5-2022

On Approximating Solitary Wave Solutions for the Classical Euler Equations

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ON APPROXIMATING SOLITARY WAVE SOLUTIONS FOR THE CLASSICAL EULER EQUATIONS

A Thesis

by

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Submitted in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

Major Subject: Mathematics

The University of Texas Rio Grande Valley

May 2022

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ABSTRACT

Páez, Julio C., On Approximating Solitary Wave Solutions for the classical Euler Equations. Master of Science (MS), May, 2022, 19 pp., references, 2 titles.

In this paper, we use a method based on Hirota substitution or the Wrońskian method to find approximate solitary wave solutions to the classical Euler equations. This method uses a small parameter λ as the basis of approximation, a parameter derived from the form of prospective solutions we consider, rather than the standard small parameters α and β. The L^∞ norm and asymptotic notation are used to measure the accuracy of the approximation rather than finding the error explicitly.

DEDICATION

This thesis is dedicated to my father, Julio César Páez, and my mother, Elizabeth Hernández, who have done everything in their power to help me through my academic endeavors.

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CHAPTER I

INTRODUCTION

The physical behavior of fluids has been of interest in the domain of mathematics for millennia. Among the many models developed to simulate fluid mechanics are the classical Euler equations, which can be used to model the flow of inviscid, incompressible fluid. This paper will study a system derived from Euler's equations applied to water. More specifically, the system models the behavior of water in a shallow channel: the velocity of the water particles within the channel and the wave the water makes on the surface.

This paper will focus on solutions to the system that produce solitary surface waves. However, no exact solution to the system will be given. Rather, we will provide approximate solutions. We will measure the accuracy of these approximations using the L^∞ norm. Given a function $f = f(\vec{x})$ for $\vec{x} \in \mathbb{R}$, this norm is typically defined as

$$\|f\|_\infty = \|f\| := \inf\{M \geq 0 : |f(\vec{x})| \leq M \text{ for almost every } \vec{x}\}.$$

As we will work only with continuous functions, the L^∞ norm of f in this context is the smallest positive number that bounds the function f, or the smallest upper bound of $|f|$. We will also use asymptotic notation for estimates, namely Big O notation.

CHAPTER II

PRELIMINARIES

To set the stage, we will follow the notation and derivation from [1], with a few additional details.

Consider a shallow channel containing an inviscid, incompressible fluid, with a surface wave that is evolving through time. Using x and y as the horizontal coordinates, z as the vertical coordinate, and t as the temporal coordinate, we will now mathematically define the different parts of this system. We will represent the wave propagating on the surface of the fluid by $\eta = \eta(x,y,t)$, the bottom of the channel by $-h(x,y)$, and the domain of the fluid in the channel at time t by Ω_t. Note that the bottom of the channel is fixed in time, as we do not consider the weathering effects of the fluid.

If $\vec{v} = \vec{v}(x,y,z,t) = u(x,y,z,t)\ \hat{\imath} + v(x,y,z,t)\ \hat{\jmath} + w(x,y,z,t)\ \hat{k}$ represents the velocity field of the fluid at time t, then the motion of the fluid is described by the classical Euler equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = -g\hat{k} \quad \text{in } \Omega_t, \tag{2.1}$$

$$\nabla\cdot\vec{v} = 0 \quad \text{in } \Omega_t, \tag{2.2}$$

where g is the acceleration due to gravity, p is the pressure field, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit vectors in the x, y, and z directions, respectively, and $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ is the gradient of the spatial coordinates.

We now suppose that the initial velocity field $\vec{v}(x,y,z,t_0)$ for some t_0 is irrotational. Since pressure and gravity are not rotational forces, Helmholtz's third theorem implies $\vec{v}(x,y,z,t)$ remains irrotational for all t. Hence, $\vec{v}(x,y,z,t)$ is a conservative vector field for all t, so there exists a potential function $\phi = \phi(x,y,z,t)$ such that

$$\vec{v} = \nabla\phi. \tag{2.3}$$

Together with (2.2), (2.3) implies ϕ is a solution to Laplace's equation

$$\nabla^2\phi = 0 \quad \text{in } \Omega_t, \tag{2.4}$$

for all t.

Assuming Ω_t is unbounded in the horizontal directions, the boundary of the domain Ω_t is the free surface at $z = \eta(x,y,t)$ and the fixed surface at $z = -h(x,y)$.

