8-2009

Integrable equations with non-smooth solitons

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INTEGRABLE EQUATIONS WITH NON-SMOOTH SOLITONS

A Thesis

by

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Submitted to the Graduate School of the
University of Texas-Pan American
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2009

Major Subject: Mathematics

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ABSTRACT

Li, Xianqi, Integrable Equations with Non-smooth Solitons. Master of Science (MS), August, 2009, 28 pp., 5 figures, 22 titles.

In this thesis, we present a class of integrable equations with non-smooth soliton solutions. In particular, we derive the bi-Hamiltonian structure and Lax pair of the equation $\rho_t = bu_x + \frac{1}{2}[(u^2 - u_x^2)\rho]_x, \rho = u - u_{xx}$, which guarantee its integrability. Another interesting integrable equation we study is $(\frac{-u_{xx}}{u})_t = 2uu_x$, which is exactly the first member of the negative KdV hierarchy. Through traveling wave setting and phase step analysis, we obtain non-smooth soliton solutions of these integrable equations under different boundary condition at infinities. These equations were shown to have peaked soliton (peakon), "W/M-shape" peakon or cusped soliton (cuspon) solutions. Furthermore, some other non-smooth soliton equations are also investigated for our future work.

ACKNOWLEDGMENTS

I will always be grateful to Dr. Zhijun Qiao, chair of my master thesis committee and thesis advisor, for all his mentoring and advice. Many thanks go to my master thesis committee members. Their advice, input, and comments on my thesis helped to ensure the quality of my intellectual work.

I would also like to thank the integrable system research group who helped me generate the initial ideal for my research. Also, I would like to acknowledge my teachers in mathematics department who participated in my focus research.

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CHAPTER 1

INTRODUCTION

Soliton theory and integrable systems play an important role in the study of nonlinear wave equations. They have many significant applications in fluid mechanics, nonlinear optics, classical and quantum fields theories etc. Particularly in recent years, more focuses have been pulled to integrable systems with non-smooth solitons, such as peakons, cuspons, since the study of well-known Camassa-Holm (CH) equation with peakon solutions [1]. Recently, considerable progress has been made in the study of shallow water waves [4, 3, 6, 5, 12, 15, 16, 20, 9, 18, 19].

In this thesis, first we shall give the bi-Hamiltonian structure and Lax pair of the equation

$$\rho_t = bu_x + \frac{1}{2}[(u^2 - u_x^2)\rho]_x, \rho = u - u_{xx}, b = \text{constant}. \tag{1.1}$$

This equation was derived by "reshuffling" Hamiltonian operators of the bi-Hamiltonian structure of the modified KdV equation and adding one more linear term bu_x [17]. Later on Qiao constructed the equation without the linear term and its Lax pair through employing the spectral problem technique [11]. This equation can be extended to an integrable hierarchy through recursion operator procedure and solving a crucial matrix equation [14].

In [17], Rosenau discussed the conditions of the peakon existence for equation (1.1). But the explicit peakon and cuspon solutions were not given. In my thesis, the equation (1.1) was shown to have both "W/M-shape"- peakon and cuspon solutions under the vanishing boundary condition at infinities. Those peakon and cuspon solutions are illustrated through their graphs [7].

Then we consider the integrable equation

$$(\frac{-u_{xx}}{u})_t = 2uu_x, \tag{1.2}$$

which is the first member of the negative KdV hierarchy [14]. It is studied under the inhomogeneous boundary conditions $u \to A$ as $x \to +\infty$ and $u \to B$ as $x \to -\infty$. This equation not only has peakon solutions but also kink solutions. Particularly its Lax representation is given here [8].

In this thesis, we also extend our procedure to two more integrable equations

$$u_t + 6u^2u_x + u_{xxx} = 0, \tag{1.3}$$

$$u_t + u_x + \beta u_{xxx} + \alpha uu_x + 3\mu\alpha^2u^2u_x = 0, \tag{1.4}$$

where α, β and μ are constants. We find these two integrable equations possess peakon solutions as well, which could be helpful to explore their other properties.

