained much interest since it possesses peakon solutions, which are solitons with a discontinuous first derivative at the peak.

1.2 Overview

We will show how to obtain the KdV hierarchy starting from the Schrodinger-KdV spectral problem. First, we find Lenard's operators and their inverse. Then, we construct the recursion operators for the positive and negative order hierarchies. Lenard's sequence is defined by powers of the recursion operator acting on the chosen seed functions. Finally, the entire KdV hierarchy is given, which includes the NKdV hierarchy. Depending on our choice of seed function, the hierarchy produces distinct nonlinear evolution equations. We show all the possible seed functions for the NKdV hierarchy, as well as the equations these produce with the setting $k = -1$. We attempt to solve one of these evolution equations given by

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = g(t_n)\big(2uu_x\partial^{-1}u^{-2}+1\big) \tag{1.12}$$

which hadn't been studied in detail.

We apply the traveling wave setting to convert the partial differential equation into an ordinary differential equation. Using some algebra and calculus we obtain the new equation

$$3cu'u'' - cuu''' - 2Auu' + u^2 = 0. \tag{1.13}$$

We assume solutions of the form

$$u(\xi) = ae^{\lambda\xi} \tag{1.14}$$

and find λ after solving a cubic equation. New solutions are given but none of these are classical solitary traveling wave solutions. We also consider several other solution types, but find that they are not solutions to our equation.

CHAPTER II

REVIEW OF LITERATURE

We review how to obtain the KdV hierarchy from the KdV spectral problem. Our choice of seed functions then produces different evolution equations in the NKdV hierarchy. Finding solutions to one evolution equation will require us to solve a depressed cubic equation, so we review the process to solving them.

2.1 KdV Hierarchy

The Schrodinger-KdV spectral problem is given by

$$L\psi = \psi_{xx} + v\psi = \lambda\psi \tag{2.1}$$

where λ is an eigenvalue, ψ is the eigenfunction, and v is a potential function. To produce the KdV hierarchy, we must find the operators K and J which satisfy the Lenard operator relation

$$K \cdot \nabla\lambda = \lambda J \cdot \nabla\lambda \tag{2.2}$$

where $\nabla\lambda = \psi^2$ is the functional gradient of the spectral problem with respect to v. First, we calculate

$$(\nabla\lambda)_x = 2\psi\psi_x \tag{2.3}$$

$$(\nabla\lambda)_{xx} = 2(\psi_x^2 + (\lambda - v)\psi^2) \tag{2.4}$$

$$(\nabla\lambda)_{xxx} = 2\big(2(\lambda - v)(\nabla\lambda)_x - v_x\nabla\lambda\big). \tag{2.5}$$

It follows from (2.5) that

$$\frac{1}{4}(\nabla\lambda)_{xxx} + v(\nabla\lambda)_x + \frac{1}{2}v_x\nabla\lambda = \lambda(\nabla\lambda)_x. \tag{2.6}$$

This can be written as

$$\left(\frac{1}{4}\partial^3 + \frac{1}{2}\big(v\partial + \partial v\big)\right)\cdot\nabla\lambda = \lambda\partial\cdot\nabla\lambda \tag{2.7}$$

where ∂ is the differential operator with respect to x. Comparing (2.2) and (2.7), we conclude that Lenard's operators are given by

$$K = \frac{1}{4}\partial^3 + \frac{1}{2}\big(v\partial + \partial v\big) \tag{2.8}$$

$$J = \partial. \tag{2.9}$$

Introducing the setting $v = -\frac{u_{xx}}{u}$, turns K into the product form

$$K = \frac{1}{4}u^{-2}\partial u^2\partial u^2\partial u^{-2} \tag{2.10}$$

(Qiao and Li [2011]). Now, the inverse operators are given by

$$K^{-1} = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2 \tag{2.11}$$

$$J^{-1} = \partial^{-1}. \tag{2.12}$$

So, the recursion operator and its inverse are given by

$$\mathscr{L} = J^{-1}K = \frac{1}{4}\partial^{-1}u^{-2}\partial u^2\partial u^2\partial u^{-2} \tag{2.13}$$

$$\mathscr{L}^{-1} = K^{-1}J = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2\partial. \tag{2.14}$$

Next, Lenard's sequence is defined as

$$G_j = \begin{cases} \mathscr{L}^j \cdot G_0, & j \geq 0 \\ \mathscr{L}^{j+1} \cdot G_{-1}, & j < 0 \end{cases} \tag{2.15}$$

where $j \in \mathbb{Z}$, and the seed functions are given by

$$G_0 \in Ker(J) = \{G \in C^\infty(\mathbb{R}) | JG = 0\} \tag{2.16}$$

$$G_{-1} \in Ker(K) = \{G \in C^\infty(\mathbb{R}) | KG = 0\}. \tag{2.17}$$

Finally, the entire KdV hierarchy is given by

$$v_{t_k} = JG_k = KG_{k-1} \tag{2.18}$$

where t is the time variable, and $k \in \mathbb{Z}$. Using $k \geq 0$ gives the positive order KdV hierarchy, while $k < 0$ gives the negative order KdV (NKdV) hierarchy.

