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Two-dimensional Wigner-Ville transforms and their basic properties

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TWO-DIMENSIONAL WIGNER-VILLE TRANSFORMS

AND THEIR BASIC PROPERTIES

A Thesis

by

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ABSTRACT

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This thesis deals with Wigner-Ville transforms and their basic properties. The Wigner-Ville transforms are a non-linear transform which constitute an important tool in nonstationary signal analysis. Wigner-Ville transforms in one dimension and their basic properties are discussed here. Special attention is given to formulation of two dimensional Wigner-Ville transform, its inversion formula and some of their basic properties. Some applications of Wigner-Ville transforms are also briefly discussed.

DEDICATION

This work is dedicated to my father, Shankara Narayana Rao, my mother, Shakuntala and my husband, Mallikarjun.

ACKNOWLEDGEMENTS

I am grateful to Dr. Lokenath Debnath chair of my thesis committee, for all his mentoring and advice. My thanks go to my thesis committee members: Dr. Dambaru Bhatta, Dr. Paul Bracken, Dr. Ramendra Krishna Bose. Also, I would like to express my thanks to Ms.Veronica Martinez for her help in formatting the thesis.

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DECLARATION

A part of the work presented in this thesis has been submitted in the following:

[9] Debnath, L. and Shankara Narayana Rao, B. V., On New Two-Dimensional Wigner-Ville Nonlinear Integral Transforms and their Basic Properties, *Integral transforms and special functions*, Accepted 2009.

CHAPTER 1

INTRODUCTION

Eugene Paul Wigner (1902-1995) was born in Budapest, Hungary on November 17, 1902. Wigner was educated at the Berlin Institute of Technology and obtained a doctrate in engineering in 1925. He moved to America in 1930 and joined the Princeton University and in 1938 was appointed to the chair of theoretical physics. Wigner remained at Princeton until his retirement in 1971.

Wigner made many fundamental contributions to quantum mechanics and nuclear physics. He determined that the nuclear force is short-range and does not involve an electric charge, using group theory to investigate atomic structure. His name has been given to several formulations, including the Breit-Wigner formula, which describes resonant nuclear reactions. During World War II he worked on the Manhattan Project, which resulted in the first atomic bomb. After beginning his association with the Atomic energy Commisssion in 1947, he served as a member of its general advisory committee. He won the 1963 Nobel prize in Physics (shared with U.S physicist Maria Goeppert-Mayer and German physicist J.H.D. Jensen) for his work on the structure of the atomic nucleus, quantum mechanics and law of conservation of parity. Wigner also received other major awards, including the Max Planck Medal of the German Physical Society, the Enrico Fermi Prize, the Albert Einstein Award, National Science Medal and Atoms for Peace Award. The Wigner distribution or the Wigner transforms was introduced by Wigner in 1932 to study quantum mechanics [1].

Although Fourier transform analysis has widespread applications in Science and Engineering, it cannot be used effectively to analyse nonstationary signals. To overcome this difficulty, other time-frequency transform such as Gabor transform, Zak transform, Wavelet transform, Wigner transform were developed. Also the need for a combined time-frequency representation stemmed from the inadequacy of either time domain or frequency domain analysis to fully describe the nature of non-stationary signals. A time frequency distribution of a signal provides information about how the spectral content of the signal evolves with time, thus providing an ideal tool to dissect, analyse and interpret non-stationary signals.

Later in 1948, Jean Ville a French Mathematician proposed that the Wigner transform can be used to analyse time frequency structures of a non stationary signal which led to the development of Wigner-Ville transforms [2]

$$W_f(t,\omega) = \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\bar{f}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau. \tag{1.1}$$

Wigner-Ville transforms are very important in the field of bilinear/quadratic time-frequency representations. From both theoretical and application points of view, the Wigner-Ville transforms plays a vital role in time-frequency signal analysis for the following reasons. For a non-stationary signal it provides a high resolution representation in terms of time and frequency. It satisfies time and frequency marginals in terms of instantaneous power in time and energy spectrum in frequency and total energy of the signal in the time and frequency plane. Also the first conditional moment of frequency at a given time is the derivative of the phase of the signal at that time. The theory of Wigner-Ville transforms was reformulated in the context of sonar and radar signal analysis and a new function called Woodward ambiguity function was introduced by Woodward in 1953 to mathematically analyse sonar and radar systems [3]. In analogy with the Heisenberg uncertainty principle in quantum mechanics, Woodward introduced the radar uncertainty principle which says that the range and velocity of a target cannot be measured precisely and simultaneously..

Even though the time-frequency analysis has its origin almost fifty years ago, there has been major developments in the past three decades. So the time frequency analysis is a widely recognized and applied subject in signal processing. The basic idea of the method is to develop a joint function of time and frequency, known as a time-frequency distribution, that can describe the energy density of a signal simultaneously in both time and frequency. In principle, the time-frequency distributions characterize phenomena in a two-dimensional time-frequency plane. So, the time-frequency signal analysis deals with time-frequency representations of signals and with problems related to their definition, estimation and interpretation, and it has evolved into a widely recognized applied discipline of signal processing. Thus based on studies of its mathematical structures and properties by several authors including Claasen and Mecklenbräuker [4], Boashash [5], Debnath [6], Wigner-Ville transforms and its applications has been brought to the attention of Mathematical and Scientific Community.

During the last fifty years, considerable attention has been given to the one-dimensional Wigner-Ville transform for the time-frequency signal analysis. However, hardly any attention is given to two-dimensional Wigner-Ville transforms. In this thesis we have formulated the two- dimensional Wigner-Ville nonlinear integral transforms , its inversion formula and some of its basic properties. Some of the most important applications of Wigner-Ville transforms are also discussed.

Chapter 1 is Introduction, the history of Wigner-Ville Transform is discussed here.

Chapter 2 deals with the motivation of Wigner.

Chapter 3 deals with one-dimensional Wigner-Ville transforms, its definition and interpretation of time-frequency marginal integrals and time-frequency energy distribution.

In Chapter 4 we will see some of its properties such as non linearity, translation, complex conjugation, modulation, translation and modulation, general modulation, dilation, multiplication, differentiation, time and frequency moments and its proof [7].

Chapter 5 deals with formulation of two- dimensional Cross and Auto Wigner-Ville transforms which are nonlinear integral transforms, its definition, inversion formula and energy density function.

In Chapter 6 some two-dimensional properties of Wigner-Ville transforms like non linearity, translation, complex conjugation, modulation, translation and modulation, general modulation, dilation, multiplication, differentiation are proved.

In Chapter 7 examples of two-dimensional Wigner-Ville transforms are discussed.

In Chapter 8 some important applications of Wigner-Ville transforms are briefly discussed.

Chapter 9 is Conclusion.

CHAPTER 2

WIGNER'S MOTIVATION

Wigner's original motivation for introducing Wigner distribution was to be able to calculate the quantum corrrection to the second viral coefficient of gas, which shows how it deviates from the ideal gas law. Classically to calculate this, we need a joint distribution of position and momentum. So Wigner devised a joint distribution that gave marginals, the quantum mechanical distributions of position and momentum. The quantum mechanics came in the distribution but the distribution was used in the classical manner. It was a hybrid method. Also, Wigner was motivated by the work of Kirkwood and Margenau who calculated this quantity but Wigner improved on it [8].

The Wigner distribution was introduced into signal analysis by Ville, some 15 years after Wigner's paper. Ville derived it by a method based on characteristic function. The same type of derivation was used by Moyal at about same time.

CHAPTER 3

WIGNER-VILLE TRANSFORMS IN ONE DIMENSION

The one-dimensional cross Wigner-Ville transform of two functions (or signals) $f, g \in L^2(\mathbb{R})$ is defined by

$$W_{f,g}(t,\omega) = \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\overline{g}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau, \tag{3.1}$$

provided the integral exists.