So, finding more integrable equations with non-smooth solitons is the main subject of this thesis.

CHAPTER 2

AN INTEGRABLE EQUATION WITH "W/M-SHAPE"-PEAKS SOLITONS

2.1 Bi-Hamiltonian Structure

The bi-Hamiltonian structure is one of the most significant properties for integrable equations. Through finding the bi-Hamiltonian structure, we can guarantee the integrable equations possess an infinite number of conservation laws. Next let us consider the bi-Hamiltonian structure of equation (1.1).

The wave equation (1.1) can be rewritten as

$$\rho_t = [bu + \frac{1}{2}(u^2 - u_x^2)\rho]_x = J\frac{\delta H_1}{\delta \rho} = K\frac{\delta H_2}{\delta \rho}, \tag{2.1}$$

where

$$J = \partial\rho\partial^{-1}\rho\partial + b\partial, \quad K = \partial - \partial^3, \tag{2.2}$$

$$H_1 = \int_\Omega \rho u \mathrm{d}x, \quad H_2 = \frac{1}{8}\int_\Omega (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 + 4bu^2)\mathrm{d}x, \tag{2.3}$$

and $\Omega = (x_0, x_0 + T)$ or $\Omega = (-\infty, +\infty)$, which sets u to be periodic with T or to approach a constant at infinities. Apparently, J and K are two Hamiltonian operators, and therefore H_1, H_2 are two Hamiltonian functions. So, equation (1.1) has the bi-Hamiltonian structure. In the verification of bi-Hamiltonian structure, $(1 - \partial^2)(\delta H_2/\delta\rho) = (\delta H_2/\delta u)$ is applied.

2.2 Lax Pair

In this subsection, we show the integrability of equation (1.1) in the sense of Lax pair. Let us consider the following spectral problem

$$\psi_x = \begin{pmatrix} -\frac{\sqrt{1-\lambda^2 b}}{2} & \frac{1}{2}\lambda\rho \\ -\frac{1}{2}\lambda\rho & \frac{\sqrt{1-\lambda^2 b}}{2} \end{pmatrix} \psi, \tag{2.4}$$

where λ is a spectral parameter, ρ is a scalar potential function periodic or decaying at infinities, and $\psi = (\psi_1, \psi_2)^T$ is the spectral function corresponding to the spectral parameter λ. Then we have

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{2.5}$$

where the functional gradient $\nabla\lambda := \frac{\delta\lambda}{\delta\rho} = \lambda(\psi_1^2 + \psi_2^2)$.

It is not hard for us to prove that equation (1.1) possesses the following Lax pair:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(\rho, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.6}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V(\rho, u, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.7}$$

where

$$U = \begin{pmatrix} -\frac{\sqrt{1-\lambda^2 b}}{2} & \frac{1}{2}\lambda\rho \\ -\frac{1}{2}\lambda\rho & \frac{\sqrt{1-\lambda^2 b}}{2} \end{pmatrix}, \quad \rho = u - u_{xx},$$

$$V = -\frac{1}{2}\begin{pmatrix} \lambda^{-2}\sqrt{1-\lambda^2 b} + \frac{\sqrt{1-\lambda^2 b}}{2}(u^2 - u_x^2) & \lambda^{-1}(\sqrt{1-\lambda^2 b}\, u_x - u_{xx}) - \frac{1}{2}\lambda\rho(u^2 - u_x^2) \\ \lambda^{-1}(\sqrt{1-\lambda^2 b}\, u_x + u_{xx}) + \frac{1}{2}\lambda\rho(u^2 - u_x^2) & -\lambda^{-2}\sqrt{1-\lambda^2 b} - \frac{\sqrt{1-\lambda^2 b}}{2}(u^2 - u_x^2) \end{pmatrix}.$$

We can easily verify that the following zero curvature equation:

$$U_t - V_x + [U, V] = 0 \tag{2.8}$$

exactly produces equation (1.1). Therefore, equation (1.1) is completely integrable in the sense of Lax pair [11].

