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An Analysis of Antichimeral Ramanujan Type Congruences for Quotients of the Rogers-Ramanujan Functions

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AN ANALYSIS OF ANTICHIMERAL RAMANUJAN TYPE CONGRUENCES FOR QUOTIENTS OF THE ROGERS-RAMANUJAN FUNCTIONS

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ABSTRACT

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This paper proves the existence of antichimeral Ramanujan type congruences for certain modular forms These modular forms can be represented in terms of Klein forms and the Dedekind eta function. The main focus of this thesis is to introduce the necessary theory to characterize these specific Ramanujan type congruences and prove their antichimerality.

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CHAPTER I

INTRODUCTION

1.1 Overview

In this chapter we will outline this thesis, with additional technical detail appearing in later chapters. We will, however, state the background, the main focus and other objectives, and the motivation of this research. We will also briefly discuss the limitations and optimizations that exist for this research. In this thesis, we study products of certain q-series that, when raised to certain powers, yield a modular form that satisfies a specific kind of congruence, which is defined as an antichimeral Ramanujan type congruence.

1.2 Background

In the early 1900s, Ramanujan discovered three patterns that emerged when looking at tables of data generated by the classical partition function,$p(n)$, given by the infinite sum

$$\sum_{n=0}^{\infty} p(n)q^n. \tag{1.1}$$

He discovered for certain values of n, $p(n)$ was always divisible by $5,7$, or 11. Put more formally,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.3}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.4}$$

Upon further research, Ramanujan was able to extend these relations to infinitely many powers of $5, 7$, or 11. He conjectured that for any $j \in \mathbb{N}^+$,

$$p(5^j n + \delta_{5,j}) \equiv 0 \pmod{5^j}, \tag{1.5}$$

$$p(7^j n + \delta_{7,j}) \equiv 0 \pmod{7^j}, \tag{1.6}$$

$$p(11^j n + \delta_{11,j}) \equiv 0 \pmod{11^j}, \tag{1.7}$$

where $\delta_{p,j} = 24^{-1} \pmod{p^j}$. It was later shown by Gupta [2], that for (1.5), this was not true for $j = 3$. The correct congruences were later proven to be

$$p(5^j n + \delta_{5,j}) \equiv 0 \pmod{5^j}, \tag{1.8}$$

$$p(7^{\lfloor \frac{j}{2} \rfloor} + \delta_{7,j}) \equiv 0 \pmod{7^j}, \tag{1.9}$$

$$p(11^j n + \delta_{11,j}) \equiv 0 \pmod{11^j}, \tag{1.10}$$

by Watson [11] and others (c.f. [9]). These congruences came to be known as Ramanujan type congruences for the partition function.

Ramanujan re-discovered two identities regarding q-series. He realized that these two functions:

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \tag{1.11}$$

and

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} \tag{1.12}$$

could be represented as:

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1.13}$$

and

$$H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.14}$$

Rogers [10] originally discovered and proved these identities in a paper published one to two decades earlier, so these functions became known as the Rogers-Ramanujan functions. Interestingly enough, these two functions are also partition generating functions under different constraints. For example, (1.12) can be interpreted as the generating function of the number of partitions of an integer such that each part is congruent to 1 or 4 modulo 5. Similarly, (1.13) can be interpreted as the generating function of the number of partitions of an integer such that each part is congruent to 2 or 3 modulo 5.

1.3 Main Focus

We broke down the classical partition function into different building blocks.

Note that

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^5;q^5)_\infty}\frac{1}{(q;q^5)_\infty}\frac{1}{(q^2;q^5)_\infty}\frac{1}{(q^3;q^5)_\infty}\frac{1}{(q^4;q^5)_\infty} = \frac{1}{(q^5;q^5)_\infty}G(q)H(q). \tag{1.15}$$

We then raised these building blocks to other specific powers. The conditions for these exponents are developed in later chapters. The resulting product was then multiplied by appropriate powers of q. Such products result in modular forms for $\Gamma_1(5)$, a subgroup of the full modular group which will be defined upon in later chapters. Certain powers caused the modular form to do one of three things:

1. The modular form may satisfy infinitely many Ramanujan type congruences modulo powers of 5^j, for any $j \in \mathbb{N}^+$.

2. The modular form may satisfy finitely many Ramanujan type congruences modulo powers of 5^j, for all natural numbers $j \leq k \in \mathbb{N}^+$. The Ramanujan type congruences generated this way were said to be **chimeral** modulo powers of 5^j.

3. The modular form may satisfy infinitely many Ramanujan type congruences modulo 5^k for all $k > j$, where $k, j \in \mathbb{N}^+$ and does not satisfy a Ramanujan type congruence modulo 5^k for $1 \leq k < j$. We term congruences generated this way as **antichimeral** modulo powers of 5^j.

This last item was of special interest to us, because we did not expect such congruences and have no prior knowledge of Ramanujan type congruences for other modular forms of this form. The main focus of our work was to understand these antichimeral Ramanujan type congruences, and rigorously prove that they satisfy the conditions given.

1.4 Other Objectives

We now describe things more concretely. We consider functions constructed in terms of

$$A(q) = (q;q)_\infty^{\frac{2}{5}} H(q) \qquad B(q) = q^{\frac{1}{5}}, (q;q)_\infty^{\frac{2}{5}} G(q), \tag{1.16}$$

where $G(q)$ and $H(q)$ are the Rogers-Ramanujan functions defined earlier. Using these two functions, we can generate a subset of modular forms of weight k of $\Gamma_1(5)$ that are infinite products of the form

$$f_{(a_0,a_1,a_2)}(\tau) = q^{\ell(a_0,a_1,a_2)}(q^5;q^5)_\infty^{a_0}(q;q^5)_\infty^{a_1}(q^4;q^5)_\infty^{a_1}(q^2;q^5)_\infty^{a_2}(q^3;q^5)_\infty^{a_2}, \quad (a_0,a_1,a_2) \in 2\mathbb{Z}\times\mathbb{Z}^2, \tag{1.17}$$

where

$$\ell(a_0,a_1,a_2) = \frac{5}{24}(a_0+2a_1+2a_2) - \frac{2}{5}a_1 - \frac{3}{5}a_2, \quad q = e^{2\pi i\tau}. \tag{1.18}$$

The definition of modular forms and other details will be explained in detail in later chapters. In our quest to understand and prove these antichimeral congruences, we had other goals that we wanted to complete along the way. In particular, we sought to

1. Find all modular forms $f_{(a_0,a_1,a_2)}(\tau)$ of the subgroup $\Gamma_1(5)$ of $SL_2(Z)$ for every even $a_0 \geq 2$.

2. Derive an algorithm that will yield all the triples (a_0,a_1,a_2) that will result in Ramanujan type congruences.

3. Find a way to determine when a triple will satisfy infinitely many Ramanujan type congruences modulo 5^j for all $j \in \mathbb{N}^+$.

4. Derive an algorithm to find lattice points lying within the bounded $\frac{p-1}{2}$-dimensional polytope that defines the answer to item 1. We were also interested in finding a method to generate

these, as well as a way to count them for higher values of a_0. Answers to these questions would allow us to determine computing time constraints.

5. Find a way to characterize the difference between a product satisfying antichimeral congruences and ones satisfying other classes of congruences. We also needed a way to prove that product satisfying antichimeral congruences does in fact satisfy infinitely many Ramanujan type congruences.

1.5 Motivation

Ramanujan type congruences are a major study in the field of analytic number theory. Associated functions show up in statistical mechanics and topology [3].

1.6 Limitations and Future Work

In order to find all the triples that we are looking for, the methods we use are largely computational. As a_0 increases, the associated polytope increases in size and the calculations get more and more labor intensive. In the future, I would like to determine the form of all antichimeral triples.

1.7 Final Remarks

In the next chapters, we'll be going into several important topics for complete understanding of the material. Chapter 2 will consist of a comprehensive review on Ramanujan type congruences and modular forms, complete with crucial definitions, theorems and conventions that will be needed to fully understand the rest of the thesis. Chapter 3 will describe in detail the relevant polytope and lattice points arising in the work. Chapter 4 will use linear algebra to prove Ramanujan type congruences for certain products. Chapter 5 will showcase our results that prove the existence of antichimeral triples.

CHAPTER II

A COMPREHENSIVE REVIEW OF MODULAR FORMS

We will now provide a comprehensive review of modular forms in order to fully understand the body of our research. This will include some basic terminology, definitions, and theorems that will be needed to fully understand our research.

2.1 Introduction

Definition 1. *Let G be a group with neutral element e, and let X be a set. G is said to* ***act*** *on X if there exists a map* $\varphi : G \times X \to X$ *such that:*

1. $\varphi(e,x) = x \qquad \forall x \in X$

2. $\varphi(g_1 g_2, x) = \varphi(g_1, \varphi(g_2.x)) \qquad \forall g_1, g_2 \in G$ *and* $\forall x \in X$

Alternatively, we say that φ *is an* ***action*** *of G on X.*

Definition 2. *An* ***analytic*** *function is a function f defined on an open set X that has a power series expansion centered at* x_0 *for every* $x_0 \in X$*, with a positive radius of convergence. That is to say, for every x in a neighborhood of* x_0

$$f(x) = \sum_{i=0}^{\infty} a_i (x - x_0)^n$$

for all $x_0 \in X$*. However, if f is a complex function, this definition is sometimes equivalently stated another way, shown in the next definition.*

Definition 3. *A* ***holomorphic*** *function is a complex-valued function that is complex differentiable in neighborhoods of all points in its domain. That is to say, if we let* $f : A \subseteq \mathbb{C} \to \mathbb{C}$*, where A is open in* $\mathbb{C}$*, the difference quotient*

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

*exists in $\mathbb{C}$ for every $z \in A$. Also, a **meromorphic** function is one that is holomorphic except at a finite number of points in its domain. These points are called **poles**.*

Note: The terms holomorphic and analytic may be equivalent for complex functions, but the definitions are not the same. In complex analysis, however, we usually use the two terms interchangably, because they both describe the same type of functions. In fact, we are usually talking about a complex differentiable function when we say "analytic," since every complex differentiable function has a power series expansion.

Remark 1. *There are a couple of useful properties of holomorphic functions that we can use. Here are a few of them. Assume that f and g are holomorphic on a set $X \subseteq \mathbb{C}$. Then,*

1. *$f+g$ is holomorphic in X and $(f+g)' = f' + g'$.*

2. *fg is holomorphic in X and $(fg)' = f'g + fg'$.*

3. *For any $\tau \in X$, if $g(\tau) \neq 0$, then f/g is holomorphic at τ, and*

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Definition 4. *The upper half plane, denoted as $\mathbb{H}$, is the set of all complex numbers z where Im(z)>0. Put explicitly,*

$$\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}.$$

Definition 5. *The **special linear** (or **modular**) group $SL_2(\mathbb{Z})$ consists of all $2x2$ matrices that have integer entries with a determinant of 1. Put explicitly,*

$$SL_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \quad a,b,c,d \in \mathbb{Z}, \quad det(A) = 1 \right\}.$$

Definition 6. *For some $\gamma \in SL_2(\mathbb{Z})$, we define the Mobius transformation as follows: for some*

$\tau \in \mathbb{C}$,

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}.$$

Through this transformation, we see the following lemma:

Lemma 1. *$SL_2(\mathbb{Z})$ acts on $\mathbb{H}$, since if $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,*

$$\gamma(\tau) = \frac{1\tau + 0}{0\tau + 1} = \frac{\tau}{1} = \tau$$

and, for any $A, B \in SL_2(\mathbb{Z})$, we have that

$$\gamma(AB, \tau) = \frac{(a_{1,1}b_{1,1} + a_{1,2}b_{2,1})\tau + (a_{1,1}b_{1,2} + a_{1,2}b_{2,2})}{(a_{2,1}b_{1,1} + a_{2,2}b_{2,1})\tau + (a_{2,1}b_{2,1} + a_{2,2}b_{2,2})}$$

and

$$\gamma(A, \gamma(B, \tau)) = \gamma(A, \frac{b_{1,1}\tau + b_{1,2}}{b_{2,1}\tau + b_{2,2}}) = \frac{a_{1,1}(\frac{b_{1,1}\tau + b_{1,2}}{b_{2,1}\tau + b_{2,2}}) + a_{1,2}}{a_{2,1}(\frac{b_{1,1}\tau + b_{1,2}}{b_{2,1}\tau + b_{2,2}}) + a_{2,2}} = \frac{a_{1,1}(b_{1,1}\tau + b_{1,2}) + a_{1,2}(b_{2,1}\tau + b_{2,2})}{a_{2,1}(b_{1,1}\tau + b_{1,2}) + a_{2,2}(b_{2,1}\tau + b_{2,2})}.$$

Thus,

$$\gamma(A, \gamma(B, \tau)) = \frac{a_{1,1}b_{1,1}\tau + a_{1,2}b_{2,1}\tau + a_{1,1}b_{1,2} + a_{1,2}b_{2,2}}{a_{2,1}b_{1,1}\tau + a_{2,2}b_{2,1}\tau + a_{2,1}b_{2,1} + a_{2,2}b_{2,2}} \tag{2.1}$$

$$= \frac{(a_{1,1}b_{1,1} + a_{1,2}b_{2,1})\tau + (a_{1,1}b_{1,2} + a_{1,2}b_{2,2})}{(a_{2,1}b_{1,1} + a_{2,2}b_{2,1})\tau + (a_{2,1}b_{2,1} + a_{2,2}b_{2,2})} = \gamma(AB, \tau). \tag{2.2}$$

Definition 7. *A* ***congruence subgroup*** *of $SL_2(\mathbb{Z})$ is a subgroup of $SL_2(\mathbb{Z})$ that contains the subgroup $\Gamma(N)$ (defined below). There are multiple congruence subgroups of $SL_2(\mathbb{Z})$, including $\Gamma_0(N)$ and $\Gamma_1(N)$, where $N \in \mathbb{N}^+$. The congruence subgroups are defined as follows:*

$$\Gamma(N) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; det(A) = 1, a \equiv d \equiv \pm 1, b \equiv c \equiv 0 \pmod N \right\},$$

and

$$\Gamma_0(N) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; det(A) = 1, c \equiv 0 (mod\, N) \right\},$$

$$\Gamma_1(N) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; det(A) = 1, a \equiv d \equiv \pm 1, c \equiv 0 \pmod{N} \right\}.$$

Specifically, we were interested in the congruence subgroups $\Gamma_0(5)$ and $\Gamma_1(5)$, which is the specific case when $N = 5$. In doing so, we can see that we can actually decompose $\Gamma_1(5)$ into cosets of $\Gamma(5)$, an important part in showing that the products we consider are modular forms. This is proven in Chapter 3.

Definition 8. *Every congruence subgroup of $SL_2(\mathbb{Z})$has a corresponding **modular curve**, which is defined as the quotient space $\Gamma \backslash \mathbb{H}$*

$$Y(\Gamma) = \Gamma \backslash \mathbb{H} = \{\Gamma\tau : \tau \in \mathbb{H}\}.$$

Theorem 1. *(Kilford, [6])*

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} (1 - \frac{1}{p^2}),$$

and

$$[SL_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p|N} (1 - \frac{1}{p^2}).$$

Definition 9. *A **modular form** on the congruence subgroup Γ of weight k is a holomorphic complex function of the upper half plane $\mathbb{H}$ that satisfies the following two conditions:*

1. *For any element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ, $f(\gamma(\tau))) = f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$.*

2. *For any element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$, $(c\tau + d)^{-k} f(\gamma(\tau))$ is bounded as the imaginary part of*

τ goes to infinity. In other words

$$\lim_{\tau \to i\infty} (c\tau + d)^{-k} f(\gamma(\tau)) < \infty.$$

*Should the function only satisfy the modular transformation law, we say that this function is **weakly modular**.*

Note: The second condition of this definition can also be written as requiring the function f to be holomorphic at ∞, in addition to the upper half plane $\mathbb{H}$, that came from the definition of modular forms. This will be used when analyzing the cusps of the compact surface $X_1(p)$, which will expanded upon in the next chapter, and this will be the convention that is used from now on, unless we expressly need the formal definition.

Remark 2. *The modular transformation law that defines modular forms can be generalized by considering the nebentypus of a modular form. The generalized modular transformation law dictates that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$,*

$$f(\gamma(\tau))) = f(\frac{a\tau + b}{c\tau + d}) = \varepsilon(a,b,c,d)(c\tau + d)^k f(\tau).$$

*The function $\varepsilon(a,b,c,d)$ is called the **nebentypus** of the modular form.*

The Dedekind eta function (defined later in this chapter) is in fact a modular form on $SL_2(\mathbb{Z})$ of weight $\frac{1}{2}$, but it has a non-unitary nebentypus (i.e. the nebentypus is not 1).

Definition 10. *The weight-k **slash operator** is denoted as follows, for any $\gamma \in SL_2(\mathbb{Z})$:*

$$f|[\gamma]_k(\tau) = (c\tau + d)^{-k} f(\gamma(\tau)).$$

For any weakly modular function f and its associated congruence subgroup Γ, we sometimes say that f is **weight-**k *Gamma*-**invariant** under the slash operator. For certain conditions on a_0, a_1, a_2 that we will define in Chapter 3, the products $f_{a_0,a_1,a_2}(\tau)$

are modular forms of weight $\frac{a_0}{2}$ for $\Gamma_1(5)$.

We say that these modular forms are of level 5.

Definition 11. *The set of all modular forms for* $\Gamma \leq SL_2(\mathbb{Z})$ *of weight k is denoted as* $M_k(SL_2(\Gamma)$.

This is a finite-dimensional vector space over $\mathbb{C}$ *[1], which is denoted as* $M_k(\Gamma)$.