On the fixed surface $z = -h(x,y)$, the impermeability condition $\vec{v}\cdot\vec{n} = 0$ is satisfied, where $\vec{n} = h_x(x,y)\hat{\imath} + h_y(x,y)\hat{\jmath} + \hat{k}$ is a vector normal to $-h(x,y)$. Recalling (2.3), we have that ϕ satisfies

$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \quad \text{on } z = -h(x,y). \tag{2.5}$$

Moreover, as the free surface is a material surface the kinematic condition $\frac{D(\eta - z)}{Dt} = 0$ is satisfied, where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ is the material derivative. Since ϕ is the potential function of $\vec{v}$, we can write

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \quad \text{on } z = \eta(x,y,t). \tag{2.6}$$

Furthermore, assuming the pressure on the free surface is equal to the ambient air pressure, [2] shows that (2.1) and (2.2) imply the Bernoulli condition is satisfied:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz = 0 \quad \text{on } z = \eta(x,y,t). \tag{2.7}$$

Next, assume the fluid in the open channel is uniform in the y direction, and the bottom of the channel is flat and horizontal. Then, h is the undisturbed depth of the fluid, and the system consisting of (2.4), (2.5), (2.6), and (2.7) simplifies to

$$\phi_{xx} + \phi_{zz} = 0 \quad \text{in } -h < z < \eta(x,t), \tag{2.8}$$

$$\phi_z = 0 \quad \text{on } z = -h, \tag{2.9}$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0 \quad \text{on } z = \eta(x,t), \tag{2.10}$$

$$\phi_t + \frac{1}{2}\Big((\phi_x)^2 + (\phi_z)^2\Big) + gz = 0 \quad \text{on } z = \eta(x,t), \tag{2.11}$$

where the undisturbed surface of the fluid is located at $z = 0$.

We now transform the system by setting $h = 1$ and by shifting and scaling the variables as follows:

$t = g^{-1/2}\,\tilde{t}\,,\quad z = \tilde{z} - 1,\quad \phi(x,z,t) = \sqrt{g}\,\tilde{\phi}\left(x,\tilde{z},\tilde{t}\right),\quad \eta(x,t) = \tilde{\eta}\left(x,\tilde{t}\right).$

Using these variables, the system (2.8)-(2.11) becomes

$$\tilde{\phi}_{xx} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0 \quad \text{in } 0 < \tilde{z} < 1 + \tilde{\eta},$$

$$\tilde{\phi}_{\tilde{z}} = 0 \quad \text{on } \tilde{z} = 0,$$

$$\tilde{\eta}_{\tilde{t}} + \tilde{\phi}_x \tilde{\eta}_x - \tilde{\phi}_{\tilde{z}} = 0 \quad \text{on } \tilde{z} = 1 + \tilde{\eta},$$

$$\tilde{\phi}_{\tilde{t}} + \frac{1}{2}\left(\left(\tilde{\phi}_x\right)^2 + \left(\tilde{\phi}_{\tilde{z}}\right)^2\right) + (\tilde{z} - 1) = 0 \quad \text{on } \tilde{z} = 1 + \tilde{\eta}.$$

If we omit the tildes for simplicity, we ultimately arrive at the system

$$\phi_{xx} + \phi_{zz} = 0 \quad \text{in } 0 < z < 1 + \eta, \tag{2.12}$$

$$\phi_z = 0 \quad \text{on } z = 0, \tag{2.13}$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0 \quad \text{on } z = 1 + \eta, \tag{2.14}$$

$$\eta + \phi_t + \frac{1}{2}\Big((\phi_x)^2 + (\phi_z)^2\Big) = 0 \quad \text{on } z = 1 + \eta. \tag{2.15}$$

CHAPTER III

STUDYING THE SYSTEM

Consider the ansatz $f(x,t) = C \cdot \log(\mathscr{W})$, where $\mathscr{W} = \mathscr{W}(x,t)$ is a linear combination of N exponential functions:

$$\mathscr{W}(x,t) = \sum_{j=1}^{N} e^{k_j x - p_j t}, \tag{3.1}$$

where k_j and p_j are real numbers for $j = 1, 2, ..., N$. Note that $\mathscr{W}$ is positive for all x and t.

We now define the parameter λ to be the absolute maximum of the $2N$ coefficients, k_j and p_j:

$$\lambda = \max_{1 \le j \le N} \{|k_j|, |p_j|\}. \tag{3.2}$$

Lemma 3.0.1. *Let m and n be non-negative integers. Then, the parameter λ satisfies*

$$\left\| \frac{\frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right]}{\mathscr{W}(x,t)} \right\|_\infty \le \lambda^{m+n}.$$