2.3 Non-smooth Soliton Solutions

2.3.1 Traveling wave setting

Let us consider the traveling wave solution of equation (1.1) through a generic setting $u(x,t) = U(x - ct)$, where c is the wave speed. Let $\xi = x - ct$, then $u(x,t) = U(\xi)$. Substituting it into equation (1.1) yields

$$c(U - U'')' + bU' + \frac{1}{2}\Big((U^2 - U'^2)(U - U'')\Big)' = 0, \tag{2.9}$$

where $U' = U_\xi$, $U'' = U_{\xi\xi}$, $U''' = U_{\xi\xi\xi}$.

Taking integration twice on both sides of (2.9) and using the vanishing condition at infinities, we obtain

$$U'^2 = U^2 + 2c \mp 2\sqrt{c^2 - bU^2}, \tag{2.10}$$

where "–" is chosen for $c > 0$ and "+" is for $c < 0$ since U decays at both infinities. Remark: In equation (2.10), for the case of $c = 0$, there is no traveling wave solution satisfying the vanishing boundary conditions at both infinities.

2.3.2 "W/M-shape"-peaks solitons

In equation (2.10), let $X = \sqrt{c^2 - bU^2}$. Then $U^2 = \frac{c^2-X^2}{b}$, and

$$U' = \pm\sqrt{\frac{c^2 - X^2}{b} + 2c \mp 2X}. \tag{2.11}$$

Replacing U in terms of X yields

$$\frac{X\mathrm{d}X}{\sqrt{bc^2 - bX^2}} = \pm\sqrt{\frac{c^2 - X^2}{b} + 2c \mp 2X}\mathrm{d}\xi, \tag{2.12}$$

which is equivalent to

$$\frac{X\mathrm{d}X}{|c - \sigma X|\sqrt{(b + c + \sigma X)^2 - b^2}} = \pm\mathrm{d}\xi,\ \sigma = \pm 1. \tag{2.13}$$

Next we will discuss the cases of $\sigma = 1$ (coresponding to $c > 0$) and $\sigma = -1$ (coresponding to $c < 0$), respectively.

CASE 1: $\sigma = 1$ $(c > 0)$

1.1 Let us first consider $b < 0, c > 0$. For the subcase of $b < 0, b \leq -c$ and $c > 0$, we can verify that no soliton solution exists. However, when $-c < b < 0$ and $c > 0$, equation (2.13) can be integrated as

$$\left(X + \alpha + \sqrt{(X+\alpha)^2 - b^2}\right)\left(\frac{X - c}{8c\alpha + (4c + 2b)(X - c) + 4\sqrt{c\alpha}\sqrt{(X+\alpha)^2 - b^2}}\right)^{\frac{1}{2\theta}} = e^{-|\xi|}, \tag{2.14}$$

where $\theta = \sqrt{\frac{\alpha}{c}}$ $(0 < \theta < 1)$ and $\alpha = b + c$. In general, equation (2.14) can not be solved for X in an explicit form. However, if taking some special θ, we can get the explicit solution of equation (2.14). For example, let us take $\theta = \frac{1}{2}$, then $b = -\frac{3}{4}c$. Under this condition, equation (2.14) can be transformed into

$$X^2 - vX + cv = 0, \tag{2.15}$$

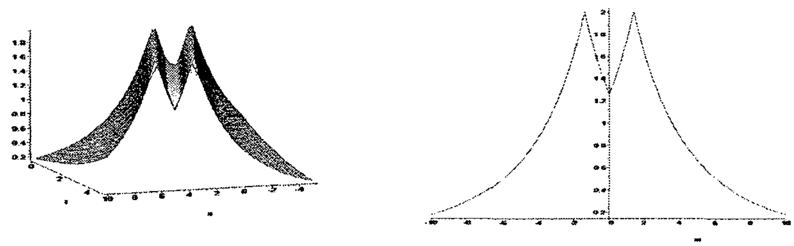

Figure 2.1. 3D and 2D graphics for "M-shape"-peaks soliton solution with $\theta = 1/2$