We may construct the graded ring of modular forms for Γ *as*

$$M(\Gamma) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma).$$

Definition 12. *For two congruence subgroups* Γ_1 *and* Γ_2 *of* $SL_2(\mathbb{Z})$*, for some* $\alpha \in GL_2^+(\mathbb{Q})$ *(where* $GL_2^+(\mathbb{Q})$ *denotes set of all invertible* 2×2 *matrices with rational entries and positive determinant) the **double coset*** $\Gamma_1 \alpha \Gamma_2$ *is the set:*

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 | \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$$

The linear map induced by the double coset operator $\Gamma_1 \alpha \Gamma_2$ *from* $M_k(\Gamma_1)$ *to* $M_k(\Gamma_2)$ *is called a **Hecke operator**. The precise definition will be introduced in chapter 4.*

We consider the operator U_5, which acts on the q-expansion of a modular form. We will show later in chapter 4 that this operator is linear and a reduction of the Hecke operator T_5 for the congruence subgroup $\Gamma_1(5)$. It maps the vector space $M_k(\Gamma_1(5))$ to itself (which will also be shown in Chapter 4). We will show later in this chapter that a modular form f has a Fourier expansion and can be written as the q-series $\sum a_n q^n$, where $q = e^{2\pi i \tau}$. The operator U_5 transforms the modular form as follows:

$$U_5(f) = U_5(\sum a_n q^n) = \sum a_{5n} q^n.$$

This effectively strips out all the fifth coefficients of the q-series, and as one might guess, is very helpful in finding congruences for the infinite products we are studying. The fact that Hecke operators map a vector space of modular forms of fixed weight to itself allows us to reduce proofs of many results to linear algebra.

Remark 3. *The vector space $M_k(\Gamma_1(5))$ can be decomposed into the direct sum of the eigenspaces of the eigenvalues λ, denoted as $E_{k,\lambda}$, for the simplified Hecke operator relative to* $\Gamma_1(5)$. *This will also be useful in proving congruences of our infinite products.*

The modular forms of level 5 that we consider can be written as products of two functions: the Dedekind eta function and Klein forms [8]. When $a_1 = a_2$ in (1.17), these modular forms are called **eta products**

Definition 13. *The* ***Dedekind eta function*** *is as follows: For some $\tau \in \mathbb{H}$,*

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}(q;q)_\infty \qquad q = e^{2\pi i\tau}.$$

As one might observe, the Dedekind eta function is closely related to the partition generating function $\sum_{n=0}^{\infty} p(n)q^n$ (outlined in (1.1)), because if one were to take the reciprocal of this, we'd have:

$$\frac{1}{\eta(\tau)} = \frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} = q^{-\frac{1}{24}} \sum_{n=0}^{\infty} p(n)q^n.$$

Definition 14. *For some $z = (\mathbb{Q}_1, \mathbb{Q}_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and $q_z = e^{2\pi i(Q_1\tau + Q_2)}$, a* ***Klein form*** *is defined by:*

$$K_{(Q_1,Q_2)} = e^{\pi i Q_2(Q_1-1)} q^{\frac{1}{2}Q_1(Q_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_zq^n)(1-q_z^{-1}q^n)(1-q^n)^{-2}.$$

Definition 15. *The product $\prod_{i=1}^{\infty}(1+a_i)$ is said to converge absolutely if and only if $\prod_{i=1}^{\infty}(1+|a_i|)$ converges. Also, if $\prod_{i=1}^{\infty}(1+a_i)$ converges absolutely, then it also converges.*

Remark 4. *([7]) A well known property of infinite products that we will need is this one: The product*

$$\prod_{i=1}^{\infty}(1+a_i)$$

converges absolutely if and only if $\sum_{i=1}^{\infty} a_i$ converges absolutely. This will help us show that the products are indeed holomorphic on $\mathbb{H}$.

Definition 16. *(The Weierstrass M test). ([7]) For continuous functions $f_n : K \to \mathbb{C}$ where K is compact, if $M_n = sup_{z \in K}|f_n(z)| < \infty$ for all n and*

$$\sum_{n=1}^{\infty} M_n < \infty,$$

then

$$\prod_{n=1}^{\infty}(1 + f_n(z))$$

converges uniformly on K.

For the products that we are studying, we also have to define the three types of Ramanujan type congruences.

Definition 17. *Let*

$$\begin{aligned} f_{(a_0,a_1,a_2)}(\tau) &= q^{\ell(a_0,a_1,a_2)}(q^5;q^5)_\infty^{a_0}(q;q^5)_\infty^{a_1}(q^4;q^5)_\infty^{a_1}(q^2;q^5)_\infty^{a_2}(q^3;q^5)_\infty^{a_2}, \quad (a_0,a_1,a_2) \in 2\mathbb{Z} \times \mathbb{Z}^2 \\ &= q^{\ell(a_0,a_1,a_2)} \textstyle\sum_{n=0}^{\infty} \mathscr{P}_{(a_0,a_1,a_2)}(n) q^n, \end{aligned}$$

where

$$\ell(a_0,a_1,a_2) = \frac{5}{24}(a_0 + 2a_1 + 2a_2) - \frac{2}{5}a_1 - \frac{3}{5}a_2, \quad q = e^{2\pi i \tau}, \tag{2.3}$$

for some fixed a_0.

- *The ordered pair (a_1,a_2) yields exponents that satisfy an* ***infinite Ramanujan Type Congruence*** *if for every $k \in \mathbb{Z}^+$, we have that for all $n \geq 1$,*

$$\mathscr{P}_{(a_0,a_1,a_2)}(5^k n - \ell(a_0,a_1,a_2)) \equiv 0 \pmod{5^k}$$

- *The ordered pair (a_1,a_2) yields exponents that satisfy an* ***chimeral Ramanujan Type Congruence*** *if there exists some $m \in \mathbb{Z}^+$ such that for every positive integer $k < m$, we have that for all $n \geq 1$,*

$$\mathscr{P}_{(a_0,a_1,a_2)}(5^k n - \ell(a_0,a_1,a_2)) \equiv 0 \pmod{5^k},$$

but for any integer $j \geq m$, we have that for some $n \geq 1$

$$\mathscr{P}_{(a_0,a_1,a_2)}(5^j n - \ell(a_0,a_1,a_2)) \not\equiv 0 \pmod{5^j}.$$

- *The ordered pair (a_1,a_2) yields exponents that satisfy an* ***antichimeral Ramanujan Type Congruence*** *if there exists some $m \in \mathbb{Z}^+$ such that for every positive integer $k < m$, we have that for some $n \geq 1$,*

$$\mathscr{P}_{(a_0,a_1,a_2)}(5^k n - \ell(a_0,a_1,a_2)) \not\equiv 0 \pmod{5^k},$$

but for any integer $j \geq m$, we have that for all $n \geq 1$

$$\mathscr{P}_{(a_0,a_1,a_2)}(5^j n - \ell(a_0,a_1,a_2)) \equiv 0 \pmod{5^j}.$$

In our research, we focused almost solely on the products that yield antichimeral Ramanujan type congruences, and it is worth noting that this specific type of congruence seems to only occur when $a_0 \equiv 6 \pmod{10}$, where $a_0 > 6$.

2.2 Basic Theorems and Necessary Results

We will need a few other things before we get started.

Lemma 2. *A nonempty set $W \subseteq \mathbb{Z}$ has a smallest element if W is bounded below on the ordered set $(\mathbb{Z}, \leq)$, where $\leq$ denotes the ordering on the integers.*

Proof. Since W is bounded below in $\mathbb{Z}$. Thus, $\exists n \in \mathbb{Z}$ such that $n \leq w$, for all $w \in W$. This implies that $w - n \geq 0$, and hence,

$$X = \{w - n | w \in W\} \subseteq \mathbb{N}.$$

Hence, by the well-ordering principle, X has a smallest element, which we denote as x_{min}. Thus, for all $w \in W$, we have that $x_{min} \leq w - n$ and there exists some $w^* \in W$ such that $x_{min} = w^* - n$. Thus,

$$w^* - n \leq w - n \implies w^* \leq w,$$

and hence, w^* is the smallest element of W, by definition of a smallest element. $\square$

Lemma 3. *(Bezout's identity) Given two integers a and b, with either $a \neq 0$ or $b \neq 0$ with greatest common divisor d, there exists integers x and y such that $ax + by = d$.*

Proof. Consider some a and b in $\mathbb{Z}$, with gcd(a,b)=d. We'll consider the set $S = \{s > 0 | ax + by = s, x, y \in \mathbb{Z}\}$. We know that $|a|$ and $|b|$ are both in S, so this set is not empty (to see this, set either x or y to ± 1, and set the other to 0). By construction, all elements of S are greater than 0, so by the previous lemma, S will have a smallest element, denoted as s_{min}. We will let $s_{min} = ax_0 + by_0$, where $x_0, y_0 \in \mathbb{Z}$. By the division algorithm, there exists integers q and r such that $\alpha = s_{min}q + r$, where $0 \leq r < s_{min}$. This implies that $r \notin S$. But,

$$r = \alpha - s_{min}q = \alpha - (ax_0 + by_0)q = a(1 - qx_0) + b(-qy_0).$$

If $r > 0$, then this is a contradiction, because we know that $1 - qx_0$ and $-qy_0$ are both integers, and $r \notin S$. So, $r = 0$, which means that $\alpha = s_{min}q$, which implies that $q|\alpha$ and that $s_{min}|\alpha$ for all $\alpha \in S$. Since $|a|$ and $|b|$ are both in S, we have that s_{min} divides both of these. By definition, gcd(a,b) also divides both $|a|$ and $|b|$, so $1 \leq s_{min} \leq$ gcd(a,b). We also can say that gcd(a,b)|$s_m in$, since $s_{min} = ax_0 + by_0$ and gcd(a,b) divides a and b. Thus, $s_{min}|$gcd(a,b) and gcd$(a,b)|s_{min}$, and hence, gcd(a,b)= s_{min}. $\square$

Theorem 2. *The special linear group $SL_2(\mathbb{Z})$ is generated by the matrices*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Let $G = \langle S, T \rangle$ be the subgroup of $SL_2(\mathbb{Z})$ that is generated by S and T. We will show that this is in fact equal to $SL_2(\mathbb{Z})$. First off, by definition, $G \subset SL_2(\mathbb{Z})$. Next, we note that for some

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$S\gamma = S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \textit{and} \quad T^n\gamma = T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

We see this because $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Now, let $\gamma \in SL_2(\mathbb{Z})$. Suppose that $c \neq 0$. If $|a| > |c|$, then we can show through the division algorithm that $a = cq + r$, for some $q, r \in \mathbb{Z}$, where $0 \leq r < |c|$. By the calculations shown above, we have that $T^{-q}\gamma = \begin{pmatrix} a-qc & b-qd \\ c & d \end{pmatrix}$. We see that this upper left entry is exactly equal to r, which is smaller in absolute value than the lower left entry, c. Multiplying this matrix by S switches the rows and changes some of the signs of the entries:

$$S(T^{-q}\gamma) = \begin{pmatrix} -c & -d \\ a-qc & b-qd \end{pmatrix}.$$

This process can be continued by using the division algorithm on $-c$ and $a - qc$ (if $a - qc \neq 0$) and multiplying by a certain power of T, and then S which will continually bring the bottom left entry's absolute value closer and closer to 0. Eventually, we will end up with a matrix of the form $\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}$ with $m \in \mathbb{Z}$, where the diagonal signs will match, and thus, the determinant of this matrix is still 1 and it still belongs to $SL_2(\mathbb{Z})$. This matrix is either T^m or $-T^{-m}$, so this shows that there is some $g \in G$ such that $g\gamma = \pm T^n$for some $n \in \mathbb{Z}$. Since $T^n \in G$, and it can be easily seen that $S^2 = -I_2$, we have that $\gamma = \pm g^{-1}T^n$, which is itself a member of G. This shows that for any arbitrary $\gamma \in SL_2(\mathbb{Z})$, γ is also in G, which means that $SL_2(\mathbb{Z}) \subset G$, which means that $SL_2(\mathbb{Z}) = G$. □

Note: Because of this theorem, the group of transformations defined by $SL_2(\mathbb{Z})$ is generated by $\tau \mapsto \tau+1$ and $\tau \mapsto \frac{-1}{\tau}$. In other words, the modular transformation law (see Definition 9) is equivalent to this (for an arbitrary modular form f of $SL_2(\mathbb{Z})$): $f(\tau) = f(\tau+1) = (\tau)^{-k} f(\frac{-1}{\tau})$ for some integer k.

Theorem 3. *$\Gamma(p)$ is a normal subgroup of $\Gamma_1(p)$ of index p.*

Proof. We first need to show that $\Gamma(p)$ is a subgroup of $\Gamma_1(p)$, which we can do by first noting that $\Gamma(p) \subset \Gamma_1(p)$, since if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(p)$, then $\gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p}$. We also know that $\Gamma(p)$ is not empty because I_2 is definitely a member. Also, for any $\gamma_1, \gamma_2 \in \Gamma(p)$,

$$\gamma_1\gamma_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p} \implies \gamma_1\gamma_2 \in \Gamma(p).$$

Also, for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(p)$, we have that

$$\gamma^{-1} = \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det(\gamma)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p} \implies \gamma^{-1} \in \Gamma_0 p,$$

since $\gamma \in SL_2(\mathbb{Z})$. Thus, $\Gamma(p)$ is a subgroup of $\Gamma_1(p)$. Moreover, if we define a map

$$\Psi : \Gamma_1(p) \to \mathbb{Z}/N\mathbb{Z}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod{p},$$

we see that for any A and B in $\Gamma_1(p)$,

$$\Psi(AB) \equiv \Psi\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) \equiv \Psi \begin{pmatrix} 1 & s+t \\ 0 & 1 \end{pmatrix} \equiv s+t \pmod{p}.$$

for some integers $0 \leq s,t < p$. Additionally,

$$s+t \pmod p = s \pmod p + t \pmod p = \Psi(A) + \Psi(B),$$

so we can clearly see that Ψ is a homomorphism. Also, for any $b \in \mathbb{Z}$, $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_1(p)$, which implies that Ψ is surjective, by definition. Due to the definition of Ψ, we also see that

$$ker(\Psi) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \gamma \in \Gamma_1(p), b \equiv 0 \pmod p \right\} = \Gamma(p).$$

In particular, this shows that $[\Gamma_1(p) : \Gamma(p)] = p$. □

Theorem 4. *The product of two modular forms of weights k_1 and k_2 from the same congruence subgroup of $SL_2(\mathbb{Z})$ is a modular form of weight $k_1 + k_2$.*

Proof. Let the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and let f and g be modular forms of some congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$, with weights k_1 and k_2, respectively. Then

$$f(\gamma(\tau)) = f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{k_1} f(\tau),$$

and

$$g(\gamma(\tau)) = g(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{k_2} g(\tau),$$

which results in this:

$$fg(\gamma(\tau)) = f(\frac{a\tau + b}{c\tau + d}) g(\frac{a\tau + b}{c\tau + d}) = [(c\tau + d)^{k_1} f(\tau)][(c\tau + d)^{k_2} g(\tau)] = (c\tau + d)^{k_1 + k_2} fg(\tau).$$

Additionally, we do need to note that f and g were holomorphic on $\mathbb{H} \cup \{\infty\}$, so since the product of two holomorphic functions is also holomorphic, according to Remark 1, fg is also holomorphic on $\mathbb{H} \cup \{\infty\}$. Thus, fg is a modular form of weight $k_1 + k_2$.

□

For the remainder of the thesis, the notation p is understood to be a positive prime integer, where $p \geq 5$.

Theorem 5. *Every modular form for $SL_2(\mathbb{Z})$ has a Fourier expansion of the form*

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad \textit{where} \quad q = e^{2\pi i \tau}.$$

Proof. From the modular transformation law (in Definition 9), we end up seeing that if we pick $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which belongs to $SL_2(\mathbb{Z})$,

$$f(\gamma(\tau)) = f\left(\frac{1\tau+1}{0\tau+1}\right) = f(\tau+1).$$

This means that all weakly modular functions in $SL_2(\mathbb{Z})$ are 1-periodic. This means that we can define a holomorphic map:

$$\phi : \mathbb{H} \to D' = 0 < |q| < 1, \quad \tau \mapsto e^{2\pi i \tau},$$

which sends points from $\mathbb{H}$ to the punctured unit disk. Since f is holomorphic and 1-periodic, we see that the map ϕ sends $f(\tau)$ to $g(q)$, where g is some holomorphic function on the punctured unit disk D'. Thus, $f(\tau)$ has a Fourier expansion around $q = 0$. □

Theorem 6. *(The Cauchy-Goursat Theorem): If a function f is analytic everywhere on the interior and boundary of a simple closed contour C, then*

$$\oint_C f(z)dz = 0.$$

Proof. If we let f be analytic in and on a simple closed contour C (i.e. f is complex differentiable on C), this result follows directly from Green's theorem. Let R be the region enclosed by C

$$\begin{aligned}\int_C f(z)\,dz = & \quad \int_C (u(x,y)+iv(x,y))(dx+idy) \\ = & \quad \int_C (udx - vdy) + i\int_C (vdx+udy) \\ & = \int\int_R (-vx - uy)dA + i\int\int_R (ux - vy)dA = 0.\end{aligned}$$

This follows because of the Cauchy-Riemann equations, since f is analytic. □

Theorem 7. *(Morera's theorem) If a function f is continuous on a region D and*

$$\oint_C f(z) = 0$$

for every closed contour C in D, then f is holomorphic (or analytic) in D.

Proof. Let D be a region where the function f is continuous. Also, let the contour integrals of $f(z)$ all be zero. We can then pick any arbitrary $z_0 \in D$ and create the complex function F, where $F(z) = \int_{z_0}^{z} f(z)dz$. It then follows that $F'(z) = f(z)$, which implies that F is holomorphic (or analytic) and has derivatives of all orders, which means that $f(z)$ is also holomorphic. Since z_0 is arbitrary, f is holomorphic (or analytic) in D. □

Theorem 8. *Suppose $f_n : V \to \mathbb{C}$ is a sequence of holomorphic functions and f_n converges to f uniformly on compact subsets of V (or f_n converges compactly to f on V). Then f is holomorphic on V.*

Proof. Let f_n be a sequence of holomorphic functions that converges compactly to f on the set V. Also, let D be any disc whose closure is contained in V. Then for any closed contour C contained in D, by the previous theorem, we have $\int_C f_n(z)dz = 0$. As $z \to \infty$, we have $\int_C f_n(z)dz \to \int_C f(z)dz$. since f_n converges compactly on V. Thus, $\int_C f(z)dz = 0$ for any closed contour C, and by Morera's theorem, f is holomorphic. □

We will now use this theorem to show that all of the products in (1.17) are all holomorphic on $\mathbb{H}$ and thus satisfy one of basic requirements of being a modular form.