Proof. Using the definition of $\mathscr{W}(x,t)$ in (3.1), we have that

$$\begin{aligned}
\left| \frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right] \right| &= \left| \frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\sum_{j=1}^{N} e^{k_j x - p_j t}\right] \right| = \left| \sum_{j=1}^{N} \frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[e^{k_j x - p_j t}\right] \right| \\
&= \left| \sum_{j=1}^{N} (k_j)^m (-p_j)^n e^{k_j x - p_j t} \right| \\
&\le \sum_{j=1}^{N} \left| (k_j)^m (-p_j)^n e^{k_j x - p_j t} \right| = \sum_{j=1}^{N} |k_j|^m |p_j|^n e^{k_j x - p_j t} \\
&\le \sum_{j=1}^{N} \lambda^m |p_j|^n e^{k_j x - p_j t} = \sum_{j=1}^{N} \lambda^m \lambda^n e^{k_j x - p_j t}.
\end{aligned}$$

Hence,

$$\left|\frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right]\right| \le \sum_{j=1}^{N} \lambda^{m+n} e^{k_j x - p_j t} = \lambda^{m+n} \sum_{j=1}^{N} e^{k_j x - p_j t} = \lambda^{m+n}\, \mathscr{W}(x,t).$$

Thus,

$$\left|\frac{\frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right]}{\mathscr{W}(x,t)}\right| = \frac{\left|\frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right]\right|}{|\mathscr{W}(x,t)|} \le \frac{\lambda^{m+n}\, \mathscr{W}(x,t)}{\mathscr{W}(x,t)} = \lambda^{m+n},$$

that is, λ^{m+n} is an upper bound of $\left|\frac{\frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right]}{\mathscr{W}(x,t)}\right|$ for all x and t.

Therefore,

$$\left\|\frac{\frac{\partial^{m+n}}{\partial x^m \partial t^n}\left[\mathscr{W}(x,t)\right]}{\mathscr{W}(x,t)}\right\|_{\infty} \le \lambda^{m+n}.$$

□

We will now use $f(x,t)$ to set

$$\phi(x,z,t) = -f_t(x,t) + \frac{z^2}{2} f_{xxt}(x,t), \text{ and} \tag{3.3}$$

$$\eta(x,t) = f_{xx}(x,t). \tag{3.4}$$

Substituting these definitions into the system (2.12) - (2.15), we get that in $0 < z < 1+\eta$,

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= \left(-f_{xxt} + \frac{z^2}{2} f_{xxxxt}\right) + (f_{xxt}) = \frac{z^2}{2} f_{xxxxt} \\ &< \frac{(1+\eta)^2}{2} f_{xxxxt} = \frac{(1+f_{xx})^2}{2} f_{xxxxt}, \text{ so} \end{aligned}$$

$$\phi_{xx} + \phi_{zz} < \frac{1}{2} f_{xxxxt} + f_{xx} f_{xxxxt} + \frac{1}{2}(f_{xx})^2 f_{xxxxt}. \tag{3.5}$$

Moreover, on $z = 0$,

$$\phi_z = z \cdot f_{xxt} = 0 \cdot f_{xxt} = 0. \tag{3.6}$$

Furthermore, on $z = 1 + \eta = 1 + f_{xx}$,

$$\begin{aligned}
\eta_t + \phi_x \eta_x - \phi_z &= (f_{xxt}) + \left(- f_{xt} + \frac{z^2}{2} f_{xxxt} \right) (f_{xxx}) - (z f_{xxt}) \\
&= (1 - z) f_{xxt} - f_{xt} f_{xxx} + \frac{z^2}{2} f_{xxx} f_{xxxt} \\
&= \big(1 - (1 + f_{xx})\big) f_{xxt} - f_{xt} f_{xxx} + \frac{(1 + f_{xx})^2}{2} f_{xxx} f_{xxxt},
\end{aligned}$$

which implies

$$\begin{aligned}
\eta_t + \phi_x \eta_x - \phi_z = -f_{xx} f_{xxt} - f_{xt} f_{xxx} + \frac{1}{2} f_{xxx} f_{xxxt} + f_{xx} f_{xxx} f_{xxxt} \\
+ \frac{1}{2} (f_{xx})^2 f_{xxx} f_{xxxt}.
\end{aligned} \tag{3.7}$$

Also on $z = 1 + \eta = 1 + f_{xx}$,

$$\begin{aligned}
\eta + \phi_t + \frac{1}{2}\Big((\phi_x)^2 + (\phi_z)^2\Big) &= (f_{xx}) + \left(- f_{tt} + \frac{z^2}{2} f_{xxtt} \right) \\
&\quad + \frac{1}{2} \left(\left(- f_{xt} + \frac{z^2}{2} f_{xxxt} \right)^2 + (z f_{xxt})^2 \right) \\
&= f_{xx} - f_{tt} + \frac{1}{2} f_{xxtt} + f_{xx} f_{xxtt} + \frac{1}{2} (f_{xx})^2 f_{xxtt} \\
&\quad + \frac{1}{2} (f_{xt})^2 - \frac{z^2}{2} f_{xt} f_{xxxt} + \frac{z^4}{8} (f_{xxxt})^2 \\
&\quad + \frac{z^2}{2} (f_{xxt})^2 \\
&= f_{xx} - f_{tt} + \frac{1}{2} (f_{xt})^2 + \frac{1}{2} f_{xxtt} \\
&\quad + f_{xx} f_{xxtt} + \frac{1}{2} (f_{xx})^2 f_{xxtt} \\
&\quad - \frac{z^2}{2} f_{xt} f_{xxxt} + \frac{z^4}{8} (f_{xxxt})^2 + \frac{z^2}{2} (f_{xxt})^2
\end{aligned}$$