Introducing a change of variable $t+\frac{\tau}{2}=x$ gives an equivalent form of (3.1)

$$\begin{aligned} W_{f,g}(t,\omega) &= 2\exp(2i\omega t)\int_{-\infty}^{\infty} f(x)\overline{g}(2t-x)\exp(-2i\omega x)\,dx \\ &= 2\exp(2i\omega t)\widetilde{f}_h(2t,2\omega), \end{aligned} \tag{3.2}$$

where $h(x)=\overline{g}(-x)$, $\widetilde{f}_h(t,\omega)$ is the continuous Gabor transform of a function $h \in L^2(\mathbb{R})$ with respect to the window $g \in L^2(\mathbb{R})$ defined by

$$\begin{aligned} \widetilde{f}_h(t,\omega) &= G[h](t,\omega) = \int_{-\infty}^{\infty} h(\tau)\overline{g}(\tau-t)e^{-i\omega\tau}d\tau = (h, g_{t,\omega}), \\ g_{t,\omega}(\tau) &= \overline{g}(\tau-t)\exp(i\omega\tau) \end{aligned} \tag{3.3}$$

so that $\|g_{t,\omega}\| = \|g\|$, and hence $g_{t,\omega} \in L^2(\mathbb{R})$.

Putting $f=g$ in (3.1), $W_{f,f}(t,\omega)=W_f(t,\omega)$ is called the auto Wigner-Ville transform defined by

$$W_f(t,\omega) = \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\overline{f}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau, \tag{3.4}$$

$$= 2\exp(2i\omega t)\int_{-\infty}^{\infty} f(x)\overline{f}(2t-x)\exp(-2i\omega x)\,dx, \tag{3.5}$$

$$= 2\exp(2i\omega t)\widetilde{f}_h(2t,2\omega), \tag{3.6}$$

where $h(x) = \overline{f}(-x)$.

It follows from (3.4) that the auto Wigner-Ville transform is the Fourier transform of the function

$$h_t(\tau) = f\left(t+\frac{\tau}{2}\right)\overline{f}\left(t-\frac{\tau}{2}\right)$$

with respect to τ. Hence, $W_f(t,\omega)$ is a complex-valued function in the time-frequency plane. In other words,

$$W_f(t,\omega) = \Im\{h_t(\tau)\} = \widehat{h}_t(\omega), \tag{3.7}$$

where $\widehat{h}_t(\omega)$ is the Fourier transform of $h_t(\tau)$ defined by

$$\widehat{h}_t(\omega) = \Im\{h_t(\tau)\} = \int_{-\infty}^{\infty} e^{-i\omega\tau} h_t(\tau)\, d\tau. \tag{3.8}$$

On the other hand, the Fourier transform of the auto Wigner-Ville transform with respect to ω is given by

$$\begin{aligned}
\widehat{W}_f(t,\sigma) &= \int_{-\infty}^{\infty} e^{-i\omega\sigma} W_f(t,\omega)\, d\omega, \\
&= \int_{-\infty}^{\infty} e^{-i\omega\sigma} d\omega \int_{-\infty}^{\infty} h_t(\tau) e^{-i\omega\tau} d\tau, \\
&= \int_{-\infty}^{\infty} h_t(\tau)\, d\tau \int_{-\infty}^{\infty} e^{-i\omega(\tau+\sigma)} d\omega, \\
&= 2\pi \int_{-\infty}^{\infty} h_t(\tau)\, \delta(\tau+\sigma)\, d\tau \\
&= 2\pi h_t(-\sigma) = 2\pi f\left(t-\frac{\sigma}{2}\right)\overline{f}\left(t+\frac{\sigma}{2}\right).
\end{aligned} \tag{3.9}$$

Or, equivalently,

$$\widehat{W}_f(t,-\sigma) = 2\pi f\left(t+\frac{\sigma}{2}\right)\overline{f}\left(t-\frac{\sigma}{2}\right). \tag{3.10}$$

The formula (3.9) can also be written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\sigma} W_f(t,\omega)\, d\omega = f\left(t-\frac{\sigma}{2}\right)\overline{f}\left(t+\frac{\sigma}{2}\right),$$

which is, by putting $\left(t+\frac{\sigma}{2}\right)=t_1$ and $\left(t-\frac{\sigma}{2}\right)=t_2$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(t_1-t_2)\omega} W_f\left(\frac{t_1+t_2}{2},\omega\right) d\omega \;=\; f(t_2)\overline{f}(t_1). \tag{3.11}$$

Putting $t_1=0$ and $t_2=t$ in (3.11) gives a representation of $f(t)$ in terms of W_f in the form

$$f(t)\overline{f}(0) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega} W_f\left(\frac{t}{2},\omega\right) d\omega, \tag{3.12}$$

provided $\overline{f}(0)\neq 0$. This is the inversion formula for the Wigner Ville transform.

In particular, putting $t_1=t_2=t$ in (3.11) leads to the inversion formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t,\omega)\, d\omega \;=\; |f(t)|^2. \tag{3.13}$$

This means that the integral of the Wigner-Ville transform over the frequency at any time t is equal to the *time energy density* of a signal $f(t)$.

Integrating (3.13) with respect to time t gives the total energy over the whole time-frequency plane (t,ω)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\, dt\, d\omega \;=\; \int_{-\infty}^{\infty} |f(t)|^2 dt = (f,f) = \|f\|_2^2. \tag{3.14}$$

If $\widehat{f}(\omega)=\mathfrak{F}\{f(t)\}$, the Wigner-Ville transform of the Fourier spectrum $\widehat{f}(\omega)$ is defined by

$$\begin{aligned} W_{\widehat{f}}(\omega,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}\left(\omega+\frac{\tau}{2}\right)\overline{\widehat{f}}\left(\omega-\frac{\tau}{2}\right) e^{it\tau} d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx \int_{-\infty}^{\infty} \overline{\widehat{f}}\left(\omega-\frac{\tau}{2}\right) \exp\left[i\tau\left(t-\frac{x}{2}\right)\right] d\tau \\ &= 2\exp(2i\omega t)\int_{-\infty}^{\infty} f(x)\, e^{-2i\omega x} dx \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\widehat{f}}(u) \exp\left[iu(x-2t)\right] du \end{aligned} \tag{3.15}$$

where $\omega-\frac{\tau}{2}=u$

$$= 2\exp(2i\omega t)\int_{-\infty}^{\infty} f(x)\overline{f}(2t-x)\exp(-2i\omega x)\, dx = W_f(t,\omega). \tag{3.16}$$

Thus, (3.16) can be used as another equivalent definition of the Wigner-Ville transform due to symmetry between time and frequency, as expressed by the relation (3.9).

It also follows from (3.15) and (3.16) that the Fourier transform of $W_f(t,\omega)$ with respect to t is given by

$$\int_{-\infty}^{\infty} e^{-it\tau} W_f(t,\omega)\,dt \;=\; \widehat{f}\left(\omega+\frac{\tau}{2}\right)\overline{\widehat{f}}\left(\omega-\frac{\tau}{2}\right). \tag{3.17}$$

Putting $\tau = 0$ in (3.17) gives

$$\int_{-\infty}^{\infty} W_f(t,\omega)\,dt \;=\; \widehat{f}(\omega)\overline{\widehat{f}}(\omega) = \left|\widehat{f}(\omega)\right|^2. \tag{3.18}$$

Integrating (3.18) with respect to frequency ω yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\,dt\,d\omega \;=\; \int_{-\infty}^{\infty} \left|\widehat{f}(\omega)\right|^2 d\omega = \left\|\widehat{f}\right\|_2^2. \tag{3.19}$$

Thus, if $f \in L^2(\mathbb{R})$, then the Wigner-Ville transform satisfies the time and frequency marginal integrals (3.13) and (3.18) respectively. Moreover, the integral of the Wigner-Ville transform over the entire time-frequency plane yields the total energy of the signal $f(t)$, that is,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\,dt\,d\omega \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widehat{f}(\omega)\right|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2\,dt. \tag{3.20}$$

Physically, the Wigner-Ville transform can be interpreted as the time-frequency energy distribution.

Both the cross Wigner-Ville transform and auto Wigner-Ville transform are referred to as Wigner-Ville transforms or Wigner-Ville distribution.