Proposition 1. *All of the products of the form*

$$f_{(a_0,a_1,a_2)}(\tau) = q^{\ell(a_0,a_1,a_2)}(q^p;q^p)_\infty^{a_0}(q;q^5)_\infty^{a_1}(q^4;q^5)_\infty^{a_1}(q^2;q^5)_\infty^{a_2}(q^3;q^5)_\infty^{a_2} \qquad (a_0,a_1,a_2) \in 2\mathbb{Z}\times\mathbb{Z}^2,$$

where

$$\ell(a_0,a_1,a_2) = \frac{5}{24}(a_0+2a_1+2a_2) - \frac{2}{5}a_1 - \frac{3}{5}a_2$$

are holomorphic on $\mathbb{H}$.

Proof. To do this, we will use the Weierstrass M-test. We will prove that the Dedekind eta function is holomorphic on $\mathbb{H}$. This will ensure that the eta quotient component of $f_{(a_0,a_1,\ldots,a_{(p-1)/2})}$ is holomorphic on $\mathbb{H}$. We will only have to prove the holomorphicity of the Klein forms on $\mathbb{H}$, which is similar to the proof we present for the Dedekind eta function, and since the product of holomorphic functions is also holomorphic, we are done.

We first should note that the Dedekind eta function has no zeros on $\mathbb{H}$. We can prove this by noting that $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$, where $q = e^{2\pi i\tau}$ for any $\tau \in \mathbb{H}$. We can immediately see that $q^{\frac{1}{24}} \neq 0$, for any value of $\tau \in \mathbb{H}$. We simply need to verify that $q^n \neq 1$ for any value of $\tau \in \mathbb{H}$. This is apparent because

$$q^n = e^{2\pi i\tau n} = e^{2\pi i(x+yi)n} = e^{2\pi xin}e^{-2\pi yn}.$$

According to Euler's formula, $e^{2\pi xin} = cos(2\pi xn) + isin(2\pi xn)$, and this expression will always have modulus 1. Since $\tau \in \mathbb{H}$, $y > 0$, so $e^{-2\pi yn} < 1$, as $-2\pi yn < 0$. Consequently, $|e^{-2\pi yn}| < 1$. Thus, it follows that $|e^{2\pi xin}e^{-2\pi yn}| < 1$, which means that $e^{2\pi xin}e^{-2\pi yn} \neq 1$, since $|1| = 1$.

Thus, $\eta(\tau) \neq 0$. We now will conduct the Weierstrass M-test to show that $\eta(\tau)$ is holomorphic for all $\tau \in \mathbb{H}$.

Let $f_n(\tau) = -e^{(2\pi i\tau)n}$. We then will show that this converges by showing that $|f_n(\tau)|$ converges. By doing so, we will satisfy the form outlined in Remark 4.

So, assume that K is a compact subset of $\mathbb{H}$. We see that for any $\tau \in K$,

$$M_n = \sup_{\tau \in K} |-e^{2\pi i \tau n}| = \sup_{(x+yi)\in K} |e^{2\pi i(x+yi)n}| = \sup_{(x+yi)\in K} |e^{2\pi xni}||e^{-2\pi yn}| = \sup_{(x+yi)\in K} (\frac{1}{e^{2\pi y}})^n.$$

We know that this is the case because by Euler's formula, $e^{2\pi xin} = cos(2\pi xn) + isin(2\pi xn)$, whose modulus is 1. Because of this, M_n only depends on the value of y. Also, for any $\tau \in K \subset \mathbb{H}$, we see that $M_n \leq 1$ because $y > 0$, and thus $e^y > 1$. We can actually show that $\sup_{(x+yi)\in K}(\frac{1}{e^{2\pi y}})^n = \alpha^n$ is strictly less than 1. To see this, assume contrarily that $\alpha = \sup_{iy\in K} \frac{1}{e^{2\pi y}} = 1$. By definition of a supremum, there exists an element $\beta_n \in K$, such that $1 - \frac{1}{n} \leq \beta_n < 1$, for some positive integer n, since K is compact. So, let β_{max} be the maximum element of K. If we choose the integers n and m such that $1 - \frac{1}{n} < \beta_{max} < 1 - \frac{1}{m}$, we then see that we have a contradiction, since every element of K is less than or equal to β_{max}, and $\beta_{max} < 1 - \frac{1}{m} < 1$. This implies that $1 - \frac{1}{m}$ is a smaller upper bound for K than 1, which would mean that 1 is no longer the supremum of K. Thus, $\sup_{iy\in K} e^{-2\pi y} \neq 1$, which implies that our original $\alpha < 1$. This means that $\sum_{n=1}^{\infty} M_n < \infty$, and as such, $|f_n(\tau)|$ converges, and thus, the product $\prod_{n=1}^{\infty} 1 + f_n(\tau)$ converges uniformly over every compact subset K of $\mathbb{H}$. This implies that the Dedekind eta function $\eta(\tau)$ is holomorphic over $\mathbb{H}$. □

Repeating the same argument for a general Klein form ensures that both components of the product $f_{(a_0,a_1,\ldots,a_{(p-1)/2})}$ are holomorphic on $\mathbb{H}$.

CHAPTER III

POLYTOPES AND LATTICES

In this chapter, we will outline some more advanced concepts of modular forms which will be used to construct polytopes and lattices that will allow us to find exponents of the products in (1.17) that will result in Ramanujan type congruences.

3.1 Introduction

In our research, we studied modular forms of level 5–that is, modular forms that belong to the congruence subgroup $\Gamma_1(5)$ of $SL_2(\mathbb{Z})$. These will be built from products of the form:

$$\sum_{n=0}^{\infty} P_{(a_0,a_1,a_2)}(n)q^n = (q^5;q^5)_\infty^{a_0}(q,q^4;q^5)_\infty^{a_1}(q^2,q^3;q^5)_\infty^{a_2}\text{‘} \qquad (a_0,a_1,a_2) \in 2\mathbb{Z}\times\mathbb{Z}^2. \tag{3.1}$$

This product form is a specific case of the more general form:

$$\sum_{n=0}^{\infty} P_{(a_0,a_1,\dots a_{(p-1)/2})}(n)q^n = (q^p;q^p)^{a_0}\prod_{i=1}^{(p-1)/2}(q,q^i;q^{p-i})^{a_i} \quad \text{where} \quad a_i \in \mathbb{Z}, \quad \text{for } p \geq 5. \tag{3.2}$$

In general, these products are not modular forms because they are not necessarily holomorphic at ∞, and may not be weakly modular. The holomorphicity at ∞ is referred to as the cuspidal condition because of the last listed requirement in the definition of modular forms.

3.2 Cusps

Recall in the definition of a modular form that a modular form candidate f had to be holomorphic at ∞ for it to be a modular form. By using the Mobius transformation $\gamma(\tau) = \frac{a\tau+b}{c\tau+d}$ for some $\gamma \in SL_2(\mathbb{Z})$, it will be shown that appropriate information about a modular form about the single point ∞ can be used to generate information about all the cusps $\mathbb{Q} \cup \infty$ of $SL_2(\mathbb{Z})$ at which the holomorphicity of a weakly modular function may break down. We'll first state exactly what cusps are.

Recall from Definition 8 that a modular curve $Y(\Gamma)$ exists for every congruence subgroup of $SL_2(\mathbb{Z})$. Since $Y(\Gamma)$ is the quotient of $\mathbb{H}$ under the action of Γ, it will inherit the standard topology of $\mathbb{H}$. This space is not compact, by itself. We can, however, compactify this set by adding in the points $\{\infty\}$ and the rationals $\mathbb{Q}$. These points are called **cusps**. We will denote the compact surface formed by modular curve $Y(\Gamma)$ as well as the quotient space formed by the action of Γ on $\mathbb{Q} \cup \infty$ as $X(\Gamma)$. For convenience's sake, we'll denote the compact surface associated with $\Gamma_1(N)$ as $X_1(N)$. Two cusps τ_1 and τ_2 are said to be **equivalent** under the action of the subgroup Γ (or simply **Γ-equivalent**) if we can find a $\gamma \in \Gamma$ such that $\gamma(\tau_1) = \frac{a\tau_1+b}{c\tau_1+d} = \tau_2$. We will now prove what we said earlier.

Lemma 4. *The group $SL_2(\mathbb{Z})$ only has $\mathbb{Q} \cup \{\infty\}$ as its cusps. All of these cusps are $SL_2(\mathbb{Z})$-equivalent to ∞.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, with $c \neq 0$. We then have that

$$\gamma(\infty) = \lim_{\tau \to \infty} \frac{a\tau + b}{c\tau + d} = \frac{a}{c}.$$

If $c = 0$, then one can see that $\gamma(\infty) = \infty$, and because we are considering $SL_2(\mathbb{Z})$, we can further see that each rational cusp is equivalent to ∞. This is apparent because we know that for any coprime a and c that we choose (this set of all unique selections of a and c will generate all the rational cusps $\frac{a}{c}$) we can find two other integers b and d such that we can form a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ due to Lemma 3. Due to this lemma, since a and c are coprime, we can find integers b and d such that $det(\gamma) = ad - bc = \gcd(a,c) = 1$. Hence all the cusps of $SL_2(\mathbb{Z})$ are equivalent to the cusp ∞. □

When we say the set of all cusps of the compact surface X, we usually are referring to the set of inequivalent cusps, which consists of unique representatives of the congruence classes under action by the congruence subgroup Γ that we are working with. For example, $SL_2(\mathbb{Z})$ has only one inequivalent cusp, which is $\{\infty\}$. **Note:** Other congruence subgroups almost always have more than 1 inequivalent cusp ($\Gamma_1(5)$ has 4 of them, which we will see in the next section). We will frequently want to consider these inequivalent cusps, but instead of calling them inequivalent cusps, we simply refer to them as cusps with respect to Γ. We now will prove something rather important:

Lemma 5. *For any congruence subgroup Γ of $SL_2(\mathbb{Z})$, $X(\Gamma)$ has a finite number of inequivalent cusps.*

Proof. Any congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ has a finite index in $SL_2(\mathbb{Z})$. So let $s \in \mathbb{Q} \cup \{\infty\}$be a cusp. It follows that there exists a γ in $SL_2(\mathbb{Z})$ such that $\gamma(\infty) = s$. If we let $h_1, h_2, \ldots, h_n$ be right coset representations of $\Gamma \in SL_2(\mathbb{Z})$, then $s = \gamma(\infty) = \lambda h_j(\infty)$ for some $\lambda \in \Gamma$, which means that all Γ-equivalent cusps are contained in the set $\{\mathbb{H} = h_1(\infty), h_2(\infty), \ldots h_r(\infty)\}$. The set $\mathbb{H}$ has a finite cardinality, and we can see that $|\mathbb{H}| \leq |SL_2(\mathbb{Z}) : \Gamma|$. □

3.3 The Cusps of $X_1(p)$

Since our main focus of our research was the congruence subgroup $\Gamma_1 5$ of $SL_2(\mathbb{Z})$, we'll now restrict the results we show to the results of modular forms in $X_1(5)$. However, most of the results can be generalized quite easily to $X_1(p)$ for some prime p. It is more advantageous to deal with this general compact surface $X_1(p)$ because then the reader can apply this knowledge to any compact surface of this type and work with any prime p that they may desire.

In the last theorem in Chapter 2, we proved that the products we were studying were in fact holomorphic on $\mathbb{H}$. We'll now go a bit out of order in the modular form conditions and show that the cuspidal condition (holomorphicity of f at all the (inequivalent) cusps of $X_1(p)$) holds for the products we are studying. Along the way, we'll end up showing that our products are also weakly modular. We will then have finally shown that our products are in fact modular forms of $\Gamma_1(5)$, as well as the fact that they have weight $k = \frac{a_0}{2}$ (this will be shown in the next section). We first need to prove something that was stated earlier.

Lemma 6. *Let* $\Gamma_1(p)$ *and* $\Gamma(p)$ *be the congruence subgroups of* $SL_2(\mathbb{Z})$ *as defined in Definition 7. Then*

$$\Gamma_1(p) = \bigcup_{j=1}^{p} \Gamma(p)\gamma_j,$$

where $\gamma_j = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j$*. In other words, we can decompose* $\Gamma_1(p)$ *into right cosets (or left cosets) of* $\Gamma(p)$ *with representatives in* $SL_2(\mathbb{Z})$*.*

Proof. We can first use the index formulas outlined in chapter 2 in Theorem 1 to see that $[\Gamma_1(p) : \Gamma(p)] = p$, by Theorem 3 in Chapter 2. Furthermore, for every $\gamma \in \Gamma(p)$ and some $\alpha \in \Gamma_1(p)$ (i.e. some $\alpha \in SL_2(\mathbb{Z})$ such that $\alpha \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{p}$ where $0 \leq b < p$) we have that

$$\gamma\alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \equiv \alpha \pmod{p},$$

due to the definition of $\Gamma(p)$. This implies that if we have $\alpha_1 \in \Gamma_1(p)$ and $\alpha_2 \in \Gamma_1(p)$ such that $\alpha_1 \equiv \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \pmod{p}$ and $\alpha \equiv \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \pmod{p}$, where $b_1 \neq b_2$, we have that for all $\gamma \in \Gamma(p)$, $\gamma\alpha_1 \not\equiv \gamma\alpha_2$. This implies that $\Gamma(p)\alpha_1 \neq \Gamma(p)\alpha_2$, and thus, $\Gamma(p)\alpha_1 \cap \Gamma(p)\alpha_2 = 0$, and the two cosets are distinct. Hence, we can now see that the coset representatives in $\Gamma_1(p)$ for $\Gamma_1(p)/\Gamma(p)$ are $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, where $b \in \mathbb{Z}$, and $0 \leq b < p$. It can be seen quite readily that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b$, so it follows that for each $\alpha \in \Gamma_1(p)$, there is some $\gamma \in \Gamma(p)$ and some integer $0 \leq b < p$ such that

$$\alpha = \gamma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j.$$

Thus, $\Gamma_1(p)$ can be rewritten as the disjoint union

$$\cup_{j=0}^{p-1}\Gamma(p)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^j.$$

□

We'll now show a couple more results needed regarding the cusps of $\Gamma_1(p)$.

Lemma 7. *Let $x = m/n$ and let $x' = m'/n'$ be two points in $\mathbb{Q}^* = \mathbb{Q}\cup\infty$ with gcd(m,n)=gcd(m',n'). Then for any $\gamma \in SL_2(\mathbb{Z})$, we have that*

$$x' = \gamma(x) \iff \begin{pmatrix}m'\\ n'\end{pmatrix} = \gamma\begin{pmatrix}m\\ n\end{pmatrix},$$

where $\begin{pmatrix}m'\\ n'\end{pmatrix}$ and $\begin{pmatrix}m\\ n\end{pmatrix}$ are column vectors.

Proof. If $x' = \gamma(x)$, then for any $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$, we have that

$$x' = \frac{ax+b}{cx+d} = \frac{a\frac{m}{n}+b}{c\frac{m}{n}+d} = \frac{am+bn}{cm+dn} = \frac{m'}{n'}.$$

Since gcd(m,n)=gcd(m',n')=g, by assumption, we have that $\frac{agr+bgs}{cgr+dgs} = \frac{g(ar+bs)}{g(cr+ds)} = \frac{gr'}{gs'}$, where $m = gr, n = gs, m' = gr'$, and $n' = gs'$ for some coprime integers r,s,r', and s'. We can then cancel out the gs and we get the equality

$$\frac{ar+bs}{cr+ds} = \frac{r'}{s'},$$

so $r' = ar + bs$ and $s' = cr + ds$. Then,

$$\gamma \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} am+bn \\ cm+dn \end{pmatrix} = \begin{pmatrix} agr+bgs \\ cgr+dgs \end{pmatrix} = \begin{pmatrix} g(ar+bs) \\ g(cr+ds) \end{pmatrix} = \begin{pmatrix} gr' \\ gs' \end{pmatrix} = \begin{pmatrix} m' \\ n' \end{pmatrix}.$$

Conversely, let

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \gamma \begin{pmatrix} m \\ n \end{pmatrix}.$$

Then, since gcd(m,n)=gcd(m',n')=g, we have that

$$\gamma \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} am+bn \\ cm+dn \end{pmatrix} = \begin{pmatrix} agr+bgs \\ cgr+dgs \end{pmatrix} = \begin{pmatrix} g(ar+bs) \\ g(cr+ds) \end{pmatrix} = \begin{pmatrix} ar+bs \\ cr+ds \end{pmatrix},$$

which is exactly equal to $\begin{pmatrix} r' \\ s' \end{pmatrix}$ where $m = gr, n = gs, m' = gr'$, and $n = gs'$ for some coprime integers r, s, r', and s'. Thus, we again see that $r' = ar + bs$ and $s' = cr + ds$. Then,

$$\gamma(x) = \frac{as+b}{cs+d} = \frac{a\frac{m}{n}+b}{c\frac{m}{n}+d} = \frac{am+bn}{cm+dn} = \frac{agr+bgs}{cgr+dgs} = \frac{g(ar+bs)}{g(cr+ds)} = \frac{ar+bs}{cr+ds} = \frac{r'}{s'} = \frac{gr'}{gs'} = \frac{m'}{n'} = x'.$$

The negative case follows if we let $x' = \frac{-(ax+b)}{-(cx+d)} = \frac{m'}{n'}$. ☐

Lemma 8. *Given $x = m/n$ and $x' = m'/n'$ in $\mathbb{Q}^*$ such that m and n are coprime, x is $\Gamma_1(p)$-equivalent to x' if and only if*

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \pm \begin{pmatrix} m+jn \\ n \end{pmatrix} \quad \text{for some } j \in \mathbb{N}^+.$$

Proof. Let x and x' be defined as above. We know that we can decompose $\Gamma_1(p)$ into cosets of $\Gamma(p)$ by Lemma 6. Thus, if x and x' are $\Gamma_1(p)$-equivalent, then the two cosets $\Gamma_1(p)x$ and $\Gamma_1(p)x'$ are equal. This is directly equivalent to saying that $x' \in \Gamma_1(p)x$, which is equivalent to saying that