Though these expressions are complicated, we can use the parameter λ and Big O notation to see that the η and ϕ defined by (3.3) satisfy

$$\phi_{xx} + \phi_{zz} \leq O\left(\lambda^5\right) \qquad \text{in } 0 < z < 1 + \eta, \quad (3.8)$$

$$\phi_z = 0 \qquad \text{on } z = 0, \quad (3.9)$$

$$\eta_t + \phi_x \eta_x - \phi_z \leq O\left(\lambda^5\right) \qquad \text{on } z = 1 + \eta, \quad (3.10)$$

$$\eta + \phi_t + \frac{1}{2}\left((\phi_x)^2 + (\phi_z)^2\right) \leq f_{xx} - f_{tt} + \frac{1}{2}(f_{xt})^2 + \frac{1}{2}f_{xxtt} + O\left(\lambda^5\right) \qquad \text{on } z = 1 + \eta. \quad (3.11)$$

Comparing the above results with the system (2.12)-(2.15), we see that if we make λ small enough, the contribution of most derivatives of f can be made negligible.

Thus, the η and ϕ defined by (3.3) and (3.4), respectively, *almost* solve the system, with the only potentially non-trivial contribution coming from the expression

$$E(x,t) = f_{xx} - f_{tt} + \frac{1}{2}(f_{xt})^2 + \frac{1}{2}f_{xxtt}. \quad (3.12)$$

If we can bound this "error function" by a high-order polynomial of λ, then the resulting η and ϕ will come very close to solving the system (2.12) - (2.15).

However, the contribution of (3.12) appears to be $O\left(\lambda^2\right)$, an order that is not as high as the $O\left(\lambda^5\right)$ present throughout the expressions.

Hence, in order to find a better bound for $E(x,t)$, we must use other methods.

CHAPTER IV

STUDYING THE ERROR: ONE-SOLITARY WAVE

It is evident that if we can make (3.12) equal to 0, we are done. The resulting equation

$$f_{xx} - f_{tt} + \frac{1}{2}(f_{xt})^2 + \frac{1}{2} f_{xxtt} = 0 \tag{4.1}$$

is a non-linear, fourth-order partial differential equation. We begin the analysis of equation (4.1) by finding a solution that yields a one-soliton surface wave η. The ansatz given by the Hirota method is

$$f(x,t) = C \log\left(1 + e^{kx-pt}\right)$$

for real, non-trivial constants C, k, and p. Substituting this form of f into (4.1) gives the equation

$$\left(k^2 - p^2\right) \frac{e^{kx-pt}}{\left(1+e^{kx-pt}\right)^2} + \frac{k^2 p^2}{2} \cdot \frac{e^{kx-pt} + (C-4)e^{2(kx-pt)} + e^{3(kx-pt)}}{\left(1+e^{kx-pt}\right)^4} = 0,$$

which can we expressed as

$$\begin{aligned} &\left(k^2 - p^2\right) \frac{e^{kx-pt} + 2e^{2(kx-pt)} + e^{3(kx-pt)}}{\left(1+e^{kx-pt}\right)^4} \\ &\qquad + \frac{k^2 p^2}{2} \cdot \frac{e^{kx-pt} + (C-4)e^{2(kx-pt)} + e^{3(kx-pt)}}{\left(1+e^{kx-pt}\right)^4} = 0. \end{aligned}$$

If we then set $C = 6$, we arrive at the equation

$$\left(k^2 - p^2 + \frac{k^2 p^2}{2}\right) \frac{e^{kx-pt} + 2e^{2(kx-pt)} + e^{3(kx-pt)}}{\left(1+e^{kx-pt}\right)^4} = 0.$$

Hence, if k and p are solutions to

$$k^2 - p^2 + \frac{k^2 p^2}{2} = 0,$$

then $f(x,t)$ solves (4.1). We can solve the above equation for $|k| < \sqrt{2}$ if we set

$$p = \pm \frac{k}{\sqrt{1 - \frac{1}{2}k^2}}.$$

Scaling k to $2k$ ultimately yields the real function

$$f(x,t) = 6 \log\left(1 + e^{2kx \pm \frac{2k}{\sqrt{1-2k^2}}t}\right), \tag{4.2}$$

which is a solution to (4.1) for any k with $|k| < \frac{\sqrt{2}}{2}$.