where $v = c(\frac{9}{4}e^{-|\xi|} + \frac{9}{16}e^{|\xi|} + \frac{7}{4})$, or $v = \frac{9}{4}\varrho c + 2c$ with $\varrho = \cosh(\frac{|\xi|}{2} - \ln 2) - \frac{1}{9}$. Solving equation (2.15) leads to $X = \frac{v - \sqrt{v^2 - 4cv}}{2} = c + \frac{9c}{8}(\varrho - \sqrt{\varrho^2 - \frac{64}{81}})$.

Since $U^2(\xi) = \frac{c^2 - X^2}{b}$, we get

$$U^2(\xi) = c(3 + \frac{27}{8}\varrho)(\varrho - \sqrt{\varrho^2 - \frac{64}{81}} - \frac{4}{3}), \tag{2.16}$$

where $\varrho = \cosh(Y) - \frac{1}{9}, Y = \frac{|x-ct|}{2} - \ln 2$. If we take very special $b = -\frac{3}{4}$ $(c = 1)$, then the solution can be expressed as

$$U(\xi) = \pm\sqrt{\frac{1}{8}(7 + 9\cosh Y)(3\cosh Y - \frac{1}{3} - \sqrt{(9\cosh Y + 7)(\cosh Y - 1)}) - \frac{4}{3}}, \tag{2.17}$$

where $Y = \frac{|x-t|}{2} - \ln 2$.

Remark:

1. Graphs for the "+" case of solution (2.17) are shown in Figure 2.1. It is "M-shape"-peaks soliton solution. If selecting the "-" case of solution (2.17), then we can get the "W"-shape-peaks soliton solution.

2. Apparently, three peaks occur at $-2\ln 2, 0, 2\ln 2$ in Figure 2.1. In fact, we have calculated both left and right derivatives at those three points: $U'(-2\ln 2^-) = \frac{\sqrt{2}}{2}, U'(-2\ln 2^+) = -\frac{\sqrt{2}}{2}, U'(0^-) \approx -0.396, U'(0^+) \approx 0.396, U'(2\ln 2^-) = \frac{\sqrt{2}}{2}, U'(2\ln 2^+) = -\frac{\sqrt{2}}{2}$.

Cuspons: If we do not consider the absolute value involved in the solution form during the solving procedure for the subcase of $-c < b < 0$ and $c > 0$, we will find another

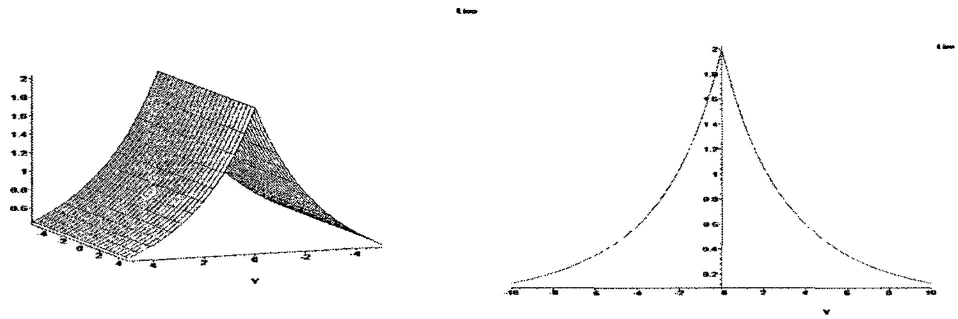

Figure 2.2. 3D and 2D graphics for positive cuspon solution with $\theta = 1/2$

new solution

$$U(\xi) = \pm\sqrt{\frac{1}{8}(7+9\cosh Y)\Big(3\cosh Y - \frac{1}{3} - \sqrt{(9\cosh Y+7)(\cosh Y-1)}\Big) - \frac{4}{3}}, \tag{2.18}$$

where $Y = \frac{x-ct}{2} = \frac{\xi}{2}$.