$x' \in \Gamma(p) \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ for some j, where $0 \leq j \leq p-1$. For any matrix γ in this coset, we have that $\gamma(x) = \frac{1x+j}{0x+1} = x + j = s'$, since x' is $\Gamma_1(p)$-equivalent to x. Thus, we see that

$$\Gamma(p)x' = \Gamma(p) \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} (x+j).$$

So, by the previous lemma, we see that since $\gamma(x) = x + j$, we have that

$$\begin{pmatrix} m' \\ n' \end{pmatrix} \equiv \pm \begin{pmatrix} m+jn \\ n \end{pmatrix} \pmod{p}.$$

□

Using this, we can find any and all inequivalent cusps of $X_1(p)$. We'll outline a simple algorithm [1] that shows us that the inequivalent cusps of $X_1(p)$ are

$$\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{2p}, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{(p-1)}.$$

To show this, we first note that we are concerned with the prime p that corresponds to our compact surface $X_1(p)$. We shall set $x = m/n$. The algorithm is as follows: First, let d =gcd(n,p). Then consider all positive values of $n \leq p$. Then, find all values of m such that m is coprime to d, for each value of n. We see that for $1 \leq n \leq p-1$, $d = 1$, so $m = 1$ for all of those values of n. These would then form the vectors:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ p-1 \end{pmatrix}.$$

However, according to Lemma 8,

$$\begin{pmatrix} m' \\ n' \end{pmatrix} \equiv \pm \begin{pmatrix} m+jn \\ n \end{pmatrix} \pmod{p},$$

and so if we let $m' = 1$ and $n' = 1$, then we shall see that there are two vectors that will be congruent to the vector

$$\begin{pmatrix} m' \\ n' \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod{p}.$$

These vectors are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod{p} \text{ and } \begin{pmatrix} 1 \\ p-1 \end{pmatrix} (\text{mod} p).$$

To see how this second vector is equivalent to the first, we simply need to let $j = 2$, $m = 1$, and $n = p-1$. We can extend this idea to any other vector $\begin{pmatrix} m' \\ n' \end{pmatrix}$ with $m', n' \in \mathbb{Z}$, to see that there are always two equivalent representative vectors in the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ p-1 \end{pmatrix} \right\}.$$

As such, the set of all of the inequivalent vectors of this form is the set:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ (p-1)/2 \end{pmatrix}.$$

If we let $n = p$, then gcd(n, p)=p, and thus, $1 \leq m \leq p-1$ are all relatively prime to n, so we get the representative vectors

$$\begin{pmatrix} 1 \\ p \end{pmatrix}, \begin{pmatrix} 2 \\ p \end{pmatrix}, \begin{pmatrix} 3 \\ p \end{pmatrix}, \ldots, \begin{pmatrix} p-1 \\ p \end{pmatrix}.$$

Like with the other set of representatives, we see that there are two sets of equivalent representatives here, because we can see that

$$\begin{pmatrix} 1 \\ p \end{pmatrix} \equiv \begin{pmatrix} p-1 \\ p \end{pmatrix} \pmod{p},$$

$$\begin{pmatrix} 2 \\ p \end{pmatrix} \equiv \begin{pmatrix} p-2 \\ p \end{pmatrix} \pmod{p},$$

and so on. To see this, we shall use Lemma 8 again to show that

$$\begin{pmatrix} 1 \\ p \end{pmatrix} \equiv \begin{pmatrix} p-1 \\ p \end{pmatrix} \pmod{p}.$$

If we consider the vector

$$\begin{pmatrix} m' \\ n' \end{pmatrix} \equiv \begin{pmatrix} 1 \\ p \end{pmatrix} \pmod{p},$$

we can set $m = 1, j = 1$, and $n = p$ and $m = p-1, j = 1$, and $n = p$ to see that

$$\begin{pmatrix} 1 \\ p \end{pmatrix} \equiv \begin{pmatrix} 1 \\ p \end{pmatrix} \equiv \begin{pmatrix} p-1 \\ p \end{pmatrix} \pmod{p}.$$

As with the other set of representatives, we can extend this idea to to any other vector $\begin{pmatrix} m' \\ n' \end{pmatrix}$ with $m', n' \in \mathbb{Z}$, to see that there are always two equivalent representative vectors in the set

$$\begin{pmatrix} 1 \\ p \end{pmatrix}, \begin{pmatrix} 2 \\ p \end{pmatrix}, \begin{pmatrix} 3 \\ p \end{pmatrix}, \ldots, \begin{pmatrix} (p-1) \\ p \end{pmatrix}.$$

Thus, the complete set of inequivalent cusp representatives can be found in the set

$$\begin{pmatrix} 1 \\ p \end{pmatrix}, \begin{pmatrix} 2 \\ p \end{pmatrix}, \begin{pmatrix} 3 \\ p \end{pmatrix}, \ldots, \begin{pmatrix} (p-1)/2 \\ p \end{pmatrix}.$$

If we recall in Lemma 8, the vector

$$\begin{pmatrix} m' \\ n' \end{pmatrix}$$

corresponds to the rational number $x' = m'/n'$ which denotes a cusp of $X_1(p)$. With this in mind, examining the set of all inequivalent cusp representatives that we just derived shows us that all inequivalent cusps of $X_1(p)$ are found in the set

$$\left\{\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{2p}, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{(p-1)}\right\}.$$

As a matter of fact, it turns out that we can do this process for any positive natural natural number N to see how many inequivalent cusps $X_1(N)$ will have, and what representatives we can use to fully delineate all cusp congruence classes. Thus, the full set of inequivalent cusps of $\Gamma_1(5)$ is

$$C_{1(5)} = \left\{\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, 1\right\}.$$

We now need to understand the behavior of certain functions called Klein forms at these cusps of $\Gamma_1(p)$, because they are an important building block of the modular forms developed here. By doing so, we will be able to refine the functions of the form in 3.1 so that they will satisfy the cuspidal condition of modular forms. We will also show that, under certain conditions, these same products satisfy the weak modularity condition of modular forms. Then, we will have finally shown that our products are modular forms.

3.4 Klein Forms and the Dedekind Eta Function

As defined in Definitions 13 and 14, the Dedekind eta function is

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}(q;q)_\infty \qquad q = e^{2\pi i\tau},$$

for some $\tau \in \mathbb{H}$, and a Klein form is defined to be

$$K_{(Q_1,Q_2)} = e^{\pi i Q_2(Q_1-1)} q^{\frac{1}{2}Q_1(Q_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_z q^n)(1-q_z^{-1}q^n)(1-q^n)^{-2},$$

for some $q_z = e^{-2\pi i(Q_1\tau+Q_2)}$, where $Q_1, Q_2 \in \mathbb{Q}-\mathbb{Z}$. The specific type of Klein forms that we will be using are of the form $K_{(j/p,0)(p\tau)}$, where $j \in \mathbb{Z}$, and $0 \le j < p$, and for ease of notation, we may refer to them as $K_{p,j}(\tau)$, or simply $K_{p,j}$. As stated earlier in Chapter 1, we can show that our products of the form

$$f_{(a_0,a_1,a_2)}(\tau) = q^{\ell(a_0,a_1,a_2)}(q^5;q^5)_\infty^{a_0}(q;q^5)_\infty^{a_1}(q^4;q^5)_\infty^{a_1}(q^2;q^5)_\infty^{a_2}(q^3;q^5)_\infty^{a_2} \qquad (a_0,a_1,a_2) \in 2\mathbb{Z}\times\mathbb{Z}^2,$$

where

$$\ell(a_0,a_1,a_2) = \frac{5}{24}(a_0+2a_1+2a_2) - \frac{2}{5}a_1 - \frac{3}{5}a_2, \quad q = e^{2\pi i\tau},$$

are exactly products of these two functions. We will now provide an in depth analysis of Klein forms. We will first need to outline a few concepts:

Definition 18. *Let X be a compact Riemann surface, and let f be meromorphic on X. For any $a \in X$, we define the* ***order*** *of f at a, denoted as $ord_a(f)$, as follows:*

- *If a is a zero of order n of f, then $ord_a(f) = n$.*
- *If a is a pole of order n of f, then $ord_a(f) = -n$.*
- *If a is neither a pole nor a zero of f, $ord_a(f) = 0$.*

This is also referred to as the ***order of vanishing****.*

By defining this order of vanishing, we can see two important things as a result:

- f is holomorphic at a when $ord_a(f)$ is nonnegative, since the poles of f are singularities. If $ord_a(f) \ge 0$ everywhere f is defined, then f is holomorphic. This is because $ord_a(f) < 0$ only when a is a pole of f.

- f "vanishes" at a when $ord_a(f)$ is positive. In other words, if f vanishes at a, a is a zero of f.

If a modular form f vanishes at all cusps, we say that f is a cusp form.

Klein forms are especially useful because their orders at inequivalent cusps are known, which is what we will outline in the following properties [8]:

Lemma 9. *Let $K_{(Q_1,Q_2)}(\tau)$ be defined as in Definition 14. Then:*

1. *For any $(Q_1,Q_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and any $(s_1,s_2)\mathbb{Z}^2$*

$$K_{(-Q_1,-Q_2)}(\tau) = -K_{(Q_1,Q_2)}(\tau)$$

and

$$K_{(Q_1,Q_2)+(s_1,s_2)}(\tau) = (-1)^{s_1s_2+s_1+s_2}e^{-\pi i Q_2(s_1-s_2)}K_{(Q_1,Q_2)}(\tau).$$

2. *For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,*

$$K_{(Q_1,Q_2)}\gamma(\tau) = (c\tau+d)^{-1}K_{(Q_1a+Q_2c,Q_1b+Q_2d)}(\tau).$$

3. *The order of vanishing at the cusps of $K_{(Q_1,Q_2)}(\tau)$ at the cusp $i\infty$ is given as*

$$ord_\infty(K_{(Q_1,Q_2)}) = \frac{1}{2}\langle Q_1\rangle(\langle Q_1\rangle - 1),$$

where $\langle Q_1\rangle$ denotes the fractional part of Q_1.

From this we can generate a number of useful results.

Lemma 10. *For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(p)$,*

$$K_{(i/p,0)}(p\gamma(\tau)) = (c\tau+d)^{-1}K_{(\frac{ia}{p},ib)}(p\tau).$$

Proof. By the second property in the previous lemma, we have that

$$K_{(i/p,0)}(p\gamma(\tau)) = K_{(i/p,0)}p\left(\frac{a\tau+b}{c\tau+d}\right) = K_{(i/p,0)}\left(\frac{ap\tau+pb}{c\tau+d}\right) = K_{(i/p,0)}\left(\frac{a(p\tau)+pb}{\frac{c}{p}(p\tau)+d}\right).$$

We can then see that the equality immediately follows because

$$K_{(i/p,0)}\left(\frac{a(p\tau)+pb}{\frac{c}{p}(p\tau)+d}\right)(c\tau+d)^{-1}K_{(\frac{i}{p}a+0c,\frac{i}{p}bp)}(p\tau) = (c\tau+d)^{-1}K_{(\frac{ia}{p},ib)}(p\tau)$$

□

Proposition 2. *Let p be an odd prime. If $a = 1+Ap$ and $b = Bp$, then $n^2(Ab+B)+n(A+b)$ is even, where $n \in \mathbb{N}$ and $0 \le n \le p-1$.*

Proof. Let a and b be defined as above. Also, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(p)$ which is a subgroup of $SL_2(\mathbb{Z})$. Thus, we see that $ad-bc = 1$. By Bezout's Identity, this implies that a and b are coprime.

Also, we know that n and n^2 have the same parity, so we simply need to show that $Ab+B$ and $A+b$ have the same parity, and the proof is complete, since the sum of two odd or two even integers is even. To do this, we note that if we let a be even, then Ap is odd, and thus, A and p are both odd. We also know that since a is even, and b is coprime to a, we have that b is odd, as well as B and p. Thus, we can see that $Ab+B$ is even, and $A+b$ is also even.

If we assume that a is odd, then Ap is even, so A is even, since p is odd. We also see that this places no restrictions on B, since $b = Bp$ can be odd or even and still be relatively prime to a. However, if B is odd, $b = Bp$ is odd, and if B is even, then b is even. Thus, if B is odd, then $Ab+B$ is odd, and $A+b$ is also odd. Similarly, if B is even, then $Ab+B$ is even, and $A+b$ is also even.

Hence, $Ab+B$ and $A+b$ always have the same parity and $n^2(Ab+B)+n(A+b)$ is even. □

We'll need this fact in this next lemma.

Lemma 11. *If $a = 1 + Ap$ and $b = Bp$, then*

$$K_{(\frac{ja}{p},jb)}(p\tau) = (-1)^{j^2(Ab+B)+j(A+b)} K_{(\frac{j}{p},0)}(p\tau) = K_{(\frac{j}{p},0)}(p\tau).$$

Proof. We let $a = 1 + Ap$ and $b = Bp$. This gives us

$$K_{(\frac{j(1+Ap)}{p},jb)}(p\tau) = K_{(\frac{j}{p}+Aj,0+jb)}(p\tau) = K_{(\frac{j}{p},0)+(Aj,jb)}(p\tau) = (-1)^{j^2Ab+Aj+jb} e^{-\pi i(\frac{-j^2b}{p})} K_{(\frac{j}{p},0)}(p\tau)$$

which we can rewrite as this:

$$(-1)^{j^2Ab+Aj+jb} e^{-\pi i(\frac{-j^2b}{p})} K_{(\frac{j}{p},0)}(p\tau) = (-1)^{j^2Ab+Aj+jb} e^{\pi i(\frac{j^2b}{p})} K_{(\frac{j}{p},0)}(p\tau)$$

Collecting the like terms, we see that

$$(-1)^{j^2Ab+Aj+jb} e^{\pi i(\frac{j^2b}{p})} K_{(\frac{j}{p},0)}(p\tau) = (-1)^{j^2Ab+Aj+jb} (-1)^{\frac{j^2b}{p}} K_{(\frac{j}{p},0)}(p\tau)$$

and that

$$(-1)^{j^2Ab+Aj+jb} (-1)^{\frac{j^2b}{p}} K_{(\frac{j}{p},0)}(p\tau) = (-1)^{j^2Ab+Aj+jb+j^2B} K_{(\frac{j}{p},0)}(p\tau).$$

We see this because $B = \frac{b}{p}$. Thus, we see that

$$K_{(\frac{j(1+Ap)}{p},jb)}(p\tau) = (-1)^{j^2(Ab+B)+j(A+b)} K_{(\frac{j}{p},0)}(p\tau) = K_{(\frac{j}{p},0)}(p\tau),$$

since $j^2(Ab+B) + j(A+b)$ is even for all natural numbers j. □

Proposition 3. *For any Klein forms of the form $K_{(\frac{j}{p},0)}(p\tau)$,*

$$K_{(\frac{j}{p},0)}(p\tau) = q^{\frac{j}{2p}(\frac{j}{p}-1)}(1 - q^{\frac{j}{p}}) \prod_{n=1}^{\infty} (1 - q^{\frac{j}{p}+n})(1 - q^{\frac{-j}{p}+n})(1 - q^n)^{-2}.$$

Proof. We see that by the definition of Klein forms in Definition 14,

$$K_{(\frac{j}{p},0)}(p\tau) = q^{\frac{j}{2p}(\frac{j}{p}-1)}(1-e^{2\pi i(\frac{j}{p}\tau+0)})\prod_{n=1}^{\infty}(1-e^{2\pi i(\frac{j}{p}\tau+0)}q^n)(1-(e^{2\pi i(\frac{j}{p}\tau+0)})^{-1}q^n)(1-q^n)^{-2},$$

which can be rewritten as

$$K_{(\frac{j}{p},0)}(p\tau) = q^{\frac{j}{2p}(\frac{j}{p}-1)}(1-e^{2\pi i\tau(\frac{j}{p})})\prod_{n=1}^{\infty}(1-e^{2\pi i\tau(\frac{j}{p})}q^n)(1-(e^{2\pi i\tau(\frac{j}{p})})^{-1}q^n)(1-q^n)^{-2}.$$

This gives us the desired equality:

$$K_{(\frac{j}{p},0)}(p\tau) = q^{\frac{j}{2p}(\frac{j}{p}-1)}(1-q^{\frac{j}{p}})\prod_{n=1}^{\infty}(1-q^{\frac{j}{p}+n})(1-q^{\frac{-j}{p}+n})(1-q^n)^{-2}$$

since $q = e^{2\pi i\tau}$. □

Theorem 9. *If* $\sum_{n=1}^{(p-1)/2}\frac{p}{2}\frac{n}{p}(\frac{n}{p}-1)a_n \in \mathbb{Z}$, *then*

$$\prod_{j=1}^{(p-1)/2} K_{(\frac{j}{p},0)}(p(\tau+1))^{a_j} = \prod_{j=1}^{(p-1)/2} K_{(\frac{j}{p},0)}(p\tau)^{a_j}.$$

Proof. Using the definition of Klein forms outlined in proposition 3, we have that

$$\prod_{j=1}^{(p-1)/2} K_{(\frac{j}{p},0)}(p(\tau+1))^{a_j} = (K_{(\frac{1}{p},0)}(p(\tau+1))^{a_1})(K_{(\frac{2}{p},0)}(p(\tau+1))^{a_2})\dots(K_{(\frac{p-1}{2p},0)}(p(\tau+1))^{a_{(p-1)/2}}).$$

This can be rewritten as:

$$\left(e^{2\pi i(p(\tau+1))\frac{1}{2p}(\frac{1}{p}-1)}(1-e^{2\pi i\frac{1}{p}(p(\tau+1))})\right)^{a_1}$$

$$\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{1}{p}(p(\tau+1))}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{1}{p}(p(\tau+1))]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_1}$$

$$\times\left(e^{2\pi i(p(\tau+1))\frac{2}{2p}(\frac{2}{p}-1)}(1-e^{2\pi i\frac{2}{p}(p(\tau+1))})\right)^{a_2}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{2}{p}(p(\tau+1))}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{2}{p}(p(\tau+1))]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_2}$$

$$\times\cdots\times\left(e^{2\pi i(p(\tau+1))\frac{p-1}{4p}(\frac{p-1}{2p}-1)}(1-e^{2\pi i\frac{p-1}{2p}(p(\tau+1))})\right)^{a_{(p-1)/2}}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{p-1}{2p}(p(\tau+1))}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{p-1}{2p}(p(\tau+1))]}e^{2\pi i(p(\tau+1))n})\right)^{a_{(p-1)/2}}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_{(p-1)/2}}$$