CHAPTER 4

PROPERTIES OF ONE-DIMENSIONAL WIGNER-VILLE TRANSFORMS

Before we discuss some basic properties of the Wigner-Ville transforms, we define the translation, modulation and dilation operators.

Translation:

$$T_a f(x) = f(x-a)$$

Modulation:

$$M_b f(x) = e^{ibx} f(x)$$

Dilation:

$$D_c f(x) = \frac{1}{\sqrt{|c|}} f\left(\frac{x}{c}\right)$$

where $a, b, c \in \mathbb{R}$ and $c \neq 0$

Properties:

Some basic properties of the one dimensional Wigner-Ville transforms are as follows:

(a) Non linearity: The Wigner-Ville transform is non linear. This follows from the definition

$$W_{f_1+f_2,g_1+g_2}(t,\omega) = W_{f_1,g_1}(t,\omega) + W_{f_1,g_2}(t,\omega) + W_{f_2,g_1}(t,\omega) + W_{f_2,g_2}(t,\omega) \quad (4.1)$$

Proof: We have, by definition $W_{f_1+f_2,g_1+g_2}(t,\omega)$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\left[f_1\left(t+\frac{\tau}{2}\right)+f_2\left(t+\frac{\tau}{2}\right)\right]\left[\overline{g}_1\left(t-\frac{\tau}{2}\right)+\overline{g}_2\left(t-\frac{\tau}{2}\right)\right]e^{-i\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} f_1\left(t+\frac{\tau}{2}\right)\overline{g}_1\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau+\int_{-\infty}^{\infty} f_1\left(t+\frac{\tau}{2}\right)\overline{g}_2\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau \\
&+ \int_{-\infty}^{\infty} f_2\left(t+\frac{\tau}{2}\right)\overline{g}_1\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau+\int_{-\infty}^{\infty} f_2\left(t+\frac{\tau}{2}\right)\overline{g}_2\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau
\end{aligned}
$$

Again using the definition of the one dimensional Wigner-Ville transform, we get

$$
= W_{f_1,g_1}(t,\omega)+W_{f_1,g_2}(t,\omega)+W_{f_2,g_1}(t,\omega)+W_{f_2,g_2}(t,\omega)
$$

(b) Translation: The time shift of signals corresponds to a time shift of the Wigner-Ville transform.

$$
W_{T_af,T_ag}(t,\omega) = W_{f,g}(t-a,\omega) \tag{4.2}
$$

Proof: We have, by definition

$$
\begin{aligned}
W_{T_af,T_ag}(t,\omega) &= \int_{-\infty}^{\infty} f\left(t-a+\frac{\tau}{2}\right)\overline{g}\left(t-a-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau \\
&= W_{f,g}(t-a,\omega)
\end{aligned}
$$

(c) Complex Conjugation:

$$
\overline{W}_{f,g}(t,\omega) = W_{g,f}(t,\omega) \tag{4.3}
$$

Proof: We have, by definition

$$
\begin{aligned}
\overline{W}_{f,g}(t,\omega) &= \int_{-\infty}^{\infty} \overline{f}\left(t+\frac{\tau}{2}\right)g\left(t-\frac{\tau}{2}\right)e^{i\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} g\left(t+\frac{x}{2}\right)\overline{f}\left(t-\frac{x}{2}\right)e^{-i\omega x}dx \\
&= W_{g,f}(t,\omega)
\end{aligned}
$$

(d) Modulation:

$$W_{M_bf,M_bg}(t,\omega) = W_{f,g}(t,\omega-b) \tag{4.4}$$

Proof: We have, by definition

$$\begin{aligned} W_{M_bf,M_bg}(t,\omega) &= \int_{-\infty}^{\infty} e^{ib\left(t+\frac{\tau}{2}\right)} f\left(t+\frac{\tau}{2}\right) e^{-ib\left(t-\frac{\tau}{2}\right)} \overline{g}\left(t-\frac{\tau}{2}\right) e^{-i\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right) \overline{g}\left(t-\frac{\tau}{2}\right) e^{-i\tau(\omega-b)} d\tau \\ &= W_{f,g}(t,\omega-b) \end{aligned}$$

(e) Translation and Modulation:

$$W_{M_bT_af,M_bT_ag}(t,\omega) = W_{T_aM_bf,T_aM_bg}(t,\omega) = W_{f,g}(t-a,\omega-b) \tag{4.5}$$

Proof: Set

$$\begin{aligned} u(t) &= M_bT_af = e^{ibt} f(t-a) \\ v(t) &= M_bT_ag = e^{ibt} g(t-a) \end{aligned}$$

Thus

$$\begin{aligned} W_{u,v}(t,\omega) &= \int_{-\infty}^{\infty} u\left(t+\frac{\tau}{2}\right) \overline{v}\left(t-\frac{\tau}{2}\right) e^{-i\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} e^{ib\left(t+\frac{\tau}{2}\right)} f\left(t-a+\frac{\tau}{2}\right) e^{-ib\left(t-\frac{\tau}{2}\right)} \overline{g}\left(t-a-\frac{\tau}{2}\right) e^{-i\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} f\left(t-a+\frac{\tau}{2}\right) \overline{g}\left(t-a-\frac{\tau}{2}\right) e^{-i\tau(\omega-b)} d\tau \\ &= W_{f,g}(t-a,\omega-b) \end{aligned}$$

(f) General Modulation: The Wigner-Ville transform of a modulated signal $f(t)\,m(t)$ is the convolution of $W_f(t,u)$ and $W_m(t,\omega)$ in the frequency variable.

$$W_{fm}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t,u)W_m(t,\omega-u)du \tag{4.6}$$

Proof: We have, by definition,

$$\begin{aligned} W_{fm}(t,\omega) &= \int_{-\infty}^{\infty} f\left(t+\frac{x}{2}\right) m\left(t+\frac{x}{2}\right) \overline{f}\left(t-\frac{x}{2}\right) \overline{m}\left(t-\frac{x}{2}\right) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} f\left(t+\frac{x}{2}\right) \overline{f}\left(t-\frac{x}{2}\right) dx. \\ & \qquad \int_{-\infty}^{\infty} m\left(t+\frac{y}{2}\right) \overline{m}\left(t-\frac{y}{2}\right) e^{-i\omega y} \delta(y-x) dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} f\left(t+\frac{x}{2}\right) \overline{f}\left(t-\frac{x}{2}\right) e^{-ixu} dx. \\ & \qquad \int m\left(t+\frac{y}{2}\right) \overline{m}\left(t-\frac{y}{2}\right) e^{-i(\omega-u)y} dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t,u)W_m(t,\omega-u)du \end{aligned}$$

(g) Dilation: If

$$D_c f(t) = \frac{1}{\sqrt{|c|}} f\left(\frac{t}{c}\right), c \neq 0$$

then

$$W_{D_c f, D_c g}(t,\omega) = W_{f,g}\left(\frac{t}{c}, c\omega\right) \tag{4.7}$$

Proof: We have, by definition

$$\begin{aligned} W_{D_c f, D_c g}(t,\omega) &= \frac{1}{|c|}\int_{-\infty}^{\infty} f\left(\frac{t}{c}+\frac{\tau}{2c}\right) \overline{g}\left(\frac{t}{c}-\frac{\tau}{2c}\right) e^{-i\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} f\left(\frac{t}{c}+\frac{x}{2}\right) \overline{g}\left(\frac{t}{c}-\frac{x}{2}\right) e^{-ic\omega x} dx \\ &= W_{f,g}\left(\frac{t}{c}, c\omega\right) \end{aligned}$$

(h) Multiplication: If

$$Mf(t) = tf(t)$$

then

$$2tW_{f,g}(t,\omega) = W_{Mf,g}(t,\omega)+W_{f,Mg}(t,\omega) \tag{4.8}$$

Proof: We have, by definition,

$$\begin{aligned} 2tW_{f,g}(t,\omega) &= \int_{-\infty}^{\infty}\left(t+\frac{\tau}{2}+t-\frac{\tau}{2}\right)f\left(t+\frac{\tau}{2}\right)\overline{g}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty}\left(t+\frac{\tau}{2}\right)f\left(t+\frac{\tau}{2}\right)\overline{g}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau \\ &\qquad +\int_{-\infty}^{\infty}\left(t-\frac{\tau}{2}\right)f\left(t+\frac{\tau}{2}\right)\overline{g}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau \\ &= W_{Mf,g}(t,\omega)+W_{f,Mg}(t,\omega) \end{aligned}$$