The surface wave defined by (4.2) is its second spatial derivative

$$f_{xx}(x,t) = 6 \, \frac{k^2 e^{2kx \pm \frac{2k}{\sqrt{1-2k^2}}t}}{\left(1 + e^{2kx \pm \frac{2k}{\sqrt{1-2k^2}}t}\right)^2}. \tag{4.3}$$

Using the linearity of the derivative, we can express the surface wave defined by (4.2) as

$$f_{xx}(x,t) = 6k^2 \, \mathrm{sech}^2\left(e^{kx \pm \frac{k}{\sqrt{1-2k^2}}t}\right). \tag{4.4}$$

CHAPTER V

STUDYING THE ERROR: A GENERAL ESTIMATE

To find higher soliton surface waves, we must bound the error function more generally than for the one-soliton case. Recall $f(x,t) = C \cdot \log(\mathscr{W})$, where $\mathscr{W} = \mathscr{W}(x,t)$ is defined as in (3.1). For simplicity, we will set the constant $C := 1$. Substituting $f = \log(\mathscr{W})$ into (3.12) yields

$$\begin{aligned}
E(x,t) &= f_{xx} - f_{tt} + \frac{1}{2}\left(f_{xt}\right)^2 + \frac{1}{2} f_{xxtt} \\
&= \frac{1}{\mathscr{W}}\left[\mathscr{W}_{xx} - \mathscr{W}_{tt} + \frac{1}{2}\mathscr{W}_{xxtt}\right] \\
&\quad + \frac{1}{\mathscr{W}^2}\left[(\mathscr{W}_t)^2 - (\mathscr{W}_x)^2 - \frac{1}{2}(\mathscr{W}_{xt})^2 - \mathscr{W}_x\mathscr{W}_{xtt} - \mathscr{W}_t\mathscr{W}_{xxt} - \frac{1}{2}\mathscr{W}_{xx}\mathscr{W}_{tt}\right] \\
&\quad + \frac{1}{\mathscr{W}^3}\left[(\mathscr{W}_x)^2\mathscr{W}_{tt} + (\mathscr{W}_t)^2\mathscr{W}_{xx} + 3\mathscr{W}_x\mathscr{W}_t\mathscr{W}_{xt}\right] \\
&\quad + \frac{1}{\mathscr{W}^4}\left[-\frac{5}{2}(\mathscr{W}_x)^2(\mathscr{W}_t)^2\right] \\
&= \frac{1}{\mathscr{W}^2}\left[\mathscr{W}\mathscr{W}_{xx} - \mathscr{W}\mathscr{W}_{tt} + \frac{1}{2}\mathscr{W}\mathscr{W}_{xxtt} + (\mathscr{W}_t)^2 - (\mathscr{W}_x)^2 - \frac{1}{2}\mathscr{W}_{xx}\mathscr{W}_{tt}\right] \\
&\quad + \frac{1}{\mathscr{W}^2}\left[-\frac{1}{2}(\mathscr{W}_{xt})^2 - \mathscr{W}_x\mathscr{W}_{xtt} - \mathscr{W}_t\mathscr{W}_{xxt}\right] \\
&\quad + \frac{1}{\mathscr{W}^3}\left[(\mathscr{W}_x)^2\mathscr{W}_{tt} + (\mathscr{W}_t)^2\mathscr{W}_{xx} + 3\mathscr{W}_x\mathscr{W}_t\mathscr{W}_{xt}\right] \\
&\quad + \frac{1}{\mathscr{W}^4}\left[-\frac{5}{2}(\mathscr{W}_x)^2(\mathscr{W}_t)^2\right]
\end{aligned}$$

We now define the functions $E_1 = E_1(x,t)$, $E_2 = E_2(x,t)$, $E_3 = E_3(x,t)$, and $E_4 = E_4(x,t)$ to be

$$E_1 = \frac{1}{\mathscr{W}^2}\left[\mathscr{W}\mathscr{W}_{xx} - \mathscr{W}\mathscr{W}_{tt} + \frac{1}{2}\mathscr{W}\mathscr{W}_{xxtt} + (\mathscr{W}_t)^2 - (\mathscr{W}_x)^2 - \frac{1}{2}\mathscr{W}_{xx}\mathscr{W}_{tt}\right], \tag{5.1}$$

$$E_2 = \frac{1}{\mathscr{W}^2}\left[-\frac{1}{2}(\mathscr{W}_{xt})^2 - \mathscr{W}_x\mathscr{W}_{xtt} - \mathscr{W}_t\mathscr{W}_{xxt}\right], \tag{5.2}$$

$$E_3 = \frac{1}{\mathscr{W}^3}\left[(\mathscr{W}_x)^2\mathscr{W}_{tt} + (\mathscr{W}_t)^2\mathscr{W}_{xx} + 3\mathscr{W}_x\mathscr{W}_t\mathscr{W}_{xt}\right], \tag{5.3}$$

$$E_4 = \frac{1}{\mathscr{W}^4}\left[-\frac{5}{2}(\mathscr{W}_x)^2(\mathscr{W}_t)^2\right]. \tag{5.4}$$