Depending on our choice of seed function G_{-1}, we can obtain different equations in the NKdV hierarchy. Since $KG_{-1} = 0$, then $G_{-1} = K^{-1}0$. This results in three possible seed functions:

$$G^1_{-1} = f(t_n)u^2 \tag{2.19}$$

$$G^2_{-1} = g(t_n)u^2\partial^{-1}u^{-2} \tag{2.20}$$

$$G^3_{-1} = h(t_n)u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2} \tag{2.21}$$

where $f(t_n), g(t_n), h(t_n)$ are functions of t but independent of x, and ∂^{-1} represents an integral with respect to x. Therefore, selecting $k=-1$ in the hierarchy (2.18) produces three distinct equations:

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = 2f(t_n)uu_x \tag{2.22}$$

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = g(t_n)\big(2uu_x\partial^{-1}u^{-2}+1\big) \tag{2.23}$$

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = h(t_n)\big(2uu_x\partial^{-1}u^{-2}\partial^{-1}u^{-2}+\partial^{-1}u^{-2}\big). \tag{2.24}$$

The first equation (2.22) has been studied before and solutions were given (Qiao and Li [2011]). We will focus our attention on the second equation (2.23) and attempt to find new traveling wave solutions.

2.2 Cubic Equations

Consider the depressed cubic equation

$$x^3+Px=Q \tag{2.25}$$

where P and Q are real constants. Like all cubic equations, equation (2.25) must have three solutions. We begin by assuming solutions of the form

$$x=a-b. \tag{2.26}$$

Plugging in (2.26) into (2.25) gives

$$(a-b)^3+P(a-b)=Q. \tag{2.27}$$

Using the identity

$$(a-b)^3+3ab(a-b)=a^3-b^3 \tag{2.28}$$

and comparing to (2.27) leads to the system of equations

$$\begin{cases} ab = \frac{P}{3} \\ a^3 - b^3 = Q. \end{cases} \tag{2.29}$$

Using substitution, we get

$$\left(\frac{P}{3b}\right)^3 - b^3 = Q. \tag{2.30}$$

Multiplying by b^3, we have

$$\left(b^3\right)^2 + Qb^3 - \left(\frac{P}{3}\right)^3 = 0 \tag{2.31}$$

which has the form of a quadratic equation. Any quadratic equation

$$ax^2 + bx + c = 0 \tag{2.32}$$

where a, b, c are constants can be solved by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.33}$$

Applying formula (2.33) to equation (2.31), we find that

$$b^3 = -\frac{Q}{2} \pm \sqrt{\Delta} \tag{2.34}$$

$$\Delta = \left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^2 \tag{2.35}$$

(Smith). Here, Δ is the discriminant, and the nature of the solutions to (2.25) will depend on the sign of Δ.

Let's first consider the simplest case, $\Delta = 0$. Here, we have

$$b^3 = -\frac{Q}{2}. \tag{2.36}$$

We can see that

$$b_1 = \sqrt[3]{-\frac{Q}{2}} \tag{2.37}$$

is a solution for b. Multiplying (2.37) by any cubic root of 1 would also be a solution for b. The three cubic roots of 1 are given by

$$\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \tag{2.38}$$

$$\omega^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \tag{2.39}$$

$$\omega^3 = e^{2\pi i} = 1. \tag{2.40}$$

It follows that

$$b_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2}} \tag{2.41}$$

$$b_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2}} \tag{2.42}$$

are the other solutions for b. For each b, we find the corresponding a by the relation

$$a = \frac{P}{3b}, \tag{2.43}$$

which is derived from (2.29). We find that

$$a_1 = \sqrt[3]{\frac{Q}{2}} \tag{2.44}$$

$$a_2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2}} \tag{2.45}$$

$$a_3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2}}. \tag{2.46}$$

Finally, the solutions to (2.25) for the case $\Delta = 0$ are given by

$$x_1 = a_1 - b_1 = \sqrt[3]{4Q} \tag{2.47}$$

$$x_2 = a_2 - b_2 = -\sqrt[3]{\frac{Q}{2}} \tag{2.48}$$

$$x_3 = a_3 - b_3 = -\sqrt[3]{\frac{Q}{2}}. \tag{2.49}$$

They are all real solutions with two of them repeated.

Next, we consider the case $\Delta > 0$. From (2.34), it follows that one solution for b is

$$b_1 = \sqrt[3]{-\frac{Q}{2} \pm \sqrt{\Delta}}. \tag{2.50}$$

Multiplying by (2.38) and (2.39) gives

$$b_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2} \pm \sqrt{\Delta}} \tag{2.51}$$

$$b_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2} \pm \sqrt{\Delta}}. \tag{2.52}$$

From using (2.43), we find that

$$a_1 = \sqrt[3]{\frac{Q}{2} \pm \sqrt{\Delta}} \tag{2.53}$$

$$a_2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2} \pm \sqrt{\Delta}} \tag{2.54}$$

$$a_3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2} \pm \sqrt{\Delta}}. \tag{2.55}$$

Then, by using (2.26), we find

$$x_1 = \sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}} \tag{2.56}$$

$$x_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) - \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) \tag{2.57}$$

$$x_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) + \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) \tag{2.58}$$

with Δ given by (2.35) These are the solutions to equation (2.25) in the case $\Delta > 0$. There is one real solution and two complex conjugate solutions.

Lastly, there is the case $\Delta < 0$. Here, we have

$$b^3 = -\frac{Q}{2} \pm i\sqrt{-\Delta}. \tag{2.59}$$

We recall that a complex number $x + iy$ can be converted to the polar form $re^{i\theta}$, where

$$r = \sqrt{x^2 + y^2} \tag{2.60}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.61}$$

So, we give (2.59) in the polar form

$$b^3 = \sqrt{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm i \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{2.62}$$

Taking the cubic root gives

$$b_1 = \sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm \frac{i}{3} \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{2.63}$$

Multiplying by (2.38) and (2.39) gives

$$b_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm \frac{i}{3} \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)} \tag{2.64}$$

$$b_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm \frac{i}{3} \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{2.65}$$

Using (2.43), we find that

$$a_1 = -\sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\mp \frac{i}{3} \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)} \tag{2.66}$$

$$a_2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\mp \frac{i}{3} \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)} \tag{2.67}$$

$$a_3 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\mp \frac{i}{3} \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{2.68}$$

We use (2.26) and convert back to Cartesian form through Euler's formula

$$e^{i\theta} = \cos\theta + i \sin\theta. \tag{2.69}$$

Simplifying, we get

$$x_1 = -2\sqrt{\frac{-P}{3}}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right) \tag{2.70}$$

$$x_2 = \sqrt{\frac{-P}{3}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right) + \sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right)\right) \tag{2.71}$$

$$x_3 = \sqrt{\frac{-P}{3}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right) - \sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right)\right) \tag{2.72}$$

where Δ is given by (2.35). These are the solutions to equation (2.25) in the case $\Delta < 0$. All are real solutions.