(i) Differentiation:

$$W_{Df,g}(t,\omega)+W_{f,Dg}(t,\omega) = 2i\omega W_{f,g}(t,\omega) \tag{4.9}$$

Proof: Applying Fourier transform to L.H.S of (4.9) w.r.t 't', we get

$$\begin{aligned} \Im\{W_{Df,g}(t,\omega)\}+\Im\{W_{f,Dg}(t,\omega)\} &= i\left(\omega+\frac{\tau}{2}\right)\hat{f}\left(\omega+\frac{\tau}{2}\right)\bar{\hat{g}}\left(\omega-\frac{\tau}{2}\right) \\ &\qquad +i\left(\omega-\frac{\tau}{2}\right)\hat{f}\left(\omega+\frac{\tau}{2}\right)\bar{\hat{g}}\left(\omega-\frac{\tau}{2}\right) \\ &= 2i\omega\hat{f}\left(\omega+\frac{\tau}{2}\right)\bar{\hat{g}}\left(\omega-\frac{\tau}{2}\right) \\ &= 2i\omega\Im\{W_{f,g}(t,\omega)\} \\ &= \Im\{2i\omega W_{f,g}(t,\omega) \end{aligned}$$

By applying inverse Fourier transform, we get

$$W_{Df,g}(t,\omega)+W_{f,Dg}(t,\omega) = 2i\omega W_{f,g}(t,\omega)$$

(j) Time and frequency moments:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_{f,g}(t,\omega)\,dt d\omega = \int_{-\infty}^{\infty} t^n f(t)\,\bar{g}(t)\,dt \tag{4.10}$$

Proof: We have, by definition

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_{f,g}(t,\omega)\,dt d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n dt d\omega. \\
&\quad \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\bar{g}\left(t-\frac{\tau}{2}\right)e^{-i\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} t^n dt \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right)\bar{g}\left(t-\frac{\tau}{2}\right)\delta(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} t f(t)\bar{g}(t)dt
\end{aligned}$$

(k) The Pseudo Wigner-Ville Distribution:

We consider a family of signals f_t and g_t defined by

$$\begin{aligned}
f_t(\tau) &= f(\tau)w_f(\tau-t) \\
g_t(\tau) &= g(\tau)w_g(\tau-t)
\end{aligned}$$

where w_f and w_g are called the window functions.

For a fixed t, we can evaluate the Wigner-Ville distribution of f_t and g_t so that, by using (4.6),

$$W_{f_t,g_t}(\tau,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_{f,g}(\tau,u)W_{w_f,w_g}(\tau-t,\omega-u)du \tag{4.11}$$

where t represents the position of the window as it moves along the time axis. Obviously, (4.11) is a family of the WVD, and a particular member of this family is obtained by putting $\tau = t$ so that

$$W_{f_t,g_t}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_{f,g}(t,u)W_{w_f,w_g}(0,\omega-u)du \tag{4.12}$$

We now define a pseudo Wigner-Ville distribution(PWVD) of f and g by (4.12) and write

$$PW_{f,g}(t,\omega) = [W_{f_t,g_t}(\tau,\omega)]_{\tau=t}. \tag{4.13}$$

This is similar to the Wigner-Ville distribution, but, in general, is not a Wigner-Ville distribution. Even though the notation does not indicate explicit dependence on the window functions. The PWVD of two functions actually depends on the window functions. It follows from (4.12) that

$$PW_{f,g}(t,\omega) = \frac{1}{2\pi}W_{f,g}(t,\omega) * W_{w_f,w_g}(t,\omega), \tag{4.14}$$

where the convolution is taken with respect to the frequency variable ω. In particular,

$$\begin{aligned} PW_f(t,\omega) &= \frac{1}{2\pi}W_f(t,\omega) * W_{w_f}(t,\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W_f(t,u)W_{w_f}(t,\omega-u)du. \end{aligned} \tag{4.15}$$

This can be interpreted in following way. The Pseudo Wigner-Ville Distribution of a signal is a smoothed version of the original WVD with respect to the frequency variable.

Moyal's Formulas:

If f_1,g_1,f_2,g_2 belong to $L^2(\mathbb{R})$, then the following Moyal's formulas hold:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{f_1,g_1}(t,\omega)\overline{W}_{f_2,g_2}(t,\omega)\,dt\,d\omega = (f_1,f_2)\overline{(g_1,g_2)} \tag{4.16}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_{f,g}(t,\omega)|^2 dt\,d\omega = \|f\|^2\|g\|^2 \tag{4.17}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\overline{W}_g(t,\omega)\,dt\,d\omega = (f,g)\overline{(f,g)} = |(f,g)|^2 \tag{4.18}$$

Proof: For a fixed t, the Fourier transform of $W_{f,g}(t,\omega)$ with respect to ω is

$$\hat{W}_f(t,\sigma) = 2\pi f\left(t-\frac{\sigma}{2}\right)\bar{g}\left(t+\frac{\sigma}{2}\right). \tag{4.19}$$

Thus it follows from Parseval's formula for the fourier transform that

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\infty}^{\infty} W_{f_1,g_1}(t,\omega)\overline{W}_{f_2,g_2}(t,\omega)d\omega \\ = & \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \hat{W}_{f_1,g_1}(t,\sigma)\overline{\hat{W}}_{f_2,g_2}(t,\sigma)d\sigma \\ = & \int_{-\infty}^{\infty} f_1\left(t-\frac{\sigma}{2}\right)\bar{g}_1\left(t+\frac{\sigma}{2}\right)\bar{f}_2\left(t-\frac{\sigma}{2}\right)g_2\left(t+\frac{\sigma}{2}\right)d\sigma \end{aligned} \tag{4.20}$$

Integrating both sides with respect to t over $\mathbb{R}$ gives

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{f_1,g_1}(t,\omega)\overline{W}_{f_2,g_2}(t,\omega)d\omega\,dt = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(t-\frac{\sigma}{2}\right)\bar{g}_1\left(t+\frac{\sigma}{2}\right)\cdot \\ & \bar{f}_2\left(t-\frac{\sigma}{2}\right)g_2\left(t+\frac{\sigma}{2}\right)d\sigma\,dt \end{aligned}$$

which is, putting $t-\frac{\sigma}{2}=x$ and $t+\frac{\sigma}{2}=y$,

$$\begin{aligned} &= \int_{-\infty}^{\infty} f_1(x)\bar{f}_2(x)dx\overline{\int_{-\infty}^{\infty} g_1(y)\bar{g}_2(y)dy} \\ &= (f_1,f_2)\overline{(g_1,g_2)}. \end{aligned}$$

Thus (4.16) is proved.

In particular, if $f_1 = f_2 = f$ and $g_1 = g_2 = g$, then (4.16) reduces to (4.17).

We use definition (3.1) to replace $W_f(t,\omega)$ and $W_g(t,\omega)$ on the left hand side of (4.18) so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\overline{W}_g(t,\omega)\,dt\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(t+\frac{r}{2}\right)\bar{f}\left(t-\frac{r}{2}\right)\cdot$$
$$\left(t+\frac{s}{2}\right) g\left(t-\frac{s}{2}\right) exp[i(s-r)\omega]\,dr\,ds\,dt\,d\omega,$$

it follows that, by replacing the ω- integral with the delta function,

$$\begin{aligned} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(t+\frac{r}{2}\right)\bar{f}\left(t-\frac{r}{2}\right)\bar{g}\left(t+\frac{s}{2}\right) g\left(t-\frac{s}{2}\right)\delta(s-r)\,dr\,ds\,dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(t+\frac{r}{2}\right)\bar{f}\left(t-\frac{r}{2}\right)\bar{g}\left(t+\frac{s}{2}\right) g\left(t-\frac{s}{2}\right)\,dr\,dt \end{aligned}$$

which is, due to change of variables $t+\frac{r}{2}=x$ and $t-\frac{r}{2}=y$,

$$\begin{aligned} &= \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx\,\overline{\int_{-\infty}^{\infty} f(y)\bar{g}(y)\,dy} \\ &= (f,g)\overline{(f,g)} = |(f,g)|^2. \end{aligned}$$

Now in the next chapter, we formulate the two dimensional Wigner-Ville transform.