Then,

$$E(x,t) = E_1(x,t) + E_2(x,t) + E_3(x,t) + E_4(x,t). \tag{5.5}$$

To bound the error $E(x,t)$, we will bound the sub-errors E_1, E_2, E_3 and E_4. The latter three can easily be bounded using the triangle inequality as well as the lemma 3.0.1:

$$\begin{aligned}
\|E_2(x,t)\| &\leq \left\|-\frac{1}{2}\left(\frac{\mathscr{W}_{xt}}{\mathscr{W}}\right)^2\right\| + \left\|-\frac{\mathscr{W}_x}{\mathscr{W}}\frac{\mathscr{W}_{xtt}}{\mathscr{W}}\right\| + \left\|-\frac{\mathscr{W}_t}{\mathscr{W}}\frac{\mathscr{W}_{xxt}}{\mathscr{W}}\right\| \\
&\leq \frac{1}{2}\left\|\frac{\mathscr{W}_{xt}}{\mathscr{W}}\right\|^2 + \left\|\frac{\mathscr{W}_x}{\mathscr{W}}\right\| \cdot \left\|\frac{\mathscr{W}_{xtt}}{\mathscr{W}}\right\| + \left\|\frac{\mathscr{W}_t}{\mathscr{W}}\right\| \cdot \left\|\frac{\mathscr{W}_{xxt}}{\mathscr{W}}\right\| \\
&\leq \frac{1}{2}(\lambda^2)^2 + \lambda\cdot\lambda^3 + \lambda\cdot\lambda^3 = \frac{5}{2}\lambda^4, \\
\|E_3(x,t)\| &\leq \left\|\left(\frac{\mathscr{W}_x}{\mathscr{W}}\right)^2\frac{\mathscr{W}_{tt}}{\mathscr{W}}\right\| + \left\|\left(\frac{\mathscr{W}_t}{\mathscr{W}}\right)^2\frac{\mathscr{W}_{xx}}{\mathscr{W}}\right\| + \left\|3\frac{\mathscr{W}_x}{\mathscr{W}}\frac{\mathscr{W}_t}{\mathscr{W}}\frac{\mathscr{W}_{xt}}{\mathscr{W}}\right\| \\
&\leq \left\|\frac{\mathscr{W}_x}{\mathscr{W}}\right\|^2\left\|\frac{\mathscr{W}_{tt}}{\mathscr{W}}\right\| + \left\|\frac{\mathscr{W}_t}{\mathscr{W}}\right\|^2\left\|\frac{\mathscr{W}_{xx}}{\mathscr{W}}\right\| + 3\left\|\frac{\mathscr{W}_x}{\mathscr{W}}\right\| \cdot \left\|\frac{\mathscr{W}_t}{\mathscr{W}}\right\| \cdot \left\|\frac{\mathscr{W}_{xt}}{\mathscr{W}}\right\| \\
&\leq (\lambda)^2\lambda^2 + (\lambda)^2\lambda^2 + 3\lambda\cdot\lambda\cdot\lambda^2 = 5\lambda^4, \text{ and} \\
\|E_4(x,t)\| &\leq \left\|-\frac{5}{2}\left(\frac{\mathscr{W}_x}{\mathscr{W}}\right)^2\left(\frac{\mathscr{W}_t}{\mathscr{W}}\right)^2\right\| = \frac{5}{2}\left\|\frac{\mathscr{W}_x}{\mathscr{W}}\right\|^2\left\|\frac{\mathscr{W}_t}{\mathscr{W}}\right\|^2 \leq \frac{5}{2}(\lambda)^2(\lambda)^2 = \frac{5}{2}\lambda^4.
\end{aligned}$$

Hence, $E(x,t)$ can be estimated by

$$\|E(x,t)\| \leq \|E_1\| + \|E_2\| + \|E_3\| + \|E_4\| \leq \|E_1\| + \frac{5}{2}\lambda^4 + 5\lambda^4 + \frac{5}{2}\lambda^4, \text{ i.e.,}$$

$$\|E(x,t)\| \leq \|E_1\| + 10\lambda^4. \tag{5.6}$$

Thus, if the sub-error $E_1(x,t)$ is estimated by a polynomial of λ with a degree of 4 or higher, then the total error $E(x,t)$ is $O(\lambda^4)$.