CHAPTER III

METHODOLOGY AND FINDINGS

We apply the traveling wave setting to equation (2.23) in the NKdV hierarchy. A new ordinary differential equation is given and we consider several possible solution types. New solutions are found but none are classical solitary traveling wave solutions.

3.1 Traveling Wave Setting

With the setting $g(t_n) = 1$, equation (2.23) becomes

$$\left(-\frac{u_{xx}}{u}\right)_t = 2uu_x\partial^{-1}u^{-2} + 1. \tag{3.1}$$

We apply the substitution

$$u^{-2} = w_x \tag{3.2}$$

to eliminate the integral in (3.1). We get

$$\left(-\frac{u_{xx}}{u}\right)_t = 2uu_x\partial^{-1}w_x + 1 \tag{3.3}$$

$$\Rightarrow \left(-\frac{u_{xx}}{u}\right)_t = 2uu_xw + 1 \tag{3.4}$$

$$\Rightarrow \left(-\frac{u_{xx}}{u}\right)_t = (u^2)_xw + 1 \tag{3.5}$$

From (3.2) we have

$$u = w_x^{-\frac{1}{2}}. \tag{3.6}$$

Substituting into (3.5) gives

$$-\left(w_x{}^{\frac{1}{2}}\left(w_x{}^{-\frac{1}{2}}\right)_{xx}\right)_t = \left(w_x{}^{-1}\right)_x w + 1 \tag{3.7}$$

$$\Rightarrow -\left(w_x{}^{\frac{1}{2}}\left(w_x{}^{-\frac{1}{2}}\right)_{xx}\right)_t = 1 - \frac{ww_{xx}}{(w_x)^2} \tag{3.8}$$

$$\Rightarrow -\left(w_x{}^{\frac{1}{2}}\left(w_x{}^{-\frac{1}{2}}\right)_{xx}\right)_t = \frac{w_x w_x - ww_{xx}}{(w_x)^2} \tag{3.9}$$

$$\Rightarrow -\left(w_x{}^{\frac{1}{2}}\left(w_x{}^{-\frac{1}{2}}\right)_{xx}\right)_t = \left(\frac{w}{w_x}\right)_x. \tag{3.10}$$

Now we introduce the traveling wave setting

$$u(x,t) = u(x-ct) = u(\xi). \tag{3.11}$$

The goal is to go from a partial differential equation with two variables x and t, into an ordinary differential equation with the variable ξ. The derivatives can be changed by

$$\frac{d}{dt} = \frac{d\xi}{dt}\frac{d}{d\xi} = -c\frac{d}{d\xi} \tag{3.12}$$

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{d}{d\xi}. \tag{3.13}$$

Applying the traveling wave setting to (3.10) gives

$$c\left((w')^{\frac{1}{2}}\left((w')^{-\frac{1}{2}}\right)''\right)' = \left(\frac{w}{w'}\right)' \tag{3.14}$$

where the derivatives are with respect to ξ. We can integrate both sides of (3.14) to get

$$c(w')^{\frac{1}{2}}\left((w')^{-\frac{1}{2}}\right)'' = \frac{w}{w'} + A \tag{3.15}$$

where A is an integration constant. Multiplying both sides of (3.15) by w' gives

$$w + Aw' = c(w')^{\frac{3}{2}}\Big((w')^{-\frac{1}{2}}\Big)'' \tag{3.16}$$

$$\Rightarrow w + Aw' = c(w')^{\frac{3}{2}}\Big(-\frac{1}{2}(w')^{-\frac{3}{2}}(w'')\Big)' \tag{3.17}$$

$$\Rightarrow w + Aw' = c(w')^{\frac{3}{2}}\left(\frac{3}{4}(w')^{-\frac{5}{2}}(w'')^2 - \frac{1}{2}(w')^{-\frac{3}{2}}(w''')\right) \tag{3.18}$$

$$\Rightarrow \frac{1}{c}(w + Aw') = \frac{3}{4}(w')^{-1}(w'')^2 - \frac{1}{2}w'''. \tag{3.19}$$

If we have derivatives of w instead of w itself, we can substitute back to u by using

$$w' = u^{-2}. \tag{3.20}$$

Differentiating both sides of (3.19) gives

$$\frac{1}{c}(w' + Aw'') = -\frac{3}{4}(w')^{-2}(w'')^3 + \frac{3}{2}(w')^{-1}w''w''' - \frac{1}{2}w''''. \tag{3.21}$$