CHAPTER 5

TWO-DIMENSIONAL WIGNER-VILLE TRANSFORMS AND ITS BASIC PROPERTIES

On two-dimensional time:

Professor Itzhak Bars of the University of Southern California in Los Angeles found an extra time dimension in M-theory in 1995. Bars claims his theory of "two time physics", which he has developed over more than a decade, can help solve problems with current theories of the cosmos. Four dimensional world consists of three dimensions of space and one of time and six dimensions consists of four of space and two of time.

Definition

If $f(\mathbf{t})$and $g(\mathbf{t})$ belong to $L^2(\mathbb{R}^2)$, the two dimensional cross Wigner-Ville transform f and g is defined by

$$W_{f,g}(\mathbf{t},\boldsymbol{\omega}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau}, \tag{5.1}$$

where $\mathbf{t} = (t_1, t_2)$, $\boldsymbol{\omega} = (\omega_1, \omega_2)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2)$ provided the double integral in (5.1) exists.

Introducing $\mathbf{t} + \frac{\boldsymbol{\tau}}{2} = \mathbf{x}$ gives an equivalent definition of the two dimensional $W_{f,g}(\mathbf{t},\boldsymbol{\omega})$ in the form

$$\begin{aligned} W_{f,g}(\mathbf{t},\boldsymbol{\omega}) &= 2\exp(2i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\bar{g}(2\mathbf{t}-\mathbf{x})\exp(-2i\boldsymbol{\omega}\cdot\mathbf{x})\,d\mathbf{x} \\ &= 2\exp(2i\boldsymbol{\omega}\cdot\mathbf{t})\widetilde{f}_h(2\mathbf{t},2\boldsymbol{\omega}), \end{aligned} \tag{5.2}$$

where $\widetilde{f}_h(2\mathbf{t},2\boldsymbol{\omega})$ is the two-dimensional Gabor transform and $h(\mathbf{x}) = \bar{g}(-\mathbf{x})$.

It follows from (5.1) that the cross Wigner-Ville transform is the two dimensional Fourier transform of the function

$$h_{\mathbf{t}}(\boldsymbol{\tau}) = f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right) \tag{5.3}$$

with respect to $\boldsymbol{\tau}$.

Hence $W_{f,g}(\mathbf{t},\boldsymbol{\omega})$is a complex-valued function in the time frequency space. In other words

$$W_{f,g}(\mathbf{t},\boldsymbol{\omega}) = F\left\{h_{\mathbf{t}}(\vec{\boldsymbol{\tau}})\right\} = \hat{h}_{\mathbf{t}}(\boldsymbol{\omega}). \tag{5.4}$$

On the other hand, the two dimensional Fourier transform of the cross Wigner-Ville transform with respect to $\boldsymbol{\omega}$ is given by

$$\begin{aligned}
\hat{W}_{f,g}(\mathbf{t},\boldsymbol{\sigma}) &= e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}} W_{f,g}(\mathbf{t},\boldsymbol{\omega})\, d\boldsymbol{\omega} \\
&= \int_{-\infty}^{\infty} e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}} d\boldsymbol{\omega} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{\mathbf{t}}(\boldsymbol{\tau})\, e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}} d\boldsymbol{\tau}, \\
&= \int_{-\infty}^{\infty} h_{\mathbf{t}}(\boldsymbol{\tau})\, d\boldsymbol{\tau} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\boldsymbol{\omega}(\boldsymbol{\tau}+\boldsymbol{\sigma})} d\boldsymbol{\omega}, \\
&= (2\pi)^2 \int_{-\infty}^{\infty} h_{\mathbf{t}}(\boldsymbol{\tau})\, \delta(\boldsymbol{\tau}+\boldsymbol{\sigma})\, d\boldsymbol{\tau} = (2\pi)^2 h_{\mathbf{t}}(-\boldsymbol{\sigma}) \\
&= (2\pi)^2 f\left(\mathbf{t}-\frac{\boldsymbol{\sigma}}{2}\right)\overline{g}\left(\mathbf{t}+\frac{\boldsymbol{\sigma}}{2}\right).
\end{aligned} \tag{5.5}$$

Or equivalently,

$$\widehat{W}_{f,g}(\mathbf{t},-\boldsymbol{\sigma}) = (2\pi)^2 f\left(\mathbf{t}+\frac{\boldsymbol{\sigma}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\sigma}}{2}\right). \tag{5.6}$$

If $f = g$ in (5.1)-(5.3), then $W_{f,f}(\mathbf{t},\boldsymbol{\omega}) = W_f(\mathbf{t},\boldsymbol{\omega})$is called the two-dimensional auto Wigner-Ville transform and is defined by

$$W_f(\mathbf{t},\boldsymbol{\omega}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau}, \tag{5.7}$$

$$= 2\exp(2i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\overline{f}(2\mathbf{t}-\mathbf{x})\exp(-2i\boldsymbol{\omega}\cdot\mathbf{x})\,d\mathbf{x}, \tag{5.8}$$

$$= 2\exp(2i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\widetilde{f}_h(2\mathbf{t},2\boldsymbol{\omega}), \tag{5.9}$$

where $h(\mathbf{x}) = \overline{f}(-\mathbf{x})$.

Obviously, results (5.4)-(5.6) hold for the two-dimensional auto Wigner-Ville transform.

Furthermore, the two dimensional Wigner-Ville transform of a real signal is an even function of the frequency vector. More precisely,

$$\begin{aligned} W_f(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau}, \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}\left(\mathbf{t}-\frac{\mathbf{x}}{2}\right)\overline{\overline{f}}\left(\mathbf{t}+\frac{\mathbf{x}}{2}\right)e^{i\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x} \\ &= W_{\overline{f}}(\mathbf{t},-\boldsymbol{\omega}). \end{aligned}$$

The result (5.5) can also be written as

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\boldsymbol{\sigma}\cdot\boldsymbol{\omega}}W_{f,g}(\mathbf{t},\boldsymbol{\omega})\,d\boldsymbol{\omega} = f\left(\mathbf{t}-\frac{\boldsymbol{\sigma}}{2}\right)\overline{g}\left(\mathbf{t}+\frac{\boldsymbol{\sigma}}{2}\right). \tag{5.10}$$

Substituting $\mathbf{t}+\frac{\boldsymbol{\sigma}}{2} = t_1$ and $\mathbf{t}-\frac{\boldsymbol{\sigma}}{2} = t_2$ in (5.10) gives

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(\mathbf{t}_1-\mathbf{t}_2)\cdot\boldsymbol{\omega}]\,W_{f,g}\left(\frac{\mathbf{t}_1+\mathbf{t}_2}{2},\boldsymbol{\omega}\right)d\boldsymbol{\omega} = f(\mathbf{t}_2)\overline{g}(\mathbf{t}_1). \tag{5.11}$$

For $\mathbf{t}_1 = \mathbf{0}$ and $\mathbf{t}_2 = \mathbf{t}$, we get a representation of $f(\mathbf{t})$ in terms of $W_{f,g}$ in the form

$$f(\mathbf{t})\overline{g}(\mathbf{0}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\mathbf{t}\cdot\boldsymbol{\omega}}W_{f,g}\left(\frac{\mathbf{t}}{2},\boldsymbol{\omega}\right)d\boldsymbol{\omega} \tag{5.12}$$

provided $\overline{g}(\mathbf{0}) \neq 0$. This is the inversion formula for the Wigner-Ville transform.