Since p and j are integers, we see that $e^{2\pi i\frac{j}{p}p}=e^{2\pi ij}=1$, and using the fact that

$$e^{2\pi i\frac{j}{p}(p(\tau+1))}=e^{2\pi ip\frac{j}{p}(\tau+1)}=e^{2\pi ip\frac{j}{p}(\tau)}e^{2\pi ij}=e^{2\pi i\tau(j)}e^{2\pi ij}=q^j=e^{2\pi i\tau(j)}=e^{2\pi i\frac{j}{p}(p\tau)},$$

the above expression can be rewritten as:

$$\left(e^{2\pi i(p(\tau+1))\frac{1}{2p}(\frac{1}{p}-1)}(1-e^{2\pi i\frac{1}{p}(p\tau)})\right)^{a_1}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{1}{p}(p\tau)}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{1}{p}(p\tau)]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_1}$$

$$\times\left(e^{2\pi i(p(\tau+1))\frac{2}{2p}(\frac{2}{p}-1)}(1-e^{2\pi i\frac{2}{p}(p\tau)})\right)^{a_2}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{2}{p}(p\tau)}e^{2\pi i(p\tau)n})(1-e^{-[2\pi i\frac{2}{p}(p\tau)]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_2}$$

$$\times\cdots\times\left(e^{2\pi i(p(\tau+1))\frac{p-1}{4p}(\frac{p-1}{2p}-1)}(1-e^{2\pi i\frac{p-1}{2p}(p\tau)})\right)^{a_{(p-1)/2}}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{p-1}{2p}(p\tau)}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{p-1}{2p}(p\tau)]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_{(p-1)/2}}.$$

Working with the first terms, we see that we can again rewrite it like this:

$$e^{2\pi i\frac{1}{2p}(\frac{1}{p}-1)pa_1}\left(q^{\frac{1}{2p}(\frac{1}{p}-1)}(1-e^{2\pi i\frac{1}{p}(p\tau)}\right)^{a_1}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{1}{p}(p\tau)}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{1}{p}(p\tau)]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_1}$$

$$\times e^{2\pi i\frac{2}{2p}(\frac{2}{p}-1)pa_2}\left(q^{\frac{2}{2p}(\frac{2}{p}-1)}(1-e^{2\pi i\frac{2}{p}(p\tau)})\right)^{a_2}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{2}{p}(p\tau)}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{2}{p}(p\tau)]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_2}$$

$$\times\cdots\times e^{2\pi i\frac{p-1}{4p}(\frac{p-1}{2p}-1)pa_{(p-1)/2}}\left(q^{\frac{p-1}{4p}(\frac{p-1}{2p}-1)}(1-e^{2\pi i\frac{p-1}{2p}(p\tau)})\right)^{a_{(p-1)/2}}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{p-1}{2p}(p\tau)}e^{2\pi i(p(\tau+1))n})(1-e^{-[2\pi i\frac{p-1}{2p}(p\tau)]}e^{2\pi i(p(\tau+1))n})(1-e^{2\pi i(p(\tau+1))n})^{-2}\right)^{a_{(p-1)/2}}.$$

Since n and p are both integers, $e^{2\pi i(p(\tau+1))n}=e^{2\pi ip\tau n}e^{2\pi ipn}=q^n*1=q^n$, we have that the expression can be further simplified to

$$e^{2p\pi i\frac{1}{2p}(\frac{1}{p}-1)a_1}\left(q^{\frac{1}{2p}(\frac{1}{p}-1)}(1-e^{2\pi i\frac{1}{p}(p\tau)})\right)^{a_1}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{1}{p}(p\tau)}q^n)(1-e^{-[2\pi i\frac{1}{p}(p\tau)]}q^n)(1-q^n)^{-2}\right)^{a_1}$$

$$\times e^{2p\pi i\frac{2}{2p}(\frac{2}{p}-1)a_2}\left(q^{\frac{2}{2p}(\frac{2}{p}-1)}(1-e^{2\pi i\frac{2}{p}(p\tau)})\right)^{a_2}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{2}{p}(p\tau)}q^n)(1-e^{-[2\pi i\frac{2}{p}(p\tau)]}q^n)(1-q^n)^{-2}\right)^{a_2}$$

$$\times\cdots\times e^{2p\pi i\frac{p-1}{4p}(\frac{p-1}{2p}-1)a_{(p-1)/2}}\left(q^{\frac{p-1}{4p}(\frac{p-1}{2p}-1)}(1-e^{2\pi i\frac{p-1}{2p}(p\tau)})\right)^{a_{(p-1)/2}}$$

$$\times\left(\prod_{n=1}^{\infty}(1-e^{2\pi i\frac{p-1}{2p}(p\tau)}q^n)(1-e^{-[2\pi i\frac{p-1}{2p}(p\tau)]}q^n)(1-q^n)^{-2}\right)^{a_{(p-1)/2}},$$

which is exactly equal to

$$e^{2\pi i\left(\sum_{n=1}^{(p-1)/2}\frac{p}{2}\frac{n}{p}\left(\frac{n}{p}-1\right)a_n\right)}\left(K_{(\frac{1}{p},0)}(p\tau)\right)^{a_1}\left(K_{(\frac{2}{p},0)}(p\tau)\right)^{a_2}\ldots\left(K_{(\frac{p-1}{2p},0)}(p\tau)\right)^{a_{(p-1)/2}}$$

$$= \prod_{n=1}^{(p-1)/2} K_{(\frac{n}{p},0)}(p\tau)^{a_n}.$$

This follows because $\sum_{n=1}^{(p-1)/2} \frac{p}{2}\frac{n}{p}(\frac{n}{p}-1)a_n \in \mathbb{Z}$, and thus, $e^{2\pi i\left(\sum_{n=1}^{(p-1)/2} \frac{p}{2}\frac{n}{p}(\frac{n}{p}-1)a_n\right)} = 1.$ □

Lemma 12. *Let* $f_m(z) = \eta(mz)$ *with* $m \in \mathbb{N}^+$ *and let* $r = \frac{-d}{c}$*, where gcd(c,d)=1 and* $c \neq 0$*. The order of* $f_m(z)$ *at the cusp r is as follows:*

$$ord_r(f_m) = \frac{1}{24m}(gcd(c,m))^2.$$

Proof. See (Kohler, 2011) for a proof. □

Lemma 13. *For some* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$ord_\infty\left(\prod_{n=1}^{(p-1)/2} K_{(\frac{n}{p},0)}(p\tau)^{a_n}[\gamma]_k\right) = \frac{gcd(c,p)^2}{2p}\sum_{n=1}^{(p-1)/2} a_n \left\langle \frac{ia}{gcd(c,p)}\right\rangle \left(\left\langle \frac{ia}{gcd(c,p)}\right\rangle - 1\right).$$

Proof. See Theorem 2.6 in (Eum et al., 2011) for a proof of this. □

Using the two above lemmas, we can compute the orders of $\prod_{n=1}^{(p-1)/2} K_{(\frac{n}{p},0)}(p\tau)^{a_n}$ at any inequivalent cusp of the compact surface $X(N)$. Recall that with respect to $\Gamma_1(p)$, these inequivalent cusps are

$$\left(\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{2p}, 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{(p-1)/2}\right).$$

Thus, we see the following proposition:

Proposition 4. *Let* $\begin{pmatrix} a & 0 \\ p & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$*, where* $1 \leq a \leq (p-1)/2$*. Then, for every cusp of* $X_1(p)$ *of the form* $\frac{a}{p}$*, we have that*

$$ord_{a/p}\left(\prod_{j=1}^{(p-1)/2} K_{p,j}^{a_j}\right) = -\sum_{j=1}^{(p-1)/2} \frac{p}{2}\left\langle\frac{ja}{p}\right\rangle\left(1 - \left\langle\frac{ja}{p}\right\rangle\right)a_j.$$

Proof. Let $1 \leq a \leq p-1$. Then, since gcd(p,p)=p, we have that using lemma 13:

$$ord_{a/p}\left(\prod_{j=1}^{(p-1)/2} K_{p,j}^{a_j}\right) = \frac{p^2}{2p}\sum_{j=1}^{(p-1)/2} a_j \left\langle \frac{ja}{p} \right\rangle \left(\left\langle \frac{ja}{p} \right\rangle - 1\right).$$

When we simplify this, we get this:

$$ord_{a/p}\left(\prod_{j=1}^{(p-1)/2} K_{p,j}^{a_j}\right) = -\sum_{j=1}^{(p-1)/2} \frac{p}{2} \left\langle \frac{ja}{p} \right\rangle \left(1 - \left\langle \frac{ja}{p} \right\rangle\right).$$

□

Now we consider the other form of cusps of $X_1(p)$, which are all the inequivalent cusps of the form $\frac{1}{c}$, where $1 \leq c \leq p-1$. Using the same equation in lemma 13, we see that

$$ord_{1/c}\left(\prod_{j=1}^{(p-1)/2} K_{p,j}^{a_j}\right) = 0.$$

This is because we see that gcd(c,p)=1, and thus,

$$ord_{1/c}\left(\prod_{j=1}^{(p-1)/2} K_{p,j}^{a_j}\right) = \frac{1}{2p}\sum_{j=1}^{(p-1)/2} \left\langle \frac{j}{1} \right\rangle \left(\left\langle \frac{j}{1} \right\rangle - 1\right) a_j = 0,$$

since $\langle j \rangle$ denotes the fractional part of j and the index j is an integer. Similar results can be derived for $\eta(p\tau)$ using lemma 12:

Proposition 5. *For each of the inequivalent cusps of $X_1(p)$ of the form $\frac{a}{p}$, where $1 \leq a \leq p-1$,*

$$ord_{a/p}\left(\eta(p\tau)^{a_0 + 2\sum_{j=1}^{(p-1)/2} a_j}\right) = \frac{p}{24}\left(a_0 + 2\sum_{j=1}^{(p-1)/2} a_j\right).$$

Proof. Using lemma 12, we have that $m = c = p$, and that

$$ord_{a/p}(\eta(p\tau)) = \frac{1}{24p}(\gcd(p,p))^2 = \frac{p}{24},$$

since gcd(p,p)=p. Thus, using the properties of orders, we have that

$$ord_{a/p}\left(\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2}a_j}\right)=\frac{a_0+2\sum_{j=1}^{(p-1)/2}a_j}{24p}(\gcd(p,p))^2=\frac{p}{24}\left(a_0+2\sum_{j=1}^{(p-1)/2}a_j\right).$$

□

Also, for the inequivalent cusps of $X_1(p)$ that have the form $\frac{1}{c}$, where $1\le c\le p-1$, using the same lemma, we see that

$$ord_{1/c}\left(\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2}a_j}\right)=\frac{a_0+2\sum_{j=1}^{(p-1)/2}a_j}{24p}(\gcd(c,p))^2=\frac{1}{24p}\left(a_0+2\sum_{j=1}^{(p-1)/2}a_j\right).$$

Using the two above propositions, and the properties of the function order, we can see that this product of Klein forms is holomorphic for all $\Gamma_1(p)$-inequivalent cusps, provided that the exponents $a_0,a_1,\dots a_{(p-1)/2}$ satisfy certain conditions (outlined in Theorem 12) and should those conditions be satisfied, is a modular form on $\Gamma_1(p)$, since it is also holomorphic on $\mathbb{H}$ (as proven in Chapter 2), and satisfies the modular transformation law on $\Gamma_1(p)$, should the condition in Theorem 9 hold. We shall now show that under certain conditions, the Dedekind eta function is also a modular form on $\Gamma_1(p)$, and because of this, we can safely say that the infinite products we are studying are in fact modular forms of $\Gamma_1(p)$.

Theorem 10. *The function* $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2}a_j}$ *is holomorphic with respect to* $\Gamma_1(p)$ *at the cusp* $\frac{a}{p}$ *if and only if* $24|a_0+2\sum_{j=1}^{(p-1)/2}a_j\ge 0$.

Proof. Let $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2}a_j}$ be holomorphic at the cusps a/p of $X_1(p)$. From the previous proposition, we have that

$$ord_{a/p}\left(\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2}a_j}\right)=\frac{p}{24}\left(a_0+2\sum_{j=1}^{(p-1)/2}a_j\right).$$

We also know that if this function is holomorphic at the cusps, then the order of vanishing at each cusp of $X_1(p)$ must be positive and integral. Thus, $24|a_0 + 2\sum_{j=1}^{(p-1)/2} a_j \geq 0$.

Conversely, suppose that $24|a_0 + 2\sum_{j=1}^{(p-1)/2} a_j \geq 0$. Then, $\frac{p}{24}(a_0 + 2\sum_{j=1}^{(p-1)/2} a_j$ is a nonnegative integer, and as such, since $\frac{p}{24}(a_0 + 2\sum_{j=1}^{(p-1)/2} a_j$ is positive, it follows that by the properties of orders of a function, we have that

$$ord_{a/p}\left(\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}\right) \geq 0,$$

and $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$ is holomorphic at the cusp $\frac{a}{p}$. □

Theorem 11. *The function* $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$ *is a modular form of* $\Gamma_1(p)$ *if and only if* $24|a_0 + 2\sum_{j=1}^{(p-1)/2} a_j \geq 0$.

Proof. We need to show that the three conditions of modular forms hold. We already know that $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$ is holomorphic on $\mathbb{H}$, as proven in chapter 2. We have also shown that $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$ is holomorphic on the cusps of $X_1(p)$ in the previous theorem. Thus, all we have to do to prove that this function is a modular form on $\Gamma_1(p)$ is show that it satisfies the modular transformation law.

To do this, let $\gamma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma_1(p)$. We can write γ like this because the bottom left entry in γ is congruent to 0 modulo p, by definition of the congruence subgroup $\Gamma_1(p)$. Thus,

$$p\frac{a\tau+b}{pc\tau+d} = \frac{a(p\tau)+pb}{c(p\tau)+d},$$

which means that

$$\eta(p\frac{a\tau+b}{pc\tau+d}) = \eta(\frac{a(p\tau)+pb}{c(p\tau)+d}).$$

We also know that $\begin{pmatrix} a & pb \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{Z})$ because $ad - pbc = ad - bpc = 1$, which we can see since

$\gamma \in \Gamma_1(p)$. According to the properties of the Dedekind eta function, if a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then,

1. $\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau)$,
2. $\eta(\frac{-1}{\tau}) = \sqrt{-i\tau}\eta(\tau)$,
3. $\eta(\frac{a\tau+b}{c\tau+d}) = \varepsilon(a,b,c,d)[-i(c\tau+d)]^{\frac{1}{2}}\eta(\tau)$ where $\varepsilon(a,b,c,d)^{24} = 1$.

It then follows that

$$\eta\left(p\frac{a\tau+b}{pc\tau+d}\right) = \eta\left(\frac{a(p\tau)+pb}{c(p\tau)+d}\right)^{a_0+\sum_{j=1}^{(p-1)/2} a_j} = \varepsilon(a,pb,c,d)[-i(c\tau+d)]^{\frac{1}{2}}\eta(p\tau)^{a_0+\sum_{j=1}^{(p-1)/2} a_j}.$$

Since the exponent is divisible by 24, we have that

$$\eta\left(p\frac{a\tau+b}{pc\tau+d}\right) = \eta\left(\frac{a(p\tau)+pb}{c(p\tau)+d}\right)^{a_0+\sum_{j=1}^{(p-1)/2} a_j} = (c\tau+d)^{12r}\eta(p\tau)^{24r},$$

where r is a nonnegative integer. This follows because $(-i)^{12} = (-i)^4 = 1$. Therefore,

$$\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$$

is a modular form for $\Gamma_1(p)$ of weight $k = 12r$, since $\eta(\tau)$ is a modular form of weight $k = \frac{1}{2}$ with a non-unitary nebentypus for $SL_2(\mathbb{Z})$. By transforming the function as we did above, through a matrix $\gamma \in \Gamma_1(p)$. Conversely, suppose that $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$ is a modular form of $\Gamma_1(p)$ with weight $k = \frac{1}{2}(a_0 + 2\sum_{j=1}^{(p-1)/2} a_j)$. It follows from the first property outlined in 3.4 that

$$\begin{aligned}\eta(p(\tau+1)) &= e^{\frac{2\pi i(p(\tau+1))}{24}} \prod_{n=1}^{\infty}(1-q^n) \\ &= e^{\frac{2\pi i(p)}{24}} e^{\frac{2\pi i(p\tau)}{24}} \\ &= e^{\frac{p\pi i}{12}} \eta(p\tau).\end{aligned}$$

Therefore, we see that

$$\eta(p(\tau+1))^{a_0+2\sum_{j=1}^{(p-1)/2} a_j} = \left(e^{\frac{p\pi i}{12}} \eta(p\tau)\right)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j},$$

which can be written as:

$$\left(e^{\frac{p\pi i}{12}} \eta(p\tau)\right)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j} = e^{\frac{2p\pi i}{24}} \left(a_0 + 2\sum_{j=1}^{(p-1)/2} a_j\right) \eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}.$$

To prove the other implication, assume that $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$ is a modular form of $\Gamma_1(p)$, which means that it satisfies the modular transformation law defined in the first condition of Definition 9. This would imply that if we let $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(p)$, then by our previously defined Mobius transformation $f(\gamma(\tau)) = \eta(p(\tau+1))$, which has to be equal to $f(\tau) = \eta(p\tau)$. However, as we just derived, this is only the case when the nebentypus $e^{\frac{2p\pi i}{24}}(a_0 + 2\sum_{j=1}^{(p-1)/2} a_j)$ is equal to 1, which implies that $a_0 + 2\sum_{j=1}^{(p-1)/2} a_j$ is divisible by 24. □

According to Proposition 4, we see that the quotient of Klein forms has order

$$-\sum_{j=1}^{(p-1)/2} \frac{p}{2} \left\langle \frac{jc}{p} \right\rangle \left(1 - \left\langle \frac{jc}{p} \right\rangle\right) a_j$$

at the cusps c/p, where a is an integer such that $1 \leq c < p$. Theorem 10 also gives us the order of vanishing of the eta quotient (which is to say, the function $\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}$) at each of the cusps c/p, which was derived to be $\frac{p}{24}\left(a_0+2\sum_{j=1}^{(p-1)/2} a_j\right)$. It then follows, by what we have shown, that the function

$$f_{a_0,a_1,\ldots,a_{(p-1)/2}}(\tau)=\sum_{n=\ell} P_{a_0,a_1,\ldots,a_{(p-1)/2}}(n-\ell)q^n=\eta(p\tau)^{a_0+2\sum_{j=1}^{(p-1)/2} a_j}\prod_{j=1}^{(p-1)/2} K_{p,j}^{a_j}$$

will have orders defined by the parameter ℓ:

$$\ell := \ell(a_0,a_1,\ldots,a_{(p-1)/2})=\frac{p}{24}\left(a_0+2\sum_{j=1}^{(p-1)/2} a_j\right)-\sum_{j=1}^{(p-1)/2}\frac{p}{2}\left\langle\frac{jc}{p}\right\rangle\left(1-\left\langle\frac{jc}{p}\right\rangle\right)a_j,$$

since the eta quotient and quotient of Klein forms that comprise the function $f_{a_0,a_1,\ldots,a_{(p-1)/2}}(\tau)$ are both modular forms should they satisfy certain conditions, as we have seen in the previous theorem. Therefore, in order for the product $f_{a_0,a_1,\ldots,a_{(p-1)/2}}(\tau)$ to be a holomorphic modular form of $\Gamma_1(p)$, there is a certain characterization for $f_{a_0,a_1,\ldots,a_{(p-1)/2}}(\tau)$ that must follow, which is outlined in this theorem. Since we are primarily dealing with the subgroup $\Gamma_1(5)$, we will show the explicit conditions for $p=5$.