Remark: Graphs for the "+" case of solution (2.18) are shown in Figure 2.2. It is a "Λ"-shape cuspon solution since $U'(0^+) = -\infty, U'(0^-) = \infty$. If selecting the "-" case of solution (2.18), then we can get the "V"-shape (anti-"Λ"-shape) cuspon solution.

1.2 For the subcase of $b > 0$ and $c > 0$, equation (2.13) can be also integrated as (2.14), but $\theta > 1$. In this subcase, we may obtain implicit peakon and cuspon solutions, whose graphs may take on the profiles similar to Figures 2.1 and 2.2.

CASE 2: $\sigma = -1$ $(c < 0)$

For this case, we just list our results since the procedure is the analysis and computing procedure is the same as the CASE 1.

2.1 Let us first consider $b > 0, c < 0$. For the subcase of $b > 0, b \geq -c$ and $c < 0$, we can verify that no soliton solution exists. However, when $0 < b < -c$ and $c < 0$, equation (2.13) could be integrated as

$$(X - \alpha + \sqrt{(X-\alpha)^2 - b^2})\Big(\frac{X+c}{8c\alpha - (4c+2b)(X+c) + 4\sqrt{c\alpha}\sqrt{(X-\alpha)^2-b^2}}\Big)^{\frac{1}{2\theta}} = e^{-|\xi|}, \tag{2.19}$$

where $\theta = \sqrt{\frac{\alpha}{c}}$ $(0 < \theta < 1)$ and $\alpha = b + c$. In general, equation (2.19) can not be solved for X in an explicit form. However, if taking some special θ, for example $\theta = 1/2$, we can also get an explicit solutions of equation (2.19) using the procedure samiliar to the subcase of $-c < b < 0$ and $c > 0$ in case 1.1. The explicit solution is same as the expression (2.17), but currently the traveling wave speed c is negative, namely, water wave travels backward.

2.2 For the subcase of $b < 0$ and $c < 0$, equation (2.13) can be also integrated as (2.14), but $\theta > 1$. In this subcase, we may obtain implicit peakon and cuspon solutions, whose graphs are similar to Figures 2.1 and 2.2.

CHAPTER 3

A NEW PEAKON EQUATION

3.1 Traveling wave solutions

3.1.1 Traveling wave setting

Let us consider the traveling wave solution of equation (1.2) through a generic setting $u(x,t) = U(x-ct)$, where c is the wave speed. Let $\xi = x - ct$, then $u(x,t) = U(\xi)$. Substituting it into equation (1.2) yields

$$c(\frac{U''}{U})' = 2UU'. \tag{3.1}$$

3.1.2 Peakon and kink Solutions

Take the integration to equation (3.1) under the boundary condition $U \to A$ as $\xi \to +\infty$, $U \to B$ as $\to -\infty$, where A and B are two constants. We obtain

$$\frac{cU''}{U} = U^2 - A^2, \tag{3.2}$$

$$\frac{cU''}{U} = U^2 - B^2. \tag{3.3}$$

Compare equation (3.2) and equation (3.3), we find $A = B$ or $A = -B$.

Next we will discuss these two cases $A = B$ and $A = -B$ separately.

Case 1: Peakon Solutions($A = B$)

Equation (3.2) can be written as

$$cU'' = U^3 - A^2U. \tag{3.4}$$

Multiply U' to the both sides of equation (3.4), do the integration again and through simplification, we get

$$\sqrt{2c}U' = \pm(U^2 - A^2), \tag{3.5}$$

where $c > 0$.

In the following we will consider $A > 0$ and $A < 0$ respectively.