In particular, if $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{t}$ in (5.11) , then the inversion formula for the Wigner-Ville transform is

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{f,g}(\mathbf{t}, \boldsymbol{\omega})\, d\boldsymbol{\omega} = f(\mathbf{t})\overline{g}(\mathbf{t}). \tag{5.13}$$

When $f = g$ in (5.13) , we obtain

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\mathbf{t}, \boldsymbol{\omega})\, d\boldsymbol{\omega} = |f(\mathbf{t})|^2. \tag{5.14}$$

This implies that the double integral of Wigner-Ville transform over the frequency at any time $\mathbf{t}$ is equal to the time energy density of a signal f.

CHAPTER 6

SOME PROPERTIES OF TWO-DIMENSIONAL WIGNER-VILLE TRANSFORMS

Properties:

Basic properties of two dimensional Wigner-Ville transforms [9]:

(a) Nonlinearity: The two-dimensional Wigner-Ville transform is nonlinear. This means that the Wigner-Ville transform of sum of two signals cannot be written as the sum of the Wigner-Ville transforms of the signals. This follows from the definition

$$\begin{aligned} W_{f_1+f_2,g_1+g_2}(\mathbf{t},\boldsymbol{\omega}) &= W_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) + W_{f_1,g_2}(\mathbf{t},\boldsymbol{\omega}) + W_{f_2,g_1}(\mathbf{t},\boldsymbol{\omega}) \\ &\quad + W_{f_2,g_2}(\mathbf{t},\boldsymbol{\omega}). \end{aligned} \tag{6.1}$$

Proof. By definition, we have

$$\begin{aligned} W_{f_1+f_2,g_1+g_2}(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f_1\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)+f_2\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right] \\ &\qquad [\overline{g}_1\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)+\overline{g}_2\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)]e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}_1\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}_2\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}_1\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}_2\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &= W_{f_1,g_1}(\mathbf{t},\boldsymbol{\omega}) + W_{f_1,g_2}(\mathbf{t},\boldsymbol{\omega}) + W_{f_2,g_1}(\mathbf{t},\boldsymbol{\omega}) + W_{f_2,g_2}(\mathbf{t},\boldsymbol{\omega}). \end{aligned}$$

(b) Translation:

$$W_{T_{\mathbf{a}}f,T_{\mathbf{a}}g}(\mathbf{t},\boldsymbol{\omega}) = W_{f,g}(\mathbf{t}-\mathbf{a},\boldsymbol{\omega}). \tag{6.2}$$

Proof. By definition,

$$\begin{aligned} W_{T_\mathbf{a}f,T_\mathbf{a}g}(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}-\mathbf{a}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\mathbf{a}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &= W_{f,g}(\mathbf{t}-\mathbf{a},\boldsymbol{\omega}). \end{aligned}$$

(c) Complex conjugation:

$$\overline{W}_{f,g}(\mathbf{t},\boldsymbol{\omega}) = W_{g,f}(\mathbf{t},\boldsymbol{\omega}). \tag{6.3}$$

Proof. By definition,

$$\overline{W}_{f,g}(\mathbf{t},\boldsymbol{\omega}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)g\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau}.$$

Now using the transformation, $\boldsymbol{\tau} = -\mathbf{x}$, we get

$$\begin{aligned} \overline{W}_{f,g}(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\mathbf{t}+\frac{\mathbf{x}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\mathbf{x}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x} \\ &= W_{g,f}(\mathbf{t},\boldsymbol{\omega}). \end{aligned}$$

(d) Modulation:

$$W_{M_\mathbf{b}f,M_\mathbf{b}g}(\mathbf{t},\boldsymbol{\omega}) = W_{f,g}(\mathbf{t},\boldsymbol{\omega}-\mathbf{b}). \tag{6.4}$$

Proof. We have, by definition,

$$\begin{aligned} W_{M_\mathbf{b}f,M_\mathbf{b}g}(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\left\{i\mathbf{b}\cdot\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right\}}f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)e^{\left\{-i\mathbf{b}\cdot\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right\}}\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{[-i\boldsymbol{\tau}\cdot(\boldsymbol{\omega}-\mathbf{b})]}d\boldsymbol{\tau} \\ &= W_{f,g}(\mathbf{t},\boldsymbol{\omega}-\mathbf{b}). \end{aligned}$$

(e) Translation and Modulation:

$$W_{M_{\mathbf{b}}T_{\mathbf{a}}f,M_{\mathbf{b}}T_{\mathbf{a}}g}(\mathbf{t},\boldsymbol{\omega}) = W_{T_{\mathbf{a}}M_{\mathbf{b}}f,T_{\mathbf{a}}M_{\mathbf{b}}g}(\mathbf{t},\boldsymbol{\omega}) = W_{f,g}(\mathbf{t}-\mathbf{a},\boldsymbol{\omega}-\mathbf{b}). \tag{6.5}$$

This follows from the joint application of translation and modulation properties.

Proof. In order to prove this property, we set

$$u(\mathbf{t}) = M_{\mathbf{b}}T_{\mathbf{a}}f = e^{i\mathbf{b}\mathbf{t}}f(\mathbf{t}-\mathbf{a}), \qquad v(\mathbf{t}) = M_{\mathbf{b}}T_{\mathbf{a}}g = e^{i\mathbf{b}\mathbf{t}}g(\mathbf{t}-\mathbf{a}).$$

Then by using the definition, we have

$$\begin{aligned} W_{u,v}(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{v}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\{i\mathbf{b}\cdot(\mathbf{t}+\frac{\boldsymbol{\tau}}{2})\}}f\left(\mathbf{t}-\mathbf{a}+\frac{\boldsymbol{\tau}}{2}\right)e^{\{-i\mathbf{b}\cdot(\mathbf{t}-\frac{\boldsymbol{\tau}}{2})\}}\bar{g}\left(\mathbf{t}-\mathbf{a}-\frac{\boldsymbol{\tau}}{2}\right) \\ &\qquad e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}-\mathbf{a}+\frac{\boldsymbol{\tau}}{2}\right)\bar{g}\left(\mathbf{t}-\mathbf{a}-\frac{\boldsymbol{\tau}}{2}\right)e^{[-i\boldsymbol{\tau}\cdot(\boldsymbol{\omega}-\mathbf{b})]}d\boldsymbol{\tau} \\ &= W_{f,g}(\mathbf{t}-\mathbf{a},\boldsymbol{\omega}-\mathbf{b}). \end{aligned}$$

(f) Dilation: If

$$D_c f(\mathbf{t}) = \frac{1}{\sqrt{|c|}} f\left(\frac{\mathbf{t}}{c}\right)$$

then

$$W_{D_c f,D_c g}(\mathbf{t},\boldsymbol{\omega}) = W_{f,g}\left(\frac{\mathbf{t}}{c},c\boldsymbol{\omega}\right). \tag{6.6}$$

Proof. By definition, we have

$$W_{D_c f,D_c g}(\mathbf{t},\boldsymbol{\omega}) = \frac{1}{|c|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\frac{\mathbf{t}}{c}+\frac{\boldsymbol{\tau}}{2c}\right)\bar{g}\left(\frac{\mathbf{t}}{c}-\frac{\boldsymbol{\tau}}{2c}\right)e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau}.$$

Now using the tranformation $\boldsymbol{\tau} = c\mathbf{x}$, we get

$$\begin{aligned} W_{D_c f, D_c g}(\mathbf{t}, \boldsymbol{\omega}) &= \frac{1}{|c|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\frac{\mathbf{t}}{c} + \frac{\mathbf{x}}{2}\right) \bar{g}\left(\frac{\mathbf{t}}{c} - \frac{\mathbf{x}}{2}\right) e^{-i(c\boldsymbol{\omega})\cdot\mathbf{x}} c\, d\mathbf{x} \\ &= W_{f,g}\left(\frac{\mathbf{t}}{c}, c\boldsymbol{\omega}\right). \end{aligned}$$

(g) Multiplication: If

$$Mf(\mathbf{t}) = \mathbf{t} f(\mathbf{t})$$

then

$$2\mathbf{t} W_{f,g}(\mathbf{t}, \boldsymbol{\omega}) = W_{Mf,g}(\mathbf{t}, \boldsymbol{\omega}) + W_{f,Mg}(\mathbf{t}, \boldsymbol{\omega}). \tag{6.7}$$