Theorem 12. *([5]) Given a positive even integer a_0, then*

$$f_{a_0,a_1,a_2}(\tau)=\sum_{n=0}^{\infty} P_{a_0,a_1,a_2}(n-\ell)q^n=\eta(5\tau)^{a_0+2a_1+2a_2}K_{1/5,0}^{a_1}K_{2/5,0}^{a_2}$$

is a holomorphic modular form of weight $k=\frac{a_0}{2}$ for the congruence subgroup $\Gamma_1(5)$ if and only if the system of congruences hold for integral values of a_1 and a_2:

$$0 \pmod{24} \equiv a_0+2a_1+2a_2 \geq 0$$

$$-\left(\frac{2}{5}a_1+\frac{3}{5}a_2\right)\in\mathbb{Z}$$

$$\frac{5}{24}(a_0+2a_1+2a_2)-\left(\frac{2}{5}a_1+\frac{3}{5}a_2\right)\geq 0$$

$$\frac{5}{24}(a_0+2a_1+2a_2)-\left(\frac{3}{5}a_1+\frac{2}{5}a_2\right)\geq 0.$$

Proof. We first note that this product is holomorphic in $\mathbb{H}$ via the Weierstrass M-test, as we showed in Chapter 2 in Proposition 1. Thus, we have to show that the other two conditions of modular forms are satisfied for these products. We can break this down into two parts: the eta quotient $\eta(5\tau)^{a_0+2a_1+2a_2}$, and the product of Klein forms $K^{a_1}_{(1/5,0)}K^{a_2}_{(2/5,0)}$, since if we show that both are modular forms for $\Gamma_1(5)$ with weights k_1 and k_2, we have that the overall product is a modular form for $\Gamma_1(5)$ (according to Theorem 4).

We first consider the eta quotient $\eta(5\tau)^{a_0+2a_1+2a_2}$. We know by Theorem 11 that by letting $p=5$, $\eta(5\tau)^{a_0+2a_1+2a_2}$ is a modular form for $\Gamma_1(5)$ if and only if $0 \pmod{24}\equiv a_0+2a_1+2a_2$, and since $\eta(\tau)$ has weight $1/2$, we see that the condition $0 \pmod{24}\equiv a_0+2a_1+2a_2\geq 0$ is equivalent to saying that $\eta(p\tau)^{a_0+2a_1+2a_2}$ is a modular form of $\Gamma_1(5)$ with weight $\frac{a_0}{2}+a_1+a_2$, which we can verify through Theorem 4.

We now deal with the Klein form product. If we can show that this product is a modular form for $\Gamma_1(5)$, then we are done. So first, we address the weak modularity condition.

We first need to establish that $K_{(\frac{1}{5},0)}(5\tau)^{a_1}K_{(\frac{2}{5},0)}(5\tau)^{a_2}$ is a weakly modular form for $\Gamma(5)$ of weight $k=-a_1-a_2$. To do this, we must prove that $K_{(\frac{j}{5},0)}(5\tau)$ is a weakly modular form for $\Gamma(5)$ of weight $k=-1$, where $j=1,2$. So, let $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma(p)$. It follows from Theorem 10 that

$$K_{(\frac{j}{p},0)}\left(p\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{-1}K_{(\frac{ja}{p},jb)}(p\tau).$$

Also, from Theorem 11, we have that

$$K_{(\frac{ja}{p},jb)}(p\tau)=(-1)^{j^2(Ab+B)+j(A+b)}K_{(\frac{j}{p},0)}(p\tau),$$

and because $j^2(Ab+B)+j(A+b)$ is always even, as shown earlier in proposition 2, we have that $K_{(\frac{ja}{5},jb)}(5\tau)=K_{(\frac{j}{5},0)}(5\tau)$, which shows that $K_{(\frac{j}{5},0)}(5\tau)$ is weakly modular for $\Gamma(5)$ with weight

$k = -1$, and hence, $K_{(\frac{1}{5},0)}(5\tau)^{a_1}$ has weight $-a_1$ and $K_{(\frac{2}{5},0)}(5\tau)^{a_2}$ has weight $-a_2$ for $\Gamma(5)$.

We now need to show that this weak modularity holds in $\Gamma_1(5)$. In other words, we need to show that $K_{(\frac{1}{5},0)}(5\tau)^{a_1}K_{(\frac{2}{5},0)}(5\tau)^{a_2}$ is weight-$(-a_1-a_2)$ $\Gamma_1(5)$-invariant under the slash operator (defined in Definition 10).

We recall the fact that $\Gamma(5)$ can be decomposed into cosets of $\Gamma_1(5)$ of the form

$$\Gamma(5)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n,$$

for $0 \leq n < 5$, as outlined in Lemma 6.

Because of this, we have that if $f(\tau) = K_{(j/5,0)}(5\tau)$, where $j = 1,2$ and $\gamma = \begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix} \in \Gamma(p)$, then

$$f|[\gamma]_{-1}(\tau) = f(\tau), \qquad or \qquad (c'\tau+d')^{-1}f\left(\frac{a'\tau+b'}{c'\tau+d'}\right) = f(\tau).$$

This means that if

$$g(\tau) = K_{(1/5,0)}(5\tau)^{a_1}K_{(2/5,0)}(5\tau)^{a_2},$$

then by the properties outlined in Theorem 4, we have that

$$(c'\tau+d')^{-a_1-a_2}f\left(\frac{a'\tau+b'}{c'\tau+d'}\right) = f(\tau)$$

According to the properties of slash operators outlined in Diamond and Shurman [1], we have that

$$f|[\gamma_1\gamma_2]_k(\tau) = (f|[\gamma_1]_k)|[\gamma_2]_k,$$

and thus, for any $\alpha \in \Gamma_1(5)$, we have that

$$f|\,[\alpha]_{(-a_1-a_2)}(\tau) = f|\left[\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right]_{(-a_1-a_2)}(\tau) = f|\left[\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^b\right]_{(-a_1-a_2)}(\tau).$$

This means that

$$f\left|\left[\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b\right]_{(-a_1-a_2)}(\tau) = \left(f\left|\left[\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right]\right)\right._{(-a_1-a_2)}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b_{(-a_1-a_2)}(\tau),$$

and thus it be rewritten as:

$$\left((c'\tau+d')^{-a_1-a_2} f\left(\frac{a'\tau+b'}{c'\tau+d'}\right)\right)\left|\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b\right]\right.,$$

which is exactly equal to

$$f(\tau)\left|\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b\right]\right.,$$

since $\gamma \in \Gamma(5)$. Using the slash operator property we outlined above, we see that

$$f(\tau)\left|\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b\right]_{(-a_1-a_2)}\right. = ((0\tau+1)^{-a_1-a_2} f(\tau+1))\left|\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{b-1}\right]_{(-a_1-a_2)}\right.,$$

which is equal to

$$(f(\tau))\left|\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{b-1}\right]_{(-a_1-a_2)}\right..$$

This last equality happens if $\frac{2}{5}a_1 + \frac{3}{5}a_2 \in \mathbb{Z}$, since $K_{(j/5,0)}(5(\tau+1)) = K_{(j/5,0)}(5\tau)$ for $j = 1,2$ if and only if $\frac{2}{5}a_1 + \frac{3}{5}a_2 \in \mathbb{Z}$ (this is exactly what occurs when we apply Theorem 9 with $p = 5$). Thus, $\frac{2}{5}a_1 + \frac{3}{5}a_2 \in \mathbb{Z}$ is equivalent to the statement $g(\tau) = K^{a_1}_{(1/5,0)}K^{a_2}_{(2/5,0)}$ is weakly modular of weight $(-a_1 - a_2)$ with regards to $\Gamma_1(5)$. Lastly, we must show that $f_{(a_0,a_1,a_2)}$ is holomorphic at the cusps of $X_1(5)$. Since the eta quotient $\eta(5\tau)^{a_0+2a_1+2a_2}$ has already been established to be a modular form if and only if $24|a_0 + 2a_1 + 2a_2 \geq 0$, we are only verifying the holomorphy of the Klein form product $K_{(1/5,0)}(5\tau)^{a_1}K_{(2/5,0)}(5\tau)^{a_2}$.

We previously showed that the set of all inequivalent cusps of $X_1(p)$ is precisely the set

$$\left(\frac{1}{p},\frac{2}{p},\ldots,\frac{p-1}{2p},1,\frac{1}{2},\frac{1}{3},\ldots,\frac{2}{p-1}\right).$$

Thus, for $X_1(5)$, we have that the set of inequivalent cusps is

$$\left(\frac{1}{5},\frac{2}{5},1,\frac{1}{2}\right).$$

Also, we proved in Proposition 4 that

$$ord_{c/p}\left(\prod_{j=1}^{(p-1)/2}K_{(j/p,0)}^{a_j}\right)=-\frac{p}{2}\left\langle\frac{jc}{p}\right\rangle\left(1-\left\langle\frac{jc}{p}\right\rangle\right)a_j,$$

for the integer c such that $1\leq c<p$, as well as the fact that the order of $\prod_{j=1}^{(p-1)/2}K_{(j/p,0)}^{a_j}$ is 0 at any other inequivalent cusp. Thus, by properties of orders,

$$ord_{1/5}(K_{(1/5,0)}(5\tau)^{a_1}K_{(2/5,0)}(5\tau)^{a_2})+ord_{1/5}(\eta(5\tau)^{a_0+2a_1+2a_2}),$$

$$ord_{2/5}(K_{(1/5,0)}(5\tau)^{a_1}K_{(2/5,0)}(5\tau)^{a_2})+ord_{2/5}(\eta(5\tau)^{a_0+2a_1+2a_2}),$$

$$ord_{1}(K_{(1/5,0)}(5\tau)^{a_1}K_{(2/5,0)}(5\tau)^{a_2})+ord_{1}(\eta(5\tau)^{a_0+2a_1+2a_2}),$$

and

$$ord_{1/2}(K_{(1/5,0)}(5\tau)^{a_1}K_{(2/5,0)}(5\tau)^{a_2})+ord_{1/2}(\eta(5\tau)^{a_0+2a_1+2a_2}),$$

are all required to be nonnegative for $f_{(a_0,a_1,a_2)}$ to be holomorphic at all the cusps of $X_1(5)$. For the cusps 1 and $1/2$, the orders of the Klein form product are both 0 by Proposition 4, and the order of the eta quotient is $\frac{1}{120}(a_0+2a_1+2a_2)$ by Proposition 5. We require this order to be nonnegative to be holomorphic at the cusps, but that's already satisfied because we've already established that the eta quotient is a modular form of $\Gamma_1(5)$ if and only if $24|a_0+2a_1+2a_2\geq 0$, and this ensures that the order in question will be nonnegative. For the cusps at $1/5$ and $2/5$, by Propositions 4 and 5, we have that the order of the eta quotient is $\frac{5}{24}(a_0+2a_1+2a_2)$ and $ord_{1/5}(K^{a_1}_{(1/5,0)}K^{a_2}_{(2/5,0)}) = -(\frac{2}{5}a_1+\frac{3}{5}a_2)$ and $ord_{2/5}(K^{a_1}_{(1/5,0)}K^{a_2}_{(2/5,0)}) = -(\frac{3}{5}a_1+\frac{2}{5}a_2)$ Thus, by properties of orders, we see that $f_{(a_0,a_1,a_2)}$ is holomorphic at the cusps of $X_1(5)$ if and only if

$$\frac{5}{24}(a_0+2a_1+2a_2) - \left(\frac{2}{5}a_1+\frac{3}{5}a_2\right) \geq 0$$

and

$$\frac{5}{24}(a_0+2a_1+2a_2) - \left(\frac{3}{5}a_1+\frac{2}{5}a_2\right) \geq 0.$$

□

We can rewrite all the congruences and inequalities in Theorem 12 as follows:

Theorem 13. *$f_{(a_0,a_1,a_2)}(\tau)$ is a modular form for $\Gamma_1(5)$ if and only if*

$$0 \pmod{24 \equiv a_0+2a_1+2a_2 \geq 0)},$$
$$0 \pmod{5} \equiv 25a_0+2a_1-22a_2 \geq 0,$$
$$0 \pmod{5} \equiv 25a_0-22a_1+2a_2 \geq 0.$$

Theorem 14. *The weight of the modular form*

$$f_{(a_0,a_1,a_2)}(\tau) = \sum_{n=0}^{\infty} P_{a_0,a_1,a_2}(n-\ell)q^n = \eta(5\tau)^{a_0+2a_1+2a_2}K^{a_1}_{1/5,0}(5\tau)K^{a_2}_{2/5,0}(5\tau)$$

is given by $k = \frac{a_0}{2}$.

Proof. By properties of modular forms, the weight of $f_{(a_0,a_1,a_2)}$ is the sum of the weights k_1+k_2, where k_1 is the weight of the eta quotient $\eta(5\tau)^{a_0+2a_1+2a_2}$ and k_2 is the weight of the Klein form product $K^{a_1}_{1/5,0}(5\tau)K^{a_2}_{2/5,0}(5\tau)$. We have established in Theorem 12 that $k_1 = \frac{a_0}{2}+a_1+a_2$, and that $k_2 = -a_1-a_2$. Thus, $k_1+k_2 = \frac{a_0}{2}$. □

Therefore, $f_{(a_0,a_1,a_2)}(\tau) = \sum_{n=0}^{\infty} P_{a_0,a_1,a_2}(n-\ell)q^n = \eta(5\tau)^{a_0+2a_1+2a_2} K_{(1/5,0)}^{a_1} K_{(2/5,0)}^{a_2}$ is a holomorphic modular form of weight $\frac{a_0}{2}$ for the subgroup $\Gamma_1(5)$ if and only if a_0, a_1, a_2 satisfies the system of congruences and inequalities in Theorem 13. Our main focus for our research, was finding $(a_1, a_2) \in \mathbb{Z}^2$, for a fixed even integer $a_0 \geq 4$, that satisfy the system of congruences and inequalities outlined in Theorem 13, as well as the products that they define. These products will be of the form

$$f_{(a_0,a_1,a_2)}(\tau) = q^{\ell(a_0,a_1,a_2)}(q^p;q^p)_\infty^{a_0}(q;q^5)_\infty^{a_1}(q^4;q^5)_\infty^{a_1}(q^2;q^5)_\infty^{a_2}(q^3;q^5)_\infty^{a_2} \qquad (a_0,a_1,a_2) \in 2\mathbb{Z}\times\mathbb{Z}^2,$$

and we are concerned with the ones that belong to $M_{a_0/2}(\Gamma_1(5))$.

By using the system of congruences and inequalities generated by Theorem 13, we can create a bounded $(5-1)/2$-dimensional region called a $\mathbb{Z}$-polyhedron, which we will refer to as a polytope. This polytope bounds a region of interior lattice points for a fixed even integer $a_0 \geq 2$ and integral values of a_1 and a_2. From the points inside the polytope that satisfy these congruences and inequalities, we can find all the modular forms for $\Gamma_1(5)$ that satisfy Ramanujan type congruences modulo powers of 5^j for any positive integer j.

3.5 Using the $\mathbb{Z}$-polyhedron to Find Modular Forms of $\Gamma_1(5)$

Now that we've established that a function $f_{(a_0,a_1,a_2)}(\tau)$ is a modular form of $\Gamma_1(5)$ of weight $\frac{a_0}{2}$ if and only if the system of inequalities and congruences outlined in Theorem 12, we now have a method to find all integral values of a_1 and a_2 that will return a modular form that gives a Ramanujan type congruence modulo powers of 5^j, for a specific integer j. For the prime $p = 5$, since the $\mathbb{Z}$-polyhedron is of dimension 2, we can use software like Desmos to plot the polytope (shown below) of possible lattice points for a given value of a_0. To actually find our desired lattice points, we can do this by implementing some computer code. In Mathematica, we shall first define the function $Sol[a_0]$ to find all solutions of the system outlined in Theorem 12.