1.1: If $A > 0$, equation (3.5) can be integrated as

$$\frac{1}{2A}\ln|\frac{U-A}{U+A}| = -\frac{1}{\sqrt{2c}}|\xi| + c_1, \tag{3.6}$$

where c_1=real const., by computation, we find the solution

$$U(\xi) = \frac{2A}{1 - \tilde{c}e^{\frac{-\sqrt{2}A|\xi|}{\sqrt{c}}}} - A, \tag{3.7}$$

where $c > 0$, $\tilde{c} \neq 0$ is a constant. When $\tilde{c} < 1$, it is a peakon solution of equation (1.2), 3D and 2D of graphs are given here. When $\tilde{c} \geq 1$, it is a singular solution of equation (1.2), but still goes to A at infinities.

1.2: If $A < 0$, equation (3.5) could be integrated as

$$\frac{1}{2A}\ln|\frac{U-A}{U+A}| = \frac{1}{\sqrt{2c}}|\xi| + c_1. \tag{3.8}$$

Similarly, we find the solution

$$U(\xi) = \frac{2A}{1 - \tilde{c}e^{\frac{\sqrt{2}A|\xi|}{\sqrt{c}}}} - A, \tag{3.9}$$

where $c > 0$, $\tilde{c} \neq 0$ is a constant. In the like manner, we can get the same conclusion as the case **1.1**.

Case 2: Kink Solutions($A = -B$)

In this case we still need to consider the positive or negative sign of A.

2.1: If $A > 0$, equation (3.5) can be integrated as

$$\frac{1}{2A} \ln |\frac{U - A}{U + A}| = -\frac{1}{\sqrt{2c}}\xi + c_1, \tag{3.10}$$

we obtain the solution

$$U(\xi) = \frac{2A}{1 - \tilde{c}e^{\frac{-\sqrt{2}A\xi}{\sqrt{c}}}} - A, \tag{3.11}$$

where $c > 0$, $\tilde{c} \neq 0$ is a constant. When $\tilde{c} < 0$, it is a kink solution of equation (1.2), 2D of graph is given here. When $\tilde{c} \geq 0$, it is a singular solution of equation (1.2).

2.2: If $A < 0$, equation (3.5) could be integrated as

$$\frac{1}{2A} \ln |\frac{U - A}{U + A}| = \frac{1}{\sqrt{2c}}\xi + c_1. \tag{3.12}$$

We find the solution

$$U(\xi) = \frac{2A}{1 - \tilde{c}e^{\frac{\sqrt{2}A\xi}{\sqrt{c}}}} - A, \tag{3.13}$$

where $c > 0$, $\tilde{c} \neq 0$ is a constant. Similarly, we can get the same conclusion as the case **2.1**.

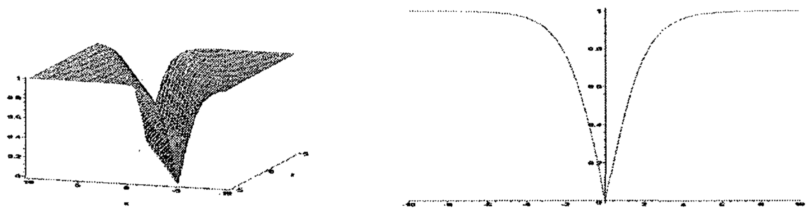

Figure 3.1. 3D and 2D graphics for peakon solution with $A=1$, $c=2$ and $\tilde{c}=-1$

3.2 Lax Representation

Now let us give the Lax representation for equation (1.2). As we mention above, equation (1.2) is the first member of negative order hierarchy of the Schrödinger-KdV spectral problem, so we discuss the Schrödinger-KdV spectral problem first.