Proof. By definition, we have

$$\begin{aligned} 2\mathbf{t} W_{f,g}(\mathbf{t}, \boldsymbol{\omega}) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) + \left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right)\right] f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \bar{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}} d\boldsymbol{\tau} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \bar{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}} d\boldsymbol{\tau} \\ &\quad + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) f\left(\mathbf{t} + \frac{\boldsymbol{\tau}}{2}\right) \bar{g}\left(\mathbf{t} - \frac{\boldsymbol{\tau}}{2}\right) e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}} d\boldsymbol{\tau} \\ &= W_{Mf,g}(\mathbf{t}, \boldsymbol{\omega}) + W_{f,Mg}(\mathbf{t}, \boldsymbol{\omega}). \end{aligned}$$

(i) Differentiation:

$$W_{Df,g}(\mathbf{t}, \boldsymbol{\omega}) + W_{f,Dg}(\mathbf{t}, \boldsymbol{\omega}) = 2i\boldsymbol{\omega} W_{f,g}(\mathbf{t}, \boldsymbol{\omega}) \tag{6.8}$$

Proof: Applying Fourier transform to $W_{Df,g}(\mathbf{t}, \boldsymbol{\omega}) + W_{f,Dg}(\mathbf{t}, \boldsymbol{\omega})$, we get

$$\Im\left\{W_{Df,g}(\mathbf{t}, \boldsymbol{\omega}) + W_{f,Dg}(\mathbf{t}, \boldsymbol{\omega})\right\}$$

Consider Wigner-Ville transform of Fourier spectrum $\hat{f}'$and $\hat{g}$,

$$\begin{aligned}
W_{\hat{f},\hat{g}}(\mathbf{t},\boldsymbol{\omega}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}'\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)e^{i\mathbf{t}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(\mathbf{x})e^{-i\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)}d\mathbf{x}\right]\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)e^{i\mathbf{t}\cdot\boldsymbol{\tau}}d\boldsymbol{\tau} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(\mathbf{x})e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)e^{i\boldsymbol{\tau}\cdot\left(\mathbf{t}-\frac{\mathbf{x}}{2}\right)}d\boldsymbol{\tau}
\end{aligned}$$

Now using the substitution $\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}=\mathbf{u}$ in the second integral, we get

$$\begin{aligned}
&= 2\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(\mathbf{x})e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\hat{g}}(\mathbf{u})\,e^{i2(\boldsymbol{\omega}-\mathbf{u})\cdot\left(\mathbf{t}-\frac{\mathbf{x}}{2}\right)}d\mathbf{u} \\
&= 2e^{2i\boldsymbol{\omega}.\mathbf{t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(\mathbf{x})e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\hat{g}}(\mathbf{u})\,e^{i\mathbf{u}.(\mathbf{x}-2\mathbf{t})}d\mathbf{u}\right\} \\
&= 2e^{2i\boldsymbol{\omega}.\mathbf{t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(\mathbf{x})\bar{g}(2\mathbf{t}-\mathbf{x})\,e^{-2i\boldsymbol{\omega}.\mathbf{x}}d\mathbf{x} \\
&= W_{f',g}(\mathbf{t},\boldsymbol{\omega})
\end{aligned}$$

Therefore , $\Im\left\{W_{f',g}(\mathbf{t},\boldsymbol{\omega})\right\} = \hat{f}'\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)$

Similarly $\Im\left\{W_{f,g'}(\mathbf{t},\boldsymbol{\omega})\right\} = \hat{f}\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\bar{\hat{g}}'\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)$

Thus

$$\begin{aligned}
\Im\left\{W_{f',g}(\mathbf{t},\boldsymbol{\omega})\right\} &= \hat{f}'\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(\mathbf{x})e^{-i\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right).\mathbf{x}}d\mathbf{x}\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right) \\
&= i\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})e^{-i\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right).\mathbf{x}}d\mathbf{x}\,\hat{g}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right) \\
&= i\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\hat{f}\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\hat{g}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)
\end{aligned}$$

Similarly $\Im\left\{W_{f,g'}(\mathbf{t},\boldsymbol{\omega})\right\} = i\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)\hat{f}\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\hat{g}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)$

Therefore

$$\begin{aligned}
\Im\left\{W_{Df,g}(\mathbf{t},\boldsymbol{\omega})+W_{f,Dg}(\mathbf{t},\boldsymbol{\omega})\right\} &= i\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\hat{f}\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\hat{g}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)\\
&\quad +i\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)\hat{f}\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\hat{g}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)\\
&= 2i\boldsymbol{\omega}\hat{f}\left(\boldsymbol{\omega}+\frac{\boldsymbol{\tau}}{2}\right)\bar{\hat{g}}\left(\boldsymbol{\omega}-\frac{\boldsymbol{\tau}}{2}\right)\\
&= 2i\boldsymbol{\omega}\Im\left\{W_{f,g}(\mathbf{t},\boldsymbol{\omega})\right\}\\
&= \Im\left\{2i\boldsymbol{\omega}W_{f,g}(\mathbf{t},\boldsymbol{\omega})\right\}
\end{aligned}$$

Now applying inverse Fourier transform, we get

$$W_{Df,g}(\mathbf{t},\boldsymbol{\omega})+W_{f,Dg}(\mathbf{t},\boldsymbol{\omega})=2i\boldsymbol{\omega}W_{f,g}(\mathbf{t},\boldsymbol{\omega}).$$

CHAPTER 7

EXAMPLES OF WIGNER-VILLE TRANSFORMS

Some examples of two-dimensional Wigner-Ville transforms:

1. The Wigner-Ville transform of a harmonic signal $f(\mathbf{t}) = A\exp(i\boldsymbol{\omega}_0 \cdot \mathbf{t})$ where A is constant. We have by definition

$$\begin{aligned}
W_f(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\,d\boldsymbol{\tau} \\
&= A\bar{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[i\boldsymbol{\omega}_0\cdot\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right]\exp\left[-i\boldsymbol{\omega}_0\cdot\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right]\exp(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\,d\boldsymbol{\tau} \\
&= A\bar{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[i\boldsymbol{\omega}_0\cdot\mathbf{t}+i\boldsymbol{\omega}_0\cdot\frac{\boldsymbol{\tau}}{2}-i\boldsymbol{\omega}_0\cdot\mathbf{t}+i\boldsymbol{\omega}_0\cdot\frac{\boldsymbol{\tau}}{2}\right]\exp(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\,d\boldsymbol{\tau} \\
&= A\bar{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i\boldsymbol{\omega}_0\cdot\boldsymbol{\tau}]\exp(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau})\,d\boldsymbol{\tau} \\
&= |A|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i\boldsymbol{\tau}\cdot(\boldsymbol{\omega}_0-\boldsymbol{\omega})]\,d\boldsymbol{\tau} \\
&= |A|^2\cdot(2\pi)^2\cdot\delta(\boldsymbol{\omega}-\boldsymbol{\omega}_0) \\
&= 4\pi^2|A|^2\delta(\boldsymbol{\omega}-\boldsymbol{\omega}_0). \qquad (7.1)
\end{aligned}$$

2. The Wigner-Ville transform of plane waves. If $f(\mathbf{t}) = A_1 e^{i\boldsymbol{\omega}_1\cdot\boldsymbol{\tau}}$, $g(\mathbf{t}) = A_2 e^{i\boldsymbol{\omega}_2\cdot\boldsymbol{\tau}}$ represent two plane waves, then

$$W_f(\mathbf{t},\boldsymbol{\omega}) = 2\pi A_1\bar{A}_2\exp[i(\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_2)\cdot\mathbf{t}]\,\delta\left(\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_1+\boldsymbol{\omega}_2}{2}\right).$$

We have, by definition

$$\begin{aligned}
W_f(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\overline{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= A_1\overline{A_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[i\boldsymbol{\omega}_1\cdot\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\right]\exp\left[-i\boldsymbol{\omega}_2\cdot\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\right] \\
&\qquad \exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= A_1\overline{A_2}\exp\left[i\left(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2\right)\cdot\mathbf{t}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-i\boldsymbol{\tau}\cdot\left(\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_1+\boldsymbol{\omega}_2}{2}\right)\right]d\boldsymbol{\tau} \\
&= A_1\overline{A_2}\exp\left[i\left(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2\right)\cdot\mathbf{t}\right]2\pi\cdot\delta\left(\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_1+\boldsymbol{\omega}_2}{2}\right) \\
&= 2\pi A_1\overline{A_2}\exp\left[i\left(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2\right)\cdot\mathbf{t}\right]\delta\left(\boldsymbol{\omega}-\frac{\boldsymbol{\omega}_1+\boldsymbol{\omega}_2}{2}\right). \qquad (7.2)
\end{aligned}$$

3. The Wigner-Ville distribution of a quadratic phase signal $f(\mathbf{t}) = A\exp\left\{\frac{1}{2}ia(\mathbf{t})^2\right\}$, where $\mathbf{t} = (t_1,t_2)$ is a two dimensional vector and $(\mathbf{t})^2 = \mathbf{t}\cdot\mathbf{t} = t_1^2+t_2^2$.