CHAPTER VI

STUDYING THE ERROR: TWO-SOLITARY WAVE

We will now attempt to find an $f(x,t)$ that produces a two-soliton surface wave. We will define $f(x,t) = C \cdot \log(\mathscr{W})$, where $\mathscr{W} = \mathscr{W}(x,t)$ is defined as in (3.1) for $N = 4$:

$$\mathscr{W}(x,t) = \sum_{j=1}^{4} e^{k_j x - p_j t} = e^{k_1 x - p_1 t} + e^{k_2 x - p_2 t} + e^{k_3 x - p_3 t} + e^{k_4 x - p_4 t}. \tag{6.1}$$

Again setting $C := 1$, we will substitute $f = \log(\mathscr{W})$ into (5.1) and express E_1 in terms of the exponential functions:

$$\begin{aligned}
E_1(x,t) &= \frac{1}{\mathscr{W}^2}\left[\mathscr{W}\mathscr{W}_{xx} - \mathscr{W}\mathscr{W}_{tt} + \frac{1}{2}\mathscr{W}\mathscr{W}_{xxtt} + (\mathscr{W}_t)^2 - (\mathscr{W}_x)^2 - \frac{1}{2}\mathscr{W}_{xx}\mathscr{W}_{tt}\right] \\
&= \frac{1}{\mathscr{W}^2}\Big[R_{1,2}\, e^{(k_1+k_2)x-(p_1+p_2)t} + R_{1,3}\, e^{(k_1+k_3)x-(p_1+p_3)t} + R_{1,4}\, e^{(k_1+k_4)x-(p_1+p_4)t} \\
&\qquad + R_{2,3}\, e^{(k_2+k_3)x-(p_2+p_3)t} + R_{2,4}\, e^{(k_2+k_4)x-(p_2+p_4)t} + R_{3,4}\, e^{(k_3+k_4)x-(p_3+p_4)t}\Big] \\
&= \frac{1}{\mathscr{W}^2}\sum_{\substack{i,j=1 \\ i<j}}^{4} R_{i,j}\, e^{(k_i+k_j)x-(p_i+p_j)t},
\end{aligned}$$

where the coefficients $R_{i,j}$ are given by

$$R_{i,j} = (k_i - k_j)^2 - (p_i - p_j)^2 + \frac{1}{2}\left(k_i^2 - k_j^2\right)\left(p_i^2 - p_j^2\right)$$

for $i, j = 1,2,3,4$ with $i < j$.

We now introduce the parameter ρ and define to be the absolute maximum of the $R_{i,j}$ coefficients:

$$\rho = \max\{|R_{1,2}|, |R_{1,3}|, |R_{1,4}|, |R_{2,3}|, |R_{2,4}|, |R_{3,4}|\}. \tag{6.2}$$

Then,

$$
\begin{aligned}
|E_1(x,t)| &= \left| \frac{1}{\mathscr{W}^2} \sum_{\substack{i,j=1 \\ i<j}}^{4} R_{i,j}\, e^{(k_i+k_j)x-(p_i+p_j)t} \right| \le \frac{1}{\mathscr{W}^2} \sum_{\substack{i,j=1 \\ i<j}}^{4} \left| R_{i,j}\, e^{(k_i+k_j)x-(p_i+p_j)t} \right| \\
&\le \frac{1}{\mathscr{W}^2} \sum_{\substack{i,j=1 \\ i<j}}^{4} \left| R_{i,j} \right| \left| e^{(k_i+k_j)x-(p_i+p_j)t} \right| \le \frac{1}{\mathscr{W}^2} \sum_{\substack{i,j=1 \\ i<j}}^{4} \rho\, e^{(k_i+k_j)x-(p_i+p_j)t} \\
&\le \frac{\rho}{\mathscr{W}^2} \sum_{\substack{i,j=1 \\ i<j}}^{4} e^{(k_i+k_j)x-(p_i+p_j)t} \le \frac{\rho}{\mathscr{W}^2} \left(\frac{\mathscr{W}^2}{2} \right) = \frac{\rho}{2}.
\end{aligned}
$$

Hence, if we choose k_i and p_i such that $\rho = O\left(\lambda^4\right)$, the total error (3.12) will be $O\left(\lambda^4\right)$.