Using (3.20) we calculate

$$w'' = -2u^{-3}u' \tag{3.22}$$

$$w''' = 6u^{-4}(u')^2 - 2u^{-3}u'' \tag{3.23}$$

$$w'''' = -24u^{-5}(u')^3 + 18u^{-4}u'u'' - 2u^{-3}u'''. \tag{3.24}$$

Plugging these into (3.21) gives

$$\frac{1}{c}(u^{-2} - 2Au^{-3}u') = -\frac{3}{4}u^4(-8u^{-9}(u')^3)+ \tag{3.25}$$

$$\frac{3}{2}u^2(-2u^{-3}u')(6u^{-4}(u')^2-2u^{-3}u'')-\frac{1}{2}(-24u^{-5}(u')^3+18u^{-4}u'u''-2u^{-3}u''')$$

$$\Rightarrow \frac{1}{c}(u^{-2}-2Au^{-3}u') = 6u^{-5}(u')^3 \tag{3.26}$$

$$-18u^{-5}(u')^3+6u^{-4}u'u''+12u^{-5}(u')^3-9u^{-4}u'u''+u^{-3}u'''$$

$$\Rightarrow \frac{1}{c}(u^{-2}-2Au^{-3}u') = -3u^{-4}u'u''+u^{-3}u''' \tag{3.27}$$

Multiplying both sides of (3.27) by u^4 gives

$$\Rightarrow \frac{1}{c}(u^2-2Auu') = -3u'u''+uu''' \tag{3.28}$$

which can be turned into

$$3cu'u''-cuu'''-2Auu'+u^2=0. \tag{3.29}$$

Equation (3.29) is a new equation for which we will find new solutions.

3.2 New Solutions

We consider solutions of the form

$$u(\xi) = ae^{\lambda\xi} \tag{3.30}$$

where a is a real constant and $\lambda \in \mathbb{C}$ is a number to be determined. First, we calculate

$$u' = a\lambda e^{\lambda\xi} \tag{3.31}$$

$$u'' = a\lambda^2 e^{\lambda\xi} \tag{3.32}$$

$$u''' = a\lambda^3 e^{\lambda\xi}. \tag{3.33}$$

Plugging in u and its derivatives into equation (3.29) gives

$$3ca^2\lambda^3e^{2\lambda\xi} - ca^2\lambda^3e^{2\lambda\xi} - 2Aa^2\lambda e^{2\lambda\xi} + a^2e^{2\lambda\xi} = 0 \tag{3.34}$$

$$\Rightarrow 2c\lambda^3 - 2A\lambda + 1 = 0 \tag{3.35}$$

This gives us the cubic equation

$$\lambda^3 - \frac{A}{c}\lambda + \frac{1}{2c} = 0, \quad c \neq 0 \tag{3.36}$$

If we select a λ that solves the cubic equation (3.36), then (3.30) will be a solution to equation (3.29). We will use the formulas derived in section 2.2 to solve for λ in all the cases. Comparing equation (3.36) to the general cubic equation (2.25), we have

$$P = -\frac{A}{c} \tag{3.37}$$

$$Q = -\frac{1}{2c}. \tag{3.38}$$

Inserting P and Q into (2.35) we obtain the discriminant

$$\Delta = -\left(\frac{A}{3c}\right)^3 + \left(\frac{1}{4c}\right)^2 \tag{3.39}$$

$$\Rightarrow \Delta = \frac{1}{16c^2}\left(1 - \frac{16A^3}{27c}\right). \tag{3.40}$$

Case 1: The case $\Delta = 0$ corresponds to

$$1 - \frac{16A^3}{27c} = 0 \tag{3.41}$$

$$\Rightarrow c = \frac{16A^3}{27} \neq 0. \tag{3.42}$$

From the formulas (2.47),(2.48),(2.49), we find that

$$\lambda_1 = -\sqrt[3]{\frac{2}{c}} \tag{3.43}$$

$$\lambda_2 = \lambda_3 = \sqrt[3]{\frac{1}{4c}}. \tag{3.44}$$

These λ are all real numbers. The solutions to equation (3.29) in the case $c = \frac{16A^3}{27} \neq 0$ are given by

$$u_1 = ae^{-\sqrt[3]{\frac{2}{c}}\xi} \tag{3.45}$$

$$u_2 = ae^{\sqrt[3]{\frac{1}{4c}}\xi}. \tag{3.46}$$

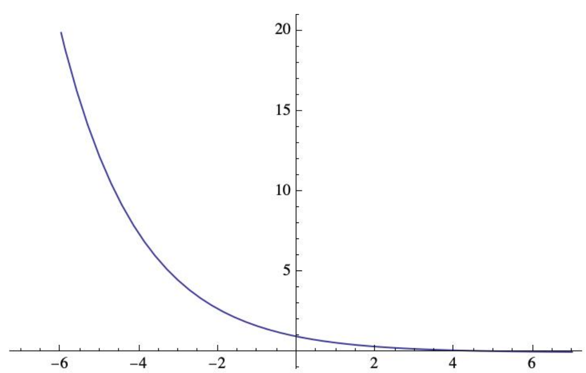

Figure 3.1: Plot of u_1 with $a = 1, A = 3, c = 16$.

Case 2: The case $\Delta < 0$ corresponds to

$$1 - \frac{16A^3}{27c} < 0 \tag{3.47}$$

$$\Rightarrow 1 < \frac{16A^3}{27c} \tag{3.48}$$

This occurs when

$$0 < c < \frac{16A^3}{27} \tag{3.49}$$

or when

$$0 > c > \frac{16A^3}{27}. \tag{3.50}$$

From the formulas (2.70),(2.71),(2.72), we find that

$$\lambda_1 = -2\sqrt{\frac{A}{3c}}\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right) \tag{3.51}$$

$$\lambda_2 = \sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right) + \sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right) \tag{3.52}$$

$$\lambda_3 = \sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right) - \sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right) \tag{3.53}$$