$$L \cdot \psi \equiv \psi_{xx} + v\psi = \lambda\psi \tag{3.14}$$

has the following Lenard eigenvalue relation:

$$K \cdot \nabla\lambda = \lambda J \cdot \nabla\lambda, \tag{3.15}$$

where $\nabla\lambda \equiv \frac{\delta\lambda}{\delta v} = \psi^2$ is the functional gradient of the spectral problem (3.14), $K = \frac{1}{4}\partial^3 + \frac{1}{2}(v\partial + \partial v)$ and $J = \partial$. By setting $v = -\frac{u_{xx}}{u}$, we have the product form of operators K, $\mathcal{L}$, L and their inverses

$$K = \frac{1}{4}u^{-2}\partial u^2\partial u^2\partial u^{-2},\ K^{-1} = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2,$$

$$\mathcal{L} = \frac{1}{4}\partial^{-1}u^{-2}\partial u^2\partial u^2\partial u^{-2},\ \mathcal{L}^{-1} = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2\partial,$$

$$L = \partial^2 + v = u^{-1}\partial u^{-2}\partial u^{-2},\ L^{-1} = u^2\partial^{-1}u^2\partial^{-1}u,$$

where $\mathcal{L} = J^{-1}K$.

Now let us choose the first seed function from the kernel of K:

$$\bar{G}^1_{-1} = f(t_n)u^2,$$

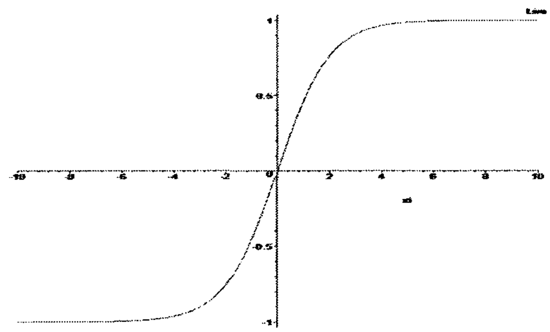

Figure 3.2. 2D graphics for kink solution with $A = 1$, $c = 2$ and $\tilde{c} = -1$

where $f(t_n)$ is arbitrarily given functions with respect to variables t_n, but independent of x. It produces one isospectral negative order hierarchy of (3.14).

$$u_{t_m} = J\mathcal{L}^{m+1} \cdot \bar{G}_{-1}^{k}, k = 1, m = -1, -2, \ldots \tag{3.16}$$

which has the standard Lax operators

$$\bar{W}_m^k = -\sum_{j=m+1}^{-1} V(\bar{G}_j^k)L^{m-j}, k = 1, m = -1, -2, \ldots \tag{3.17}$$

where $V(\bar{G}_j^k) = -\frac{1}{4}(\bar{G}_j^k)_x + \frac{1}{2}\bar{G}_j^k\partial$.

So, the standard Lax representation for equation (1.2) is $L_{t_{-1}} = [\bar{G}_{-1}^1, L]$ with $\bar{G}_{-1}^1 = (\frac{1}{4}(\bar{G}_{-1}^1)_x + \frac{1}{2}\bar{G}_{-1}^1\partial)L^{-1}$.

CHAPTER 4

MORE INTEGRABLE EQUATIONS WITH NON-SMOOTH SOLITONS

In this section we present more integrable equations which have non-smooth solitons.

4.1 Modified KdV Equation with Peakon Solution

First let us consider the modified KdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{4.1}$$

By using the same travelling setting, equation (4.1) is transformed into

$$U'(\xi) = \pm U\sqrt{c - \frac{1}{2}U^2}. \tag{4.2}$$

Then following the same procedure, we obtain

$$U(\xi) = \frac{16ce^{-\sqrt{c}|\xi|}}{e^{-2\sqrt{c}|\xi|} + 32c}, \tag{4.3}$$

where $c > 0$.

It is easy to verify that this is a peaked soliton solution, which is a new solution for modified KdV equation.