To this end, we will define all the k_j parameters in terms of a single parameter, k. For three given constants a_2, a_3, and a_4, and with $a_1 = 1$, let

$$
\begin{aligned}
k_1 &= a_1 k = k \\
k_2 &= a_2 k \\
k_3 &= a_3 k \\
k_4 &= a_4 k
\end{aligned}
$$

Moreover, we will order k_1, k_2, k_3, and k_4 such that

$$|k_1| \ge |k_2| \ge |k_3| \ge |k_4|.$$

Next, we use the relation derived in chapter IV to define the p_j coefficients:

$$p_j = \pm \frac{k_j}{\sqrt{1 - \frac{1}{2} k_j^2}}.$$

Moreover, we will define all the p_j coefficients as either

$$p_j = +\frac{k_j}{\sqrt{1-\frac{1}{2}k_j^2}} \quad \text{or all} \quad p_j = -\frac{k_j}{\sqrt{1-\frac{1}{2}k_j^2}}.$$

Note that for the p_j to be well-defined, we must restrict the k_j's to the interval $\left(-\sqrt{2},\sqrt{2}\right)$.

By the definition of ρ, we have that for some i and j with $i \neq j$,

$$\begin{aligned}
\rho = \left|R_{i,j}\right| &= \left|(k_i-k_j)^2-(p_i-p_j)^2+\frac{1}{2}\left(k_i^2-k_j^2\right)\left(p_i^2-p_j^2\right)\right| \\
&= \left|k_i^2-2\,k_ik_j+k_j^2-p_i^2+2\,p_ip_j-p_j^2+\frac{1}{2}\,k_i^2p_i^2-\frac{1}{2}\,k_i^2p_j^2-\frac{1}{2}\,k_j^2p_i^2+\frac{1}{2}\,k_j^2p_j^2\right| \\
&= \left|k_i^2-p_i^2+\frac{1}{2}\,k_i^2p_i^2-2\,k_ik_j+2\,p_ip_j-\frac{1}{2}\,k_i^2p_j^2-\frac{1}{2}\,k_j^2p_i^2+k_j^2-p_j^2+\frac{1}{2}\,k_j^2p_j^2\right| \\
&= \left|-2\,k_ik_j+2\,p_ip_j-\frac{1}{2}\,k_i^2p_j^2-\frac{1}{2}\,k_j^2p_i^2\right|.
\end{aligned}$$

Since $k_i = a_ik$ and $k_j = a_jk$, we see that $k_j = ak_i$, where $a = \frac{a_j}{a_i}$. Substituting ak_i for k_j, we get that

$$\begin{aligned}
\rho &= \left|-2ak_i^2+2\,p_ip_j-\frac{a^2k_i^4}{2-a^2k_i^2}-\frac{a^2k_i^4}{2-k_i^2}\right| \\
&= \left|-2ak_i^2+2\frac{ak_i^2}{\sqrt{1-\frac{1}{2}k_i^2}\sqrt{1-\frac{a^2}{2}k_i^2}}-\frac{4a^2k_i^4-a^2(a^2+1)k_i^6}{4-2(a^2+1)k_i^2+a^2k_i^4}\right|.
\end{aligned}$$

We now expand $-2ak_i^2+2\frac{ak_i^2}{\sqrt{1-\frac{1}{2}k_i^2}\sqrt{1-\frac{a^2}{2}k_i^2}}-\frac{4a^2k_i^4-a^2(a^2+1)k_i^6}{4-2(a^2+1)k_i^2+a^2k_i^4}$ into its Maclaurin series with respect to k_i:

$$-2ak_i^2+2\frac{ak_i^2}{\sqrt{1-\frac{1}{2}k_i^2}\sqrt{1-\frac{a^2}{2}k_i^2}}-\frac{4a^2k_i^4-a^2(a^2+1)k_i^6}{4-2(a^2+1)k_i^2+a^2k_i^4} = \frac{a(a-1)^2}{2}k_i^4+O\left(k_i^5\right).$$

Thus,

$$\rho \le \frac{|a|(a-1)^2}{2}\lambda^4+O\left(\lambda^5\right), \tag{6.3}$$

CHAPTER VII

CONCLUSION

With this method, we found a form for a one-soliton solution that solves the system to an order of $O\left(\lambda^5\right)$, as well as a form that leads to two-soliton solutions to the system to an order of $O\left(\lambda^4\right)$. As the approach for estimating $E(x,t)$ in chapter V generalizes for any N chosen to define $\mathscr{W}$, we can further increase the number of exponentials used in the sum to find greater soliton solutions of order $O\left(\lambda^4\right)$.

Moreover, we can consider more terms in the approximation to perhaps increase the order of a two-soliton to $O\left(\lambda^5\right)$. Another path for future work is assessing the particle trajectories given by the potential function, as this paper is focuses on the behavior of the surface wave.

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BIOGRAPHICAL SKETCH

Julio César Páez was born in Brownsville, Texas, on November 5, 1998. He graduated from Brownsville Early College High School in 2017, having taken classes at the University of Texas Rio Grande Valley during his junior and senior year.

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