These λ are all real numbers. The solutions to equation (3.29), in the cases $0 < c < \frac{16A^3}{27}$ or $0 > c > \frac{16A^3}{27}$, are given by

$$u_1 = ae^{-2\sqrt{\frac{A}{3c}}\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\xi} \tag{3.54}$$

$$u_2 = ae^{\sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)+\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right)\xi} \tag{3.55}$$

$$u_3 = ae^{\sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)-\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right)\xi}. \tag{3.56}$$

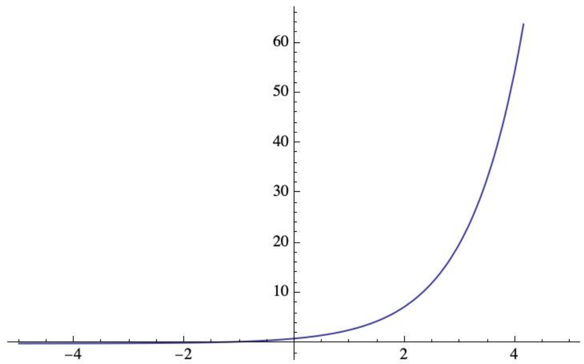

Figure 3.2: Plot of u_2 with $a = 1$, $A = 1$, $c = 1/2$.

Case 3: The case $\Delta > 0$ corresponds to

$$1 - \frac{16A^3}{27c} > 0 \tag{3.57}$$

$$\Rightarrow 1 > \frac{16A^3}{27c} \tag{3.58}$$

This occurs when

$$c > \frac{16A^3}{27}, \quad c > 0 \tag{3.59}$$

or when

$$c < \frac{16A^3}{27}, \quad c < 0. \tag{3.60}$$

From the formulas (2.56),(2.57),(2.58), we find that

$$\lambda_1 = -\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)} - \sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)} \tag{3.61}$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}(1+\sqrt{\sigma})} + \sqrt[3]{\frac{1}{4c}(1-\sqrt{\sigma})}\right) + \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}(1+\sqrt{\sigma})} - \sqrt[3]{\frac{1}{4c}(1-\sqrt{\sigma})}\right) \tag{3.62}$$

$$\lambda_3 = \frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}(1+\sqrt{\sigma})} + \sqrt[3]{\frac{1}{4c}(1-\sqrt{\sigma})}\right) - \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}(1+\sqrt{\sigma})} - \sqrt[3]{\frac{1}{4c}(1-\sqrt{\sigma})}\right) \tag{3.63}$$

where

$$\sigma = 1 - \frac{16A^3}{27c}. \tag{3.64}$$

Here λ_1 is real, while λ_2 and λ_3 are complex. Pluggin in λ into (3.30) gives

$$u_1 = ae^{\left(-\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}-\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)\xi} \tag{3.65}$$

$$u_2 = ae^{\left(\frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}+\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)+\frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}-\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)\right)\xi} \tag{3.66}$$

$$u_3 = ae^{\left(\frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}+\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)-\frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}-\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)\right)\xi}. \tag{3.67}$$

Using Euler's formula (2.69) we can separate u_2 and u_3 into their real and imaginary parts. For example, setting $a = 1, A = -3, c = 2$, we get

$$Re(u_2) = Re(u_3) = e^{\frac{1}{2}\left(\sqrt[3]{\frac{1}{2}}-\sqrt[3]{\frac{1}{4}}\right)\xi}\cos\left(\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2}}+\sqrt[3]{\frac{1}{4}}\right)\xi\right) \tag{3.68}$$

$$Im(u_2) = -Im(u_3) = e^{\frac{1}{2}\left(\sqrt[3]{\frac{1}{2}}-\sqrt[3]{\frac{1}{4}}\right)\xi}\sin\left(\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2}}+\sqrt[3]{\frac{1}{4}}\right)\xi\right). \tag{3.69}$$

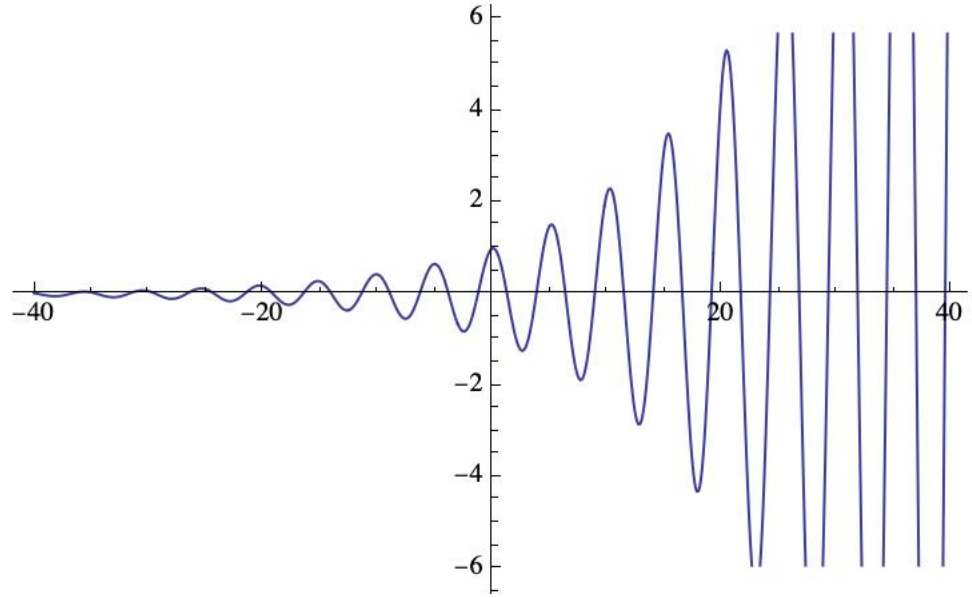

Figure 3.3: Plot of $Re(u_2)$ with $a = 1$, $A = -3$, $c = 2$.