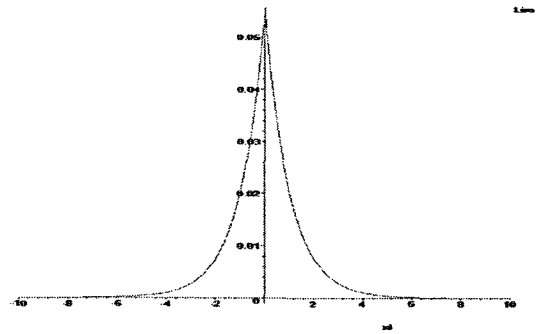

Figure 4.1. 2D graphics for peakon solution with $c = 2$

4.2 Gardner Equation with Peakon Solution

Next let's consider the Gardner equation under homogeneous boundary condition.

$$u_t + u_x + \beta u_{xxx} + \alpha u u_x + 3\mu\alpha^2 u^2 u_x = 0, \tag{4.4}$$

where α, β and μ are constants.

Through a simple scaling, equation (4.4) could be transformed into the following equation

$$u_t + u_x + u_{xxx} - 2uu_x + 12u^2 u_x = 0 \tag{4.5}$$

Following the same travelling setting as equation (1.1) and similar computation, equation (4.5) becomes

$$U'(\xi) = \pm U\sqrt{-2U^2 + 2/3U + c - 1}. \tag{4.6}$$

By some complicated computation, we obtain

$$U(\xi) = \frac{36(c-1)e^{-\sqrt{c-1}|\xi|}}{(e^{-\sqrt{c-1}|\xi|} - 3)^2 + 648(c-1)}, \tag{4.7}$$

where $c > 1$.

Apparently, this is a peaked soliton solution too. Its graph is given here.

CHAPTER 5

CONCLUSIONS AND ITS APPLICATIONS

In this thesis, we present several integrable equations with non-smooth solitons. Following the idea from [10], we provide the Lax representation, bi-Hamiltonian structure, and "M/W-shape" peakons as well as new cuspons for equation (1.1). "M/W-shape" peakon and cuspon solutions are given in explicit forms, which confirms the peakon existence for equation (1.1) in Rosenau's paper [17].

Equation (1.1) has a physical background as said in [17]. On the other hand, we believe that it could be derived from the two dimensional Euler equation. After the traveling wave setting, it can be cast into the following Newton equation $U'^2 = P(U)$ of a particle with a new potential $P(U) = U^2 + 2c \mp 2\sqrt{c^2 - bU^2}$ (take "−" if $c > 0$ and "+" if $c < 0$). In this thesis, we successfully solve this Newton equation with new cuspons and "M-shape"/"W-shape"-peaks solitons. Those peaked and cusped solutions may be applied to neuroscience for providing a mathematical model and explaining electrophysiological responses of visceral nociceptive neurons and sensitization of dorsal root reflexes [2].

No smooth solitons are found for equation (1.1), but it is completely integrable. Furthermore, we may think about a more general case: $\rho_t = bu_x + \gamma u_{xxx} + \frac{1}{2}[(u^2 - u_x^2)\rho]_x, \rho = u - u_{xx}$, where b, γ are two constants.

We just showed a few of the integrable equations with non-smooth solitons in this thesis. Actually more this kind of equations have been discovered nowadays, such as two-component CH equation. With the development of integrable system, we believe more and more such equations will be explored in future.

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BIOGRAPHICAL SKETCH

Xianqi Li was born in October 20, 1983, in Liaocheng, Shandong Province. In 2003 he graduated from the Shenxian experimental high school, then he got admission to the Liaoning University, where he enrolled as mathematics major because of his great interest. During his undergraduate study, he received the First Scholarship every year because of his comprehensive performance. And in 2005 he also received the National Scholarship, awarded by the Chinese Government. He is the only one who received this award in the department of mathematics. In 2007 he graduated from Liaoning University with B.S. in mathematics. Later that year he was admitted to the M.S. program in mathematics at University of Texas-Pan American. During these two year he has written two papers "An Integrable Equation with Non-smooth Solitons " and " A New Integrable Peakon Equation " which have been accepted for publication. He also received an award from the Sixth IMACS International Conference and gave a presentation in this conference. He expects to get his master degree with 4.00 GPA in August, 2009.