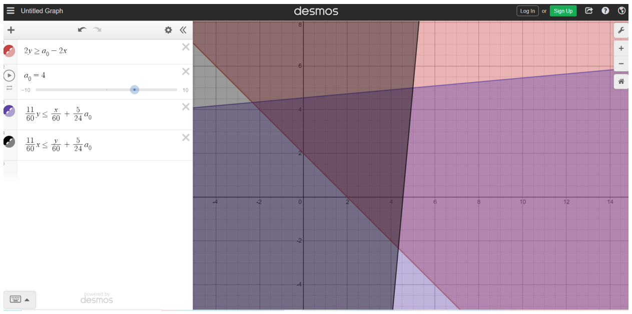

Figure 3.1: Solution region of the system with $a_0 = 4$

$$
\begin{aligned}
Sol[a_0] := Reduce[Mod[a_0+2a_1+2a_2,24] == 0, Mod[-2a_1-3a_2,5] == 0,\\
a_0+2a_1+2a_2 >= 0, 5/24(a_0+2a_1+2a_2)+(-2a_1-3a_2)/5 >= 0,\\
5/24(a_0+2a_1+2a_2)+(-3a_1-2a_2)/5 >= 0, a_1, a_2, Integers];
\end{aligned} \tag{3.3}
$$

This simply tells us all the values of a_1 and a_2 which will give us all the modular forms $f_{(a_0,a_1,a_2)}(\tau)$ of $\Gamma_1(5)$ that could potentially satisfy a Ramanujan type congruence. For example, one of these solutions when $a_0 = 4$ is $a_1 = -1$ and $a_2 = -1$. When using these exponents in the function $f_{(a_0,a_1,a_2)}(\tau)$, we can see that our result can be expressed as the q-series:

$$
\begin{aligned}
q^1 * (1+q^1+2q^2+3q^3+5q^4+2q^5+q^6+5q^7+7q^8+5q^9+12q^{10}+6q^{11}+\\
12q^{12}+6q^{13}+10q^{14}+11q^{15}+20q^{16}+16q^{17}+7q^{18}+15q^{19}+12q^{20}+12q^{21}\\
+22q^{22}+10q^{23}+25q^{24}+12q^{25}+20q^{26}+18q^{27}+30q^{28}+10q^{29}+\dots)
\end{aligned}
$$

=

$$
\begin{aligned}
q^1+q^2+2q^3+3q^4+5q^5+2q^6+6q^7+5q^8+7q^9+5q^{10}+12q^{11}+6q^{12}+\\
12q^{13}+6q^{14}+10q^{15}+11q^{16}+20q^{17}+16q^{18}+7q^{19}+15q^{20}+12q^{21}+\\
12q^{22}+22q^{23}+10q^{24}+25q^{25}+12q^{26}+20q^{27}+18q^{28}+30q^{29}+10q^{30}+\dots
\end{aligned} \tag{3.4}
$$

So, for this particular solution, it appears that this q-series satisfies a Ramanujan type congruence modulo powers of 5, and potentially modulo powers of 25, and possibly modulo higher powers of 5^j

```
In[3]:= Fin[a0_, r_] := Module[{W = {}, K = Sol[a0]},
         Do[
          If[
           Table[Mod[SeriesCoefficient[f[a0, K[[i]][[1]][[2]], K[[i]][[2]][[2]]],
                {q, 0, 5^r j}], 5^r], {j, 0, 3}].
             Table[Mod[SeriesCoefficient[f[a0, K[[i]][[1]][[2]], K[[i]][[2]][[2]]],
                {q, 0, 5^r j}], 5^r], {j, 0, 3}] == 0, AppendTo[W, {K[[i]]}]],
          {i, 1, Length[K] }]; W];
```

Figure 3.2: Do loop for triples with multiple congruences of 5-powers

for some integer $j > 2$. It is important to note that this function $Sol[a_0]$ does not eliminate the need to prove that Ramanujan type congruences are satisfied. So, we can use more computer algebra to determine if a product may satisfy a congruence modulo a higher power of 5. We define another function in Mathematica (shown below), denoted as $Fin[a_0, r]$ which carries out an algorithm that examines each entry in $Sol[a_0]$ for Ramanujan type congruences modulo 5^r, for a specific positive integer r: For example, the function $Fin[16, 1]$ returns the result

$$\{(-19, 11), (-14, 6), (3, 13), (6, -14), (8, 8), (11, -19), (13, 3), (14, 14)\}.$$

This means that when the exponents in $f_{(a_0,a_1,a_2)}(\tau)$ are $a_1 = -19$ and $a_2 = 11$, or when $a_1 = -14$ and $a_2 = 6$, or when $a_1 = 3$ and $a_2 = 13$, and so on, the modular form defined by those two parameters a_1 and a_2 may satisfy a Ramanujan type congruence modulo powers of 5.

The most important result that came from the computational experimentation was that, in some cases, there appeared to be more lattice points corresponding to products appearing to satisfy Ramanujan type congruences modulo 5^j for $j > 2$ than $j = 1$. This led to the study of antichimeral Ramanujan type congruences in this thesis.

We now have some of what we need to prove that the claimed properties hold for products satisfying antichimeral congruences, since we've derived the conditions under which a specific ordered pair (a_1, a_2), $a_1, a_2 \in \mathbb{Z}$ generates a modular form $f_{(a_0,a_1,a_2)}(\tau)$ of $\Gamma_1(5)$ (it satisfies the congruences in Theorem 13). We'll now present a method to verify if an ordered pair (a_1, a_2) generates a modular form $f_{(a_0,a_1,a_2)}(\tau)$ that satisfies infinitely many Ramanujan type congruences.

CHAPTER IV

HECKE OPERATORS AND QUINTIC RAMANUJAN TYPE CONGRUENCES

We've now theoretically developed a method to predict whether or not a specific solution to the system outlined in Theorem 13 will satisfy a Ramanujan type congruence modulo powers of 5^j, for some specific $j \in \mathbb{Z}^+$. Practically, however, this method quickly breaks down, as for higher and higher values of a_0 and r, computation time increases significantly, and the calculations themselves do not constitute a proof. In fact, this method only works to verify a finite number of Ramanujan type congruences. One of our key goals in our research was to develop a method to prove that a specific solution (a_1, a_2) satisfied infinitely many Ramanujan type congruences. To do that, we relied on some linear algebra. Specifically, we required the use of Hecke operators, as defined in Chapter 2. In this section, we will analyze these operators more thoroughly, and show how to use them to verify infinite Ramanujan type congruences.

4.1 Hecke Operators

To build upon what we defined in Chapter 2, we first consider the notion of double cosets. Let there be an element $\alpha \in GL_2^+(\mathbb{Q})$. A double coset of $GL_2^+(\mathbb{Q})$ is defined to be:

$$\Gamma_1 \alpha \Gamma_2 = \{g_1 \alpha g_2 : g_1 \in \Gamma_1, g_2 \in \Gamma_2\}.$$

We can use these double cosets to map modular forms to modular forms through the use of the **double coset operator**, but we need two things before we can state this definition:

Lemma 14. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let α be an element of $GL_2^+(\mathbb{Q})$. Then, $\alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ is also a congruence subgroup of $SL_2(\mathbb{Z})$.*

Proof. We know that due to the definition of a congruence subgroup of $SL_2(\mathbb{Z})$, there exists a positive integer N' such that $\Gamma(N') \subset \Gamma$ and that $N'\alpha, N'\alpha \in M_2(\mathbb{Z})$, where $M_2(\mathbb{Z})$ denotes the ring of all 2×2 matrices with integer entries. So, if we let $N = (N')^3$, we see that

$$\alpha\Gamma(N)\alpha^{-1} \subset \alpha(I + N'^3 M_2(\mathbb{Z}))\alpha^{-1}.$$

By simplifying the set on the right hand side, we see that

$$\alpha(I + (N')^3 M_2(\mathbb{Z}))\alpha^{-1} = I + N' * N'\alpha M_2(\mathbb{Z}) N'\alpha^{-1} \subset I + N' M_2(\mathbb{Z}).$$

We know that $\alpha\Gamma(N)\alpha^{-1}$ consists of matrices that have determinant 1, and this implies that $\alpha\Gamma(N)\alpha^{-1} \subset \Gamma(N')$. Thus, $\Gamma(N) \subset \alpha^{-1}\Gamma(N')\alpha \subset \alpha^{-1}\Gamma\alpha$. By intersecting this set with $SL_2(\mathbb{Z})$ we achieve the desired result. □

Lemma 15. *Let Γ_1 and Γ_2 be congruence subgroups of $SL_2(\mathbb{Z})$, and let $\alpha \in GL_2^+(\mathbb{Q})$. Let $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Left multiplication by α, or*

$$\Gamma_2 \to \Gamma_1\alpha\Gamma_2 \quad \textit{given by} \quad \gamma_2 \mapsto \alpha\gamma_2$$

induces a natural bijection from the coset space $\Gamma_3\backslash\Gamma_2$ to the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. In other words, $\{\gamma_{2,j}\}$ is a set of coset representatives for $\Gamma_3\backslash\Gamma_2$ if and only if $\{\beta_j\} = \{\alpha\gamma_{2,j}\}$ is a set of orbit representatives for $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.

Proof. The map $\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ taking γ_2 to $\Gamma_1\alpha\gamma_2$ is a surjection, and we can see that it takes two elements γ_2, γ_2' to the same orbit when $\Gamma_1\alpha\gamma_2 = \Gamma_2\alpha\gamma_2'$, which implies that $\gamma_2'\gamma_2^{-1} \in \alpha^{-1}\Gamma_1\alpha$. We can clearly see that $\gamma_2'\gamma_2^{-1} \in \Gamma_2$, as well. So, if we define $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$, we see that a bijection arises from $\Gamma_3\backslash\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ from cosets $\Gamma_3\gamma_2$ to orbits $\Gamma_1\alpha\gamma_2$. □

It can be shown that for any two congruence subgroups G_1 and G_2 of $SL_2(\mathbb{Z})$, the indices $[G_1 : G_1 \cap G_2]$ and $[G_2 : G_1 \cap G_2]$ are finite [1], which implies that since $\alpha^{-1}\Gamma_1\alpha \cap SL_2(\mathbb{Z})$ is a congruence subgroup of $SL_2(\mathbb{Z})$, by Lemma 14, the coset space $\Gamma_3\backslash\Gamma_2$ defined in Lemma 15 is finite, and thus, so is the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. These two lemmas ensure that the double coset $\Gamma_1\alpha\Gamma_2$ can act on modular forms. We then can define the following double coset operator:

Definition 19. *For congruence subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{Z})$ and $\alpha \in GL_2^+(\mathbb{Q})$, the **weight**-$k$ $\Gamma_1\alpha\Gamma_2$ **operator** takes functions $f \in M_k(\Gamma_1)$ to*

$$f|[\Gamma_1\alpha\Gamma_2]_k = \sum_j f|[\beta_j]_k,$$

where $\{\beta_j\}$ are orbit representatives for the space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. In other words, $\Gamma_1\alpha\Gamma_2$ can be represented as the disjoint union $\cup_j\Gamma_1\beta_j$.

We need to show that this double coset operator is well-defined, and that this action sends modular forms from $M_k(\Gamma_1)$ to modular forms from $M_k(\Gamma_2)$.

To show that this operator is well-defined, suppose that we have two representatives β_1, β_2 of the same orbit in $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, which means that $\Gamma_1\beta_1 = \Gamma_1\beta_2$. We can write these representatives β_i as $\gamma_{i,1}\alpha\gamma_{i,2}$, for some $\gamma_{i,j} \in \Gamma_j$. We see that $\alpha\gamma_{1,2} \in \Gamma_1\alpha\gamma_{2,2}$. Because f is a modular form for Γ_1, it is weight k-invariant under the action of elements of Γ_1, and we end up seeing that

$$f|[\beta_1]_k = f|[\alpha\gamma_{1,2}]_k = f|[\alpha\gamma_{2,2}]_k = f|[\gamma_{2,1}\alpha\gamma_{2,2}]_k = f|[\beta_2]_k.$$

We can remove the element $\gamma_{1,1}$ from β_1 and add the element $\gamma_{2,1}$ to form β_2 because both of these elements are in Γ_1, and thus, due to the invariance of f, act trivially. Thus, the double coset operator is independent of which orbit representative β_j is chosen, and it is therefore well-defined.

So, now we need to show that the double coset operator sends modular forms of Γ_1 to modular forms of Γ_2. We first must show that $f|[\Gamma_1\alpha\Gamma_2]_k$ is weight-k invariant under Γ_2. To show this, we first need to realize that any $\gamma_2 \in \Gamma_2$ permutes the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ by right multiplication. In other words, the map $\gamma_2 : \Gamma_1\backslash\Gamma_1\alpha\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ given by $\Gamma_1\beta \mapsto \Gamma_1\beta\gamma_2$ is well-defined and bijective. As a result of this, if $\{\beta_j\}$ is a set of orbit representatives of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, then the set $\{\beta_j \gamma_2\}$ is also a set of orbit representatives. Thus,

$$(f|[\Gamma_1\alpha\Gamma_2]_k)|[\gamma_2]_k = \sum_j f|[\beta_j\gamma_2]_k = \sum_j f|[\beta_j]_k = f|[\Gamma_1\alpha\Gamma_2]_k,$$

and thus, $f|[\Gamma_1\alpha\Gamma_2]_k$ is weight-k invariant under Γ_2. Thus, all we need to show is that holomorphy at the cusps is preserved.

We first need to see that for any $f \in M_k(\Gamma_1)$ and for any $\gamma \in GL_2^+(\mathbb{Q})$, the function $g = f|[\gamma]_k$ is holomorphic at infinity, which means that it has a Fourier expansion

$$g(\tau) = \sum_{n\geq 0} a_n(g) e^{2\pi i n\tau/h}$$

for some period $h \in \mathbb{Z}^+$. Also, if we have functions $g_1, g_2, g_3, \ldots g_d : \mathbb{H} \to \mathbb{C}$ that are all holomorphic at infinity, we have that every g_j has a similar Fourier expansion, and therefore, the sum $g_1 + g_2 + g_3 + \cdots + g_d$ is holomorphic at infinity, as well. For any matrix $\delta \in SL_2(\mathbb{Z})$, the function $(f|[\Gamma_1\alpha\Gamma_2]_k)|[\delta]_k$ is a sum of functions $g_j = f|[\gamma_j]_k$ where $\gamma_j = \beta_j\delta \in GL_2^+(\mathbb{Q})$. This implies that $(f|[\Gamma_1\alpha\Gamma_2]_k)|[\delta]_k$ is holomorphic at infinity. Since δ is arbitrary, we see that $f|[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic at the cusps. Thus, $f|[\Gamma_1\alpha\Gamma_2]_k$ is a modular form for Γ_2.

It can be shown [6] that every double coset operator is given by a combination of one or more of three special cases:

- $\Gamma_1 \subset \Gamma_2$. The double coset operator map projects $M_k(\Gamma_1)$ onto $M_k(\Gamma_2)$
- $\Gamma_1 = \alpha^{-1}\Gamma_2\alpha$. The double coset operator map gives an isomorphism between between $M_k(\Gamma_1)$ and $M_k(\Gamma_2)$
- $\Gamma_2 \subset \Gamma_1$. A modular form for Γ_1 will also be a modular form for $Gamma_2$, and the double coset operator map becomes the inclusion map from $M_k(\Gamma_1)$ to $M_k(\Gamma_2)$.

There are two "standard" types of these double coset (or Hecke) operators over $\Gamma_1(N)$, which we define now, the "diamond" operator $\langle d\rangle$, and the operator T_p.

Definition 20. *Let N be a positive integer and let $f \in M_k(\Gamma_1(N))$. Also, let* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *We define the diamond operator $\langle d \rangle$ to be the double coset operator $f|[\Gamma_1(N)\alpha\Gamma_1(N)]_k$ such that*

$$f|[\Gamma_1 \alpha \Gamma_1]_k = f|[\alpha]_k,$$

which is also in $M_k(\Gamma_1(N))$. Thus, this operator defines a map:

$$\langle d \rangle : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$$

given by

$$\langle d \rangle f = f|[\alpha]_k \quad \text{for any} \quad \alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N), \quad \text{where} \quad \delta \equiv d (mod N).$$

Because $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$, the the action of α is solely determined by d modulo N because the subgroup $\Gamma_1(N)$ acts trivially on f.

Theorem 15. *Let N be a positive integer, let $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, and let* $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. *The operator $T_p = f|[\Gamma_1 \alpha \Gamma_1(N)]_k$ on $M_k(\Gamma_1(N))$ is given by*

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} f \left| \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right. & if p|N \\ \sum_{j=0}^{p-1} f \left| \left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right]_k \right. + f \left| \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right. & if p \not| N, where \quad mp - nN = 1. \end{cases} \tag{4.1}$$

Proof. See Diamond and Shurman [1] section 5.2 for the double coset computation and analysis that yields this explicit representation of the action of this double coset. □

Before we prove the last important result we need, we also need to introduce the notion of a character.

Definition 21. *A **Dirichlet character** modulo N is a function $\chi : \mathbb{Z} \to \mathbb{C}$ that satisfies the following conditions for any $a, b \in \mathbb{Z}$:*

1. $\chi(ab) = \chi(a)\chi(b)$.
2. *If a and N are coprime, then* $\chi(a) \neq 0$*. Otherwise,* $\chi(a) = 0$.
3. χ *is periodic modulo N. In other words,* $\chi(a+N) = \chi(a)$.

We sometimes equip a space of modular forms $M_k(N)$ with a character, and we denote it as $M_k(N, \chi)$

For our purposes, we will limit our contact with the Dirichlet character functions to the principal character modulo N $\mathbf{1}_N$, where $\mathbf{1}_N(p) = 0$ if $p|N$, and $\mathbf{1}_N(p) = 1$ otherwise.

With this, we are finally ready to prove the last theorem we need in order to effectively use Hecke operators in our research.

Theorem 16. *Let $f \in M_k(\Gamma_1(N))$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(N)$, f has a period 1, and thus, has a Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i \tau}.$$

Then, if $\mathbf{1}_N : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ is given by the principal character modulo N, $T_p f$ has Fourier expansion

$$\begin{aligned}(T_p f)(\tau) &= \sum_{n=0}^{\infty} a_{np}(f) q^n + \mathbf{1}_N(p) p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p \rangle f) q^{np} \\ &= \sum_{n=0}^{\infty} [a_{np}(f) + \mathbf{1}_N(p) p^{k-1} a_{n/p}(\langle p \rangle f)] q^n,\end{aligned}$$

where $a_{n/p} = 0$ if $n/p \notin \mathbb{N}$.