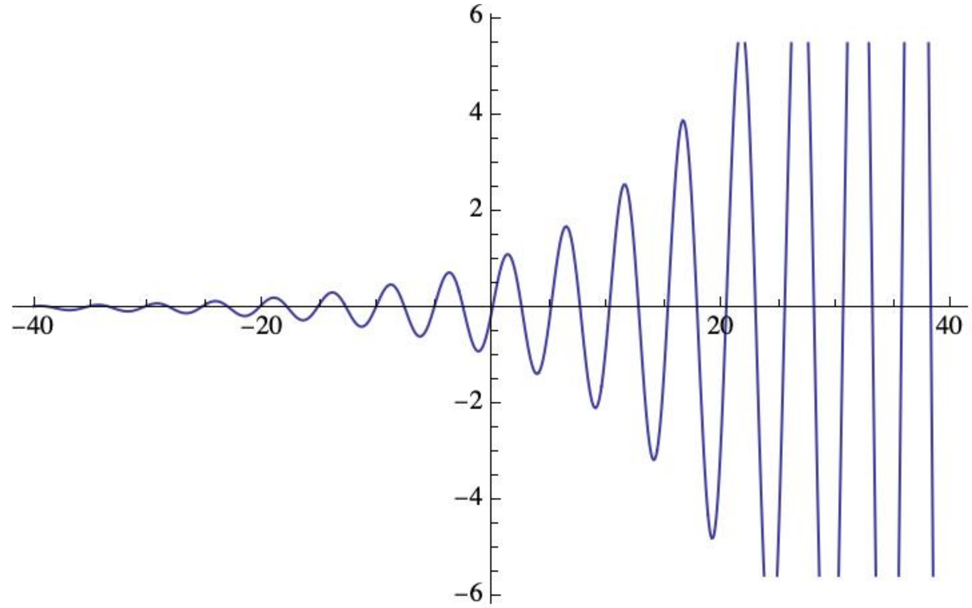

Figure 3.4: Plot of $Im(u_2)$ with $a = 1$, $A = -3$, $c = 2$.

3.3 Other Solution Types

We consider the possibility of solutions with the form

$$u(\xi) = ae^{\alpha\xi} + be^{\beta\xi} \tag{3.70}$$

where a, b are constants and α, β are numbers to be determined. First, we calculate

$$u' = a\alpha e^{\alpha\xi} + b\beta e^{\beta\xi} \tag{3.71}$$

$$u'' = a\alpha^2 e^{\alpha\xi} + b\beta^2 e^{\beta\xi} \tag{3.72}$$

$$u''' = a\alpha^3 e^{\alpha\xi} + b\beta^3 e^{\beta\xi} \tag{3.73}$$

$$u'u'' = a^2\alpha^3 e^{2\alpha\xi} + ab(\alpha^2\beta + \alpha\beta^2)e^{(\alpha+\beta)\xi} + b^2\beta^3 e^{2\beta\xi} \tag{3.74}$$

$$uu''' = a^2\alpha^3 e^{2\alpha\xi} + ab(\alpha^3 + \beta^3)e^{(\alpha+\beta)\xi} + b^2\beta^3 e^{2\beta\xi} \tag{3.75}$$

$$uu' = a^2\alpha e^{2\alpha\xi} + ab(\alpha+\beta)e^{(\alpha+\beta)\xi} + b^2\beta e^{2\beta\xi} \tag{3.76}$$

$$u^2 = a^2 e^{2\alpha\xi} + 2abe^{(\alpha+\beta)\xi} + b^2 e^{2\beta\xi} \tag{3.77}$$

Plugging these into equation (3.29) gives

$$a^2 e^{2\alpha\xi}\big(2c\alpha^3 - 2A\alpha + 1\big) + abe^{(\alpha+\beta)\xi}\Big(3c\big(\alpha^2\beta + \beta^2\alpha\big) - c\big(\alpha^3 + \beta^3\big) - 2A\big(\alpha+\beta\big) + 2\Big)$$

$$+b^2 e^{2\beta\xi}\big(2c\beta^3 - 2A\beta + 1\big) = 0. \tag{3.78}$$

This leads to three seperate equations:

$$\alpha^3 - \frac{A}{c}\alpha + \frac{1}{2c} = 0 \tag{3.79}$$

$$\beta^3 - \frac{A}{c}\beta + \frac{1}{2c} = 0 \tag{3.80}$$

$$3c\big(\alpha^2\beta + \beta^2\alpha\big) - c\big(\alpha^3 + \beta^3\big) - 2A\big(\alpha + \beta\big) + 2 = 0 \tag{3.81}$$

where $c \neq 0$. We must find the relationship between α and β, so that all three equations are solved. Consider

$$\alpha = s\beta \tag{3.82}$$

where s is a constant. Plugging in (3.82) into the third equation (3.81) gives

$$3c(s^2\beta^3 + s\beta^3) - c(s^3\beta^3 + \beta^3) - 2A(s\beta + \beta) + 2 = 0 \tag{3.83}$$

$$\Rightarrow c\beta^3(-s^3 + 3s^2 + 3s - 1) - 2A\beta(s+1) + 2 = 0 \tag{3.84}$$

$$\Rightarrow c\beta^3(s^2 - 4s + 1)(s+1) + 2A\beta(s+1) - 2 = 0 \tag{3.85}$$

$$\Rightarrow \beta^3 + \frac{A}{c}\beta\left(\frac{2}{s^2 - 4s + 1}\right) - \frac{2}{c(s+1)(s^2 - 4s + 1)} = 0. \tag{3.86}$$