We have, by definition,

$$\begin{aligned}
W_f(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= A\bar{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{ia}{2}\left\{\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)^2-\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)^2\right\}\right]\exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= |A|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[ia\mathbf{t}\cdot\boldsymbol{\tau}\right]\exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= |A|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[i\boldsymbol{\tau}\cdot\left(a\mathbf{t}-\boldsymbol{\omega}\right)\right]d\boldsymbol{\tau} \\
&= |A|^2(2\pi)^2\,\delta\left(\boldsymbol{\omega}-a\mathbf{t}\right) \\
&= 4\pi^2|A|^2\,\delta\left(\boldsymbol{\omega}-a\mathbf{t}\right). \qquad (7.3)
\end{aligned}$$

4. If $f(\mathbf{t}) = g(\mathbf{t})\exp\left\{\frac{ia}{2}(\mathbf{t})^2\right\}$ then $W_f(\mathbf{t},\boldsymbol{\omega}) = W_g(\mathbf{t},\boldsymbol{\omega}-a\mathbf{t})$ where a is the curvature.

It follows from the definition that

$$
\begin{aligned}
W_f(\mathbf{t},\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{f}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp\left[\frac{ia}{2}\left\{\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)^2-\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)^2\right\}\right] \\
&\qquad \exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp\left[\frac{ia}{2}\cdot 2\mathbf{t}\cdot\boldsymbol{\tau}\right]\exp\left(-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}\right)d\boldsymbol{\tau} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\mathbf{t}+\frac{\boldsymbol{\tau}}{2}\right)\bar{g}\left(\mathbf{t}-\frac{\boldsymbol{\tau}}{2}\right)\exp\left[i\boldsymbol{\tau}\cdot\left(a\mathbf{t}-\boldsymbol{\omega}\right)\right]d\boldsymbol{\tau} \\
&= W_g\left(\mathbf{t},\boldsymbol{\omega}-a\mathbf{t}\right) \qquad (7.4)
\end{aligned}
$$

CHAPTER 8

APPLICATIONS OF WIGNER-VILLE TRANSFORMS

In recent years, the Wigner-Ville transform has served as a useful analytic tool in many fields as diverse as quantum mechanics, optics, acoustics, communications, biomedical engineering, signal processing, and image processing. It has also been used as a method for analysing seismic data, and the phase distortion involved in a wide variety of audio engineering problems. In addition it has also been suggested as a method for investigating many important topics including instantaneous frequency estimation, spectral analysis of non-stationary random signals, detection and classification of signals, algorithms for computer implementation, speech signals, and pattern recognition.

In sonar and radar systems, a real signal is transmitted and its echo is processed in order to find out the position and velocity of a target. In many situations, the recieved signal is different from the original one only by a time translation and the doppler ferquency shift. in the context of the mathematical analysis of radar information, Woodward (1953) reformulated the theory of the Wigner-Ville distribution. He introduced a new function $A_f(t,\omega)$ of two independent variables t, ω obtained from a radar signal f in the form

$$A_f(t,\omega) = \int_{-\infty}^{\infty} f\left(\tau+\frac{t}{2}\right)\overline{f}\left(\tau-\frac{t}{2}\right)e^{-i\omega\tau}d\tau. \tag{8.1}$$

This function is now known as the Woodward ambiguity function and plays a central role in radar signal analysis and radar design. The ambiguity function has been widely used for describing the correlation between a radar signal and its Doppler-shifted and time-translated version.

In spite of some remarkable features, its energy distribution is not nonnegative and it often possesses severe cross terms, or interference terms between different time-frequency regions, leading to undesirable properties. In order to overcome some of the inherent weakness of the Wigner-Ville distribution, there has been considerable recent interest in more general time-frequency distributions as a mathematical method for time frequency signal analysis. The Wigner-Ville distribution has been modified by smoothing in one or two dimensions. In 1966, Cohen introduced a general class of bilinear shift-invariant, quadratic time frequency distributions.

The spectral characteristics of heart rate variability (HRV) are related to the modulation of the autonomic nervous system. As the physiological condition is changed by such external stimuli such as drugs, anesthesia, or by internal deregulation such as in syncope, adjective autonomic responses could alter HRV characteristics. Time-frequency analysis is commonly used to investigate the time related HRV characteristics. Time-frequency methods including the shorttime Fourier transform, the Choi-Williams distribution, the smoothed pseudo Wigner–Ville distribution (SPWVD), the filtering SPWVD compensation, and the discrete wavelet transform are used. One simulated signal and two heart rate signals during general anesthesia and postural change are used for this assessment. The result demonstrates that the filtering SPWVD compensation and the discrete wavelet transform have small spectrum interference from the transient component [10].

Inverse synthetic-aperture radar (ISAR) is a technique for improving the cross-range resolution of coherent imaging radars. ISAR exploits any relative rotational motion between target and radar and is becoming increasingly popular in the airborne maritime radar surveillance role for ship classification. In order to produce well-focused images suitable for classification any linear acceleration between target and radar must be measured and compensated for. The Wigner-Ville time-frequency transform is one method that has been applied to successfully focus ISAR images of maritime targets [11].

Electromigration noise analysis has shown great potential for the nondestructive evaluation of electromigration performance of a metal strip. However, contradictory conclusions have been published from the electromigration noise analysis. These contradictory conclusions mainly stem from the complex dynamics of the atomic movement during electromigration, rendering the electromigration noise as a nonstationary signal, and, hence, the standard Fourier transform is not adequate. Among the various nonstationary signal analysis tools, Wigner–Ville distribution is used for the analysis of electromigration noise data for the first time. It is found that much "hidden" and useful information in the noise data can be revealed by using this distribution [12].

CHAPTER 9

CONCLUSION

The time-frequency representation of a signal f is known as Wigner-Ville distribution (WVD) which is one of the fundamental methods that have been developed over the years for time-frequency signal analysis. Besides other time-frequency representations, Wigner-Ville distribution plays a central role in the field of bilinear/quadratic time-frequency representations. Wigner-Ville distribution has wide range of applications in optics, acoustics, communications, biomedical engineering, analysing seismic data. We have extended Wigner-Ville distribution in two-dimensions and have proved some of its properties such as non-linearity, translation, dilation, modulation, translation and modulation, dilation, multiplication, differentiation. Some examples of two-dimensional Wigner-Ville distribution are also discussed. Many other important properties of the two-dimensional Wigner-Ville transforms and their applications will be discussed in future work.

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BIOGRAPHICAL SKETCH

Shankara Narayana Rao, Bheemaiah Veena, was born in Karnataka, India. She obtained her Bachelors in Mathematics from Bangalore University, India in 2001 and her Masters in Mathematics from Bangalore University in the year 2003 with the specialization in Graph theory and Finite Element Method and its application. She has teaching experience of four years and has taught undergraduate courses for students in Engineering College, Bangalore, India. Presently she is persuing her Masters in the University of Texas Pan-American in the Mathematics Department.