In order for equation (3.86) to be consistent with equation (3.80) we require

$$\frac{2}{s^2 - 4s + 1} = -1 \tag{3.87}$$

$$\Rightarrow s^2 - 4s + 3 = 0 \tag{3.88}$$

$$\Rightarrow s = 1, \quad s = 3 \tag{3.89}$$

We also require

$$-\frac{2}{(s+1)(s^2-4s+1)}=\frac{1}{2} \tag{3.90}$$

$$\Rightarrow s^3-3s^2-3s+5=0 \tag{3.91}$$

$$\Rightarrow s=1, \quad s=1+\sqrt{6}, \quad s=1-\sqrt{6} \tag{3.92}$$

The only value for s that meets both requirements is $s=1$. From (3.82) this means that $\alpha=\beta$. Therefore, the solution type (3.70) becomes

$$u(\xi)=(a+b)e^{\alpha\xi}. \tag{3.93}$$

This form is identical to the form (3.30) which was already considered, so we don't obtain any new solutions.

Next, we consider solutions of the form

$$u(\xi)=a\,\mathrm{sech}(b\xi) \tag{3.94}$$

where a and b are constants. First, we calculate

$$u'=-ab\tanh(b\xi)\,\mathrm{sech}(b\xi) \tag{3.95}$$

$$u''=ab^2\big(\tanh^2(b\xi)\,\mathrm{sech}(b\xi)-\mathrm{sech}^3(b\xi)\big) \tag{3.96}$$

$$u'''=ab^3\big(5\tanh(b\xi)\,\mathrm{sech}^3(b\xi)-\tanh^3(b\xi)\,\mathrm{sech}(b\xi)\big) \tag{3.97}$$

$$u'u''=a^2b^3\big(\tanh(b\xi)\,\mathrm{sech}^4(b\xi)-\tanh^3(b\xi)\,\mathrm{sech}^2(b\xi)\big) \tag{3.98}$$

$$uu''' = a^2b^3\big(5\tanh(b\xi)\,\mathrm{sech}^4(b\xi) - \tanh^3(b\xi)\,\mathrm{sech}^2(b\xi)\big) \tag{3.99}$$

$$uu' = -a^2b\tanh(b\xi)\,\mathrm{sech}^2(b\xi) \tag{3.100}$$

$$u^2 = a^2\,\mathrm{sech}^2(b\xi) \tag{3.101}$$

Plugging these into equation (3.29) gives

$$-2ca^2b^3\tanh(b\xi)\,\mathrm{sech}^4(b\xi) - 2ca^2b^3\tanh^3(b\xi)\,\mathrm{sech}^2(b\xi) \tag{3.102}$$

$$+2Aa^2b\tanh(b\xi)\,\mathrm{sech}^2(b\xi) + a^2\,\mathrm{sech}^2(b\xi) = 0.$$

Dividing by $-a^2\tanh(b\xi)\,\mathrm{sech}^2(b\xi)$ with $b \neq 0$, we get

$$2cb^3\big(\mathrm{sech}^2(b\xi) + \tanh^2(b\xi)\big) - 2Ab - \coth(b\xi) = 0. \tag{3.103}$$

Using $\mathrm{sech}^2(b\xi) + \tanh^2(b\xi) = 1$, we get

$$2cb^3 - 2Ab = \coth(b\xi). \tag{3.104}$$

Since the right hand side of (3.104)is a function of ξ, there is no value of b that solves this equation. Therefore (3.94) is not a solution to equation (3.29).

Next, we consider solutions of the form

$$u(\xi) = a\tanh(b\xi) \tag{3.105}$$

where a and b are constants. First, we calculate

$$u' = ab\,\mathrm{sech}^2(b\xi) \tag{3.106}$$

$$u'' = -2ab^2 \tanh(b\xi)\,\mathrm{sech}^2(b\xi) \tag{3.107}$$

$$u''' = 2ab^3\big(2\tanh^2(b\xi)\,\mathrm{sech}^2(b\xi) - \mathrm{sech}^4(b\xi)\big) \tag{3.108}$$

$$u'u'' = -2a^2b^3 \tanh(b\xi)\,\mathrm{sech}^4(b\xi) \tag{3.109}$$

$$uu''' = 2a^2b^3\big(2\tanh^3(b\xi)\,\mathrm{sech}^2(b\xi) - \tanh(b\xi)\,\mathrm{sech}^4(b\xi)\big) \tag{3.110}$$

$$uu' = a^2b\tanh(b\xi)\,\mathrm{sech}^2(b\xi) \tag{3.111}$$

$$u^2 = a^2\tanh^2(b\xi) \tag{3.112}$$

Plugging these into (3.29) gives

$$-4ca^2b^3\tanh(b\xi)\,\mathrm{sech}^4(b\xi) - 4ca^2b^3\tanh^3(b\xi)\,\mathrm{sech}^2(b\xi) \tag{3.113}$$

$$-2Aa^2b\tanh(b\xi)\,\mathrm{sech}^2(b\xi) + a^2\tanh^2(b\xi) = 0. \tag{3.114}$$

Dividing by $-a^2\tanh(b\xi)\,\mathrm{sech}^2(b\xi)$ with $b \neq 0$, we get

$$4cb^3\big(\mathrm{sech}^2(b\xi) + \tanh^2(b\xi)\big) + 2Ab - \tanh(b\xi)\cosh^2(b\xi) = 0. \tag{3.115}$$

Using $\mathrm{sech}^2(b\xi) + \tanh^2(b\xi) = 1$, we get

$$4cb^3 + 2Ab - \sinh(b\xi)\cosh(b\xi) = 0. \tag{3.116}$$

Using $\sinh(b\xi)\cosh(b\xi) = \frac{1}{2}\sinh(2b\xi)$, we get

$$8cb^3 + 4Ab = \sinh(2b\xi). \tag{3.117}$$

Since the right hand side of (3.117) is a function of ξ there is no value of b that solves this equation. Therefore, (3.105) is not a solution to equation (3.29).

Finally, we consider peakon solutions of the form

$$u(\xi) = ae^{-\lambda|\xi|} \tag{3.118}$$

where a is a constant and λ is a number to be determined. First, we define

$$|\xi|' = sgn(\xi) = \begin{cases} -1, & \xi < 0 \\ 0, & \xi = 0 \\ 1, & \xi > 0 \end{cases} \tag{3.119}$$

and

$$sgn(\xi)' = 2\delta(\xi) \tag{3.120}$$

with

$$\delta(\xi) = \begin{cases} 0, & \xi \neq 0 \\ \infty, & \xi = 0 \end{cases}. \tag{3.121}$$

Now, we can calculate

$$u' = -\lambda sgn(\xi)ae^{-\lambda|\xi|}. \tag{3.122}$$

Using $sgn^2(\xi) = 1$, we calculate

$$\begin{aligned} u'' &= \big(-2\lambda\delta(\xi) + \lambda^2 sgn^2(\xi)\big)ae^{-\lambda|\xi|} \\ &= \big(\lambda^2 - 2\lambda\delta(\xi)\big)ae^{-\lambda|\xi|}. \end{aligned} \tag{3.123}$$

Using $sgn(\xi)\delta(\xi) = 0$, we calculate

$$u''' = \Big(-2\lambda\delta'(\xi) - \lambda sgn(\xi)\big(\lambda^2 - 2\lambda\delta(\xi)\big)\Big) ae^{-\lambda|\xi|}$$

$$= \big(-2\lambda\delta'(\xi) - \lambda^3 sgn(\xi)\big) ae^{-\lambda|\xi|}. \tag{3.124}$$

Plugging in u and its derivatives into (3.29), we get

$$-3c\lambda sgn(\xi)\big(\lambda^2 - 2\lambda\delta(\xi)\big) + c\big(2\lambda\delta'(\xi) + \lambda^3 sgn(\xi)\big) + 2A\lambda sgn(\xi) + 1 = 0. \tag{3.125}$$

Using $sgn(\xi)\delta(\xi) = 0$, we get

$$-2c\lambda^3 sgn(\xi) + 2c\lambda\delta'(\xi) + 2A\lambda sgn(\xi) + 1 = 0 \tag{3.126}$$

$$\Rightarrow \lambda^3 sgn(\xi) - \lambda\left(\delta'(\xi) - \frac{A}{c} sgn(\xi)\right) - \frac{1}{2c} = 0 \tag{3.127}$$

$$\Rightarrow \lambda^3 - \lambda\left(\frac{A}{c} + \delta'(\xi) sgn(\xi)\right) - \frac{1}{2c} sgn(\xi) = 0 \tag{3.128}$$

The term $\delta'(\xi)sgn(\xi)$ in (3.128) is undefined, so the existence of solutions of the type (3.118) is unknown. We leave this as an open problem for the future.

CHAPTER IV

SUMMARY AND CONCLUSION

Starting from the KdV spectral problem we showed how to produce the KdV hierarchy. Then, we gave the different evolution equations that arise after using the different possible seed functions. After applying the traveling wave setting to equation (2.23), we integrated to find the new ordinary differential equation

$$3cu'u'' - cuu''' - 2Auu' + u^2 = 0.$$

Assuming solutions of the form

$$u = ae^{\lambda\xi}$$

led us to a cubic equation (3.36) in λ. After solving the cubic equation we gave new solutions for all the different cases. For the cases $\Delta = 0$ and $\Delta < 0$ we found that all λ are real, so the solutions are not particularly interesting. For $\Delta > 0$ we have complex λ, so we can separate u into its real and imaginary parts. Plotting gives oscillatory wave solutions which eventually become unbounded.

We considered other solution forms which are common in these types of evolution equations. We found that

$$u = ae^{\alpha\xi} + be^{\beta\xi}$$

does not allow other solutions besides the trivial $\alpha = \beta$. We also found that

$$u = a\cosh(b\xi) \tag{4.1}$$

$$u = a\tanh(b\xi) \tag{4.2}$$

cannot be solutions to our equation. After considering peakon solutions of the form

$$u = ae^{-\lambda|\xi|}$$

we arrived at an equation with an undefined term. Therefore, we still don't know whether our equation allows for peakon solutions.

This research was able to find new solutions to an equation in the NKdV hierarchy. None of these solutions were classical solitary traveling wave solutions. In the future, we will continue to investigate the existence of peakon solutions in the NKdV hierachy, as well as other integrable systems. Meanwhile, our upcoming research will study the multi-peakon solutions of the fifth-order Camassa-Holm (FOCH) equation.

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BIOGRAPHICAL SKETCH

Miguel Rodriguez earned a Bachelor of Science in Physics and a Minor in Applied Mathematics from The University of Texas Rio Grande Valley (UTRGV) in May 2018. A year later, he was awarded UTRGV's Presidential Graduate Research Assistantship. At this time, he started doing research in integrable systems with Dr. Zhijun Qiao. In August 2021, he earned a Master of Science in Mathematics with a concentration in Applied Mathematics from UTRGV. Miguel has since been accepted into the Ph.D in Mathematics and Statistics with Interdisciplinary Applications Program at UTRGV. He can be reached at miguel.a.rodriguez01@utrgv.edu.