Proof. First, we recall definition of the operator T_p outlined in Definition 15, as well as the slash operator, which we defined in Definition 10 in Chapter 2. Using this operator, we take $0 \leq j < p$ and compute this:

$$f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k (\tau) = p^{k-1}(0\tau+p)^{-k} f\left(\frac{\tau+j}{p}\right) = \frac{1}{p}\sum_{n=0}^{\infty} a_n(f) e^{2\pi i n(\tau+j)/p},$$

which can be written as $\frac{1}{p}\sum_{n=0}^{\infty} a_n(f) q_p^n \mu_p^{nj}$ where $q_p = e^{2\pi i\tau/p}$ and $\mu_p = e^{2\pi i/p}$.

It's a well known result in number theory that the geometric sum $\sum_{j=0}^{p-1} \mu_p^{nj} = p$, when $p|n$, and 0 otherwise, so using this in the evaluation of the slash operator gives us this:

$$\sum_{j=0}^{p-1} f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right] = \sum_{n\equiv 0(p)} a_n(f) q_p^n = \sum_{n=0}^{\infty} a_{np}(f) q^n.$$

This is $(T_p f)(\tau)$ when $p|N$. When $p \nmid N$, $(T_p f)(\tau)$ contains also includes another term, which we apply the slash operator to, as well:

$$f\left[\begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k (\tau) = (\langle p\rangle f)\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k (\tau) = p^{k-1}(0\tau+1)^{-k}(\langle p\rangle f)\left(\frac{p\tau+0}{0\tau+1}\right),$$

which ends up simplifying to $p^{k-1}(\langle p\rangle f)(p\tau)$, which has the Fourier expansion

$$p^{k-1}\sum_{n=0}^{\infty} a_n(\langle p\rangle f) q^{np},$$

since the function argument was $p\tau$, not τ. Thus, utilizing the principal character function modulo N, we can combine the two results for all values of N, and this gives the desired equality. □

A direct consequence of this theorem is that if $p|N$, then for some $f \in M_k(\Gamma_1(N))$ with

Fourier expansion

$$\sum_{n=0}^{\infty} a_n(f)q^n,$$

$T_p(f)$ has a Fourier expansion

$$\sum_{n=0}^{\infty} b_n(f)q^n,$$

where $b_n = a_{np}$, which means we now have a way to effectively strip out every pth coefficient of f to check for Ramanujan type congruences. We now use this concept with the congruence subgroup $\Gamma_1(5)$.

4.2 Infinite Quintic Ramanujan Type Congruences

If we recall in chapter 3, we fixed a positive even integer a_0, and found all the solutions (a_1,a_2) that satisfied all the congruences and inequalities in Theorem 13. These solutions were then input into an algorithm to see which of the ordered triples (a_0,a_1,a_2) would yield a modular form that looked like this:

$$f_{(a_0,a_1,a_2)}(\tau) = q^{\ell(a_0,a_1,a_2)}(q^5;q^5)_\infty^{a_0}(q;q^5)_\infty^{a_1}(q^4;q^5)_\infty^{a_1}(q^2;q^5)_\infty^{a_2}(q^3;q^5)_\infty^{a_2} \qquad (a_0,a_1,a_2) \in 2\mathbb{Z}\times\mathbb{Z}^2, \tag{4.2}$$

where

$$\ell(a_0,a_1,a_2) = \frac{5}{24}(a_0+2a_1+2a_2) - \frac{2}{5}a_1 - \frac{3}{5}a_2, \quad q = e^{2\pi i\tau},$$

which satisfied Ramanujan type congruences modulo powers of 5^j, for some specific positive integer j. Using the knowledge of Hecke operators shown in the previous section, we can automate the process considerably. To do this, though, we do need to realize that the set of modular forms of weight $a_0/2$ of $\Gamma_1(5)$ creates a vector space with a finite number of bases [1]. For each of these vector spaces, these two functions can be used to generate all of their bases [4].

$$A(q) = q^{1/5}\frac{\left(\prod_{n=1}^{\infty} 1-q^{5n}\right)^{2/5}}{\prod_{n=0}^{\infty} 1-q^{5n+2} \quad \prod_{n=0}^{\infty} 1-q^{5n+3}} \tag{4.3}$$

and

$$B(q) = \frac{(\prod_{n=1}^{\infty} 1-q^{5n} \quad)^{2/5}}{\prod_{n=0}^{\infty} 1-q^{5n+1} \quad \prod_{n=0}^{\infty} 1-q^{5n+4}}. \tag{4.4}$$

Each vector space has $a_0/2+1$ bases (that is, the dimension of the vector space of modular forms $M_{a_0/2}(\Gamma_1(5))$ is equal to $a_0/2+1$), and if we say that $a_0/2 = n$. Since the monomials in A and B can be shown to be linearly independent, we can define the basis elements as follows:

$$\left[A^{5n}, A^{5n-5}B^5, A^{5n-10}B^{10}, \ldots A^5B^{5n-5}, B^{5n}\right]. \tag{4.5}$$

Therefore, we can represent every modular form in $\Gamma_1(5)$ outlined in 4.2, denoted as $f_{a_0,a_1,a_2}(\tau)$ as a linear combination of one or more of these basis elements. We'll explain how to calculate these basis representations shortly.

We first consider the Hecke operator T_5. Because we are considering modular forms of $\Gamma_1(5)$, we see that for any modular form f in $\Gamma_1(5)$, the Hecke operator T_5 simplifies to $T_5(f) = \sum_{n=0}^{\infty} a_{5n}(f)q^n$. We'll denote this operator as U_5. It then follows that if we apply the operator U_5 to f, then if all coefficients of the Fourier expansion of $U_5(f)$ are divisible by 5, then the values of a_0, a_1, and a_2 correspond to a modular form that satisfies a Ramanujan type congruence. As we've shown before, the simplified Hecke operator U_5 maps modular forms from $\Gamma_1(5)$ to modular forms of $\Gamma_1(5)$. This means that we can apply U_5 to the modular form f and still return a modular form in $\Gamma_1(5)$. This leads to a method to verify infinitely many Ramanujan type congruences for a specific modular form $f_{a_0,a_1,a_2}(\tau)$, which as we've shown in chapter 3 in Theorem 14, has weight $a_0/2$. We will illustrate this method with an example. A simple case is when $a_0 = 8$. This means that using the bases $A(q)$ and $B(q)$, the basis that we desire for $M_4(\Gamma_1(5))$ will be:

$$\left[A^{20}, A^{15}B^5, A^{10}B^{10}, A^5B^{15}, B^{20}\right]. \tag{4.6}$$

So

$$U_5 : M_4(\Gamma_1(5)) \to M_4(\Gamma_1(5)) \tag{4.7}$$

is given by

$$\sum_{n=1}^{\infty} a_n q^n \mapsto \sum_{n=1}^{\infty} a_{5n} q^n, \tag{4.8}$$

and we thus have a matrix of linear transformation for the operator U_5 such that:

$$\begin{pmatrix} U_5(A^{20}) \\ U_5(A^{15}B^5) \\ U_5(A^{10}B^{10}) \\ U_5(A^5B^{15}) \\ U_5(B^{20}) \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{pmatrix} \begin{pmatrix} A^{20} \\ A^{15}B^5 \\ A^{10}B^{10} \\ A^5B^{15} \\ B^{20} \end{pmatrix}. \tag{4.9}$$

Through the use of computers, we can solve for the matrix A and compute its eigenvectors and eigenvalues. After these computations, we get the results shown in Figure 4.1. Though the upper

```
In[34]:= Solve[Table[SeriesCoefficient[a1 A[q]^20 + a2 A[q]^15 B[q]^5 + a3 A[q]^10 B[q]^10 + a4 A[q]^5 B[q]^15 + a5 B[q]^20, {q, 0, l}] ==
           SeriesCoefficient[Sum[SeriesCoefficient[A[q]^20, {q, 0, 5 n}] q^n, {n, 0, 10}], {q, 0, l}], {l, 0, 6}], {a1, a2, a3, a4, a5}]
         Solve[Table[SeriesCoefficient[a1 A[q]^20 + a2 A[q]^15 B[q]^5 + a3 A[q]^10 B[q]^10 + a4 A[q]^5 B[q]^15 + a5 B[q]^20, {q, 0, l}] ==
           SeriesCoefficient[Sum[SeriesCoefficient[A[q]^15 B[q]^5, {q, 0, 5 n}] q^n, {n, 0, 10}], {q, 0, l}], {l, 0, 6}], {a1, a2, a3, a4, a5}]
         Solve[Table[SeriesCoefficient[a1 A[q]^20 + a2 A[q]^15 B[q]^5 + a3 A[q]^10 B[q]^10 + a4 A[q]^5 B[q]^15 + a5 B[q]^20, {q, 0, l}] ==
           SeriesCoefficient[Sum[SeriesCoefficient[A[q]^10 B[q]^10, {q, 0, 5 n}] q^n, {n, 0, 10}], {q, 0, l}], {l, 0, 6}], {a1, a2, a3, a4, a5}]
         Solve[Table[SeriesCoefficient[a1 A[q]^20 + a2 A[q]^15 B[q]^5 + a3 A[q]^10 B[q]^10 + a4 A[q]^5 B[q]^15 + a5 B[q]^20, {q, 0, l}] ==
           SeriesCoefficient[Sum[SeriesCoefficient[A[q]^5 B[q]^15, {q, 0, 5 n}] q^n, {n, 0, 10}], {q, 0, l}], {l, 0, 6}], {a1, a2, a3, a4, a5}]
         Solve[Table[SeriesCoefficient[a1 A[q]^20 + a2 A[q]^15 B[q]^5 + a3 A[q]^10 B[q]^10 + a4 A[q]^5 B[q]^15 + a5 B[q]^20, {q, 0, l}] ==
           SeriesCoefficient[Sum[SeriesCoefficient[B[q]^20, {q, 0, 5 n}] q^n, {n, 0, 10}], {q, 0, l}], {l, 0, 6}], {a1, a2, a3, a4, a5}]
Out[34]= {{a1 -> 1, a2 -> -1356, a3 -> 1462, a4 -> -8, a5 -> 0}}
Out[35]= {{a1 -> 0, a2 -> 115, a3 -> -110, a4 -> 10, a5 -> 0}}
Out[36]= {{a1 -> 0, a2 -> -10, a3 -> 15, a4 -> 10, a5 -> 0}}
Out[37]= {{a1 -> 0, a2 -> 10, a3 -> 110, a4 -> 115, a5 -> 0}}
Out[38]= {{a1 -> 0, a2 -> 8, a3 -> 1462, a4 -> 1356, a5 -> 1}}
```

Figure 4.1: Computations for the linear transformation matrix of the operator U_5 for $a_0 = 8$

limits in this computation are 200, they suffice for our computational purposes. We use 200 as a large value in order to minimize the computing time on the system. The results are still accurate, as the computed coefficients match those for the infinite series.

In any case, our matrix of linear transformation for the operator U_5 is

$$A = \begin{pmatrix} 1 & -1356 & 1462 & -8 & 0 \\ 0 & 115 & -110 & 10 & 0 \\ 0 & -10 & 15 & 10 & 0 \\ 0 & 10 & 110 & 115 & 0 \\ 0 & 8 & 1462 & 1356 & 1 \end{pmatrix}. \tag{4.10}$$

Now, if the product of the matrix of linear transformation A for U_5 with the basis representation of a solution (a_0, a_1, a_2) to the system outlined in Theorem 13 causes all the coefficients to be divisible by 5, then the exponents a_1 and a_2, with a fixed a_0 will yield a modular form $f_{(a_0,a_1,a_2)}(\tau)$ that satisfies Ramanujan type congruences modulo powers of 5. We prove congruences for other powers of 5 using the eigendecomposition of A. In other words, if the basis representation of a solution (a_0, a_1, a_2) can be described as a linear combination of the eigenvectors of A corresponding to the eigenvalues that are powers of 5 or divisible by 5 (excluding 1), then this solution gives exponents that yield a modular form that satisfies infinitely many Ramanujan type congruences.

We're not quite ready to find the basis representation just yet, so for now, we will take the eigensystem of the matrix A^T, and we get that the set of eigenvalues of A^T, denoted as E_{v_1} is:

$$E_{v_1} = \left[125, 125, -5, 1, 1\right], \tag{4.11}$$

and the eigenvectors e_1, e_2, e_3, e_4, and e_5 of the matrix A^T are

$$e_1 = \begin{pmatrix} 0, & 1, & 0, & 1, & 0 \end{pmatrix}^T,$$

$$e_2 = \begin{pmatrix} 0, & -1, & 1, & 0, & 0 \end{pmatrix}^T,$$

$$e_3 = \begin{pmatrix} 0, & -1, & -11, & 1, & 0 \end{pmatrix}^T,$$

$$e_4 = \begin{pmatrix} -773, & -9504, & -3454, & 0, & 91 \end{pmatrix}^T,$$

$$e_5 = \begin{pmatrix} -62, & -773, & -314, & 91, & 0 \end{pmatrix}^T. \tag{4.12}$$

Since the first 3 eigenvectors are all either powers of 5 or divisible by 5, if we can express the basis representation of a solution (a_0, a_1, a_2) as a linear combination of the first 3 eigenvectors of A, we are done, and this solution gives exponents for a modular form that will satisfy infinitely many Ramanujan type congruences modulo powers of 5^j, for all $j \in \mathbb{Z}^+$. Now, we just need an example solution (a_0, a_1, a_2) and its basis representation, which we will show how to calculate here. We will select the exponents $(a_0, a_1, a_2) = (8, 4, 4)$, because prior inspection has shown it to give a modular form that satisfies infinitely many Ramanujan-type congruences.

We first start off by evaluating the function $f_{(8,4,4)}(\tau)$ as outlined in 4.2, and that gives us the following q-series:

$$f_{(a_0,a_1,a_2)}(\tau) = q * (q^5 : q^5)^8_\infty (q^2 : q^5)^4_\infty (q^3 : q^5)^4_\infty (q : q^5)^4_\infty (q^4 : q^5)^4_\infty,$$

where $q = e^{2\pi i \tau}$. We need to write this q-series in terms of $A^{5i}B^{5j}$, where $i + j = 4$, and $i, j \geq 0$, and we can thankfully run this through some computer software to make this process less time consuming. Using Mathematica, we see this:

So, as we can see in Figure 4.2, this can be represented by the vector v_s:

$$v_s = \begin{pmatrix} 0 \\ -1 \\ -11 \\ 1 \\ 0 \end{pmatrix}. \tag{4.13}$$

```
In[52]:= Solve[
    Table[
     SeriesCoefficient[a1 A[q]^20 + a2 A[q]^15 B[q]^5 + a3 A[q]^10 B[q]^10 + a4 A[q]^5 B[q]^15 + a5 B[q]^20,
       {q, 0, l}] ==
      SeriesCoefficient[q (Product[(1 - q^(5 n)), {n, 1, 200}])^8 (Product[(1 - q^(5 n+1)), {n, 0, 200}])^4 (Product[(1 - q^(5 n+4)), {n, 0, 200}])^4
        (Product[(1 - q^(5 n+2)), {n, 0, 200}])^4 (Product[(1 - q^(5 n+3)), {n, 0, 200}])^4, {q, 0, l}], {l, 0, 6}], {a1, a2, a3, a4, a5}]

Out[52]= {{a1 → 0, a2 → -1, a3 → -11, a4 → 1, a5 → 0}}
```

Figure 4.2: Solution of the basis representation of the infinite product given by the solution (8,4,4)

This in turn corresponds to the products $A^{15}B^5$,$A^{10}B^{10}$, and A^5B^{15}, which means that the basis representation of $f_{(8,4,4)}$ is $-A^{15}B^5 - 11A^{10}B^{10} + A^5B^{15}$. So, we want to see if we can take this vector v_s and represent it using a linear combination of the three eigenvectors that correspond to eigenvalues of powers of 5 (again, not including 1).

This means that, since this particular transformation is solely generated from the eigenvector that corresponds to the eigenvalue -5, for the original matrix of our linear transformation A for the operator U_5, we end up with the following:

$$\begin{pmatrix} 1 & -1356 & 1462 & -8 & 0 \\ 0 & 115 & -110 & 10 & 0 \\ 0 & -10 & 15 & 10 & 0 \\ 0 & 10 & 110 & 115 & 0 \\ 0 & 8 & 1462 & 1356 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -11 \\ 1 \\ 0 \end{pmatrix} = -5 \begin{pmatrix} 0 \\ -1 \\ -11 \\ 1 \\ 0 \end{pmatrix} \tag{4.14}$$

This means that:

$$U_5(-A^{15}B^5 - 11A^{10}B^{10} + A^5B^{15}) = -5(-A^{15}B^5 - 11A^{10}B^{10} + A^5B^{15}). \tag{4.15}$$

If we apply the operator U_5 n times

$$U_5^n(-A^{15}B^5 - 11A^{10}B^{10} + A^5B^{15}) = (-5)^n(-A^{15}B^5 - 11A^{10}B^{10} + A^5B^{15}), \tag{4.16}$$

which implies that we are multiplying the basis representation

$$\begin{pmatrix} 0 \\ -1 \\ -11 \\ 1 \\ 0 \end{pmatrix}$$

by -5 every time we apply the operator U_5, and -5 is divisible by 5. This implies that every q-power with exponent of 5^j in this product will have a coefficient that is congruent to 0 modulo 5^j, for any positive integer j. Thus, the solution $(a_0, a_1, a_2) = (8, 4, 4)$ yields a modular form that that satisfies infinitely many Ramanujan type congruences modulo powers of 5^j.

In our original evaluation of $f_{(8,4,4)}(\tau)$ defined in 4.2, it was observed that $\ell(8,4,4)$ was equal to 1. This means that $f_{(a_0,a_1,a_2)}$ can be expressed like this:

$$\frac{q * (\prod_{n=1}^{\infty} 1 - q^{5n} \quad)^8 (\prod_{n=0}^{\infty} 1 - q^{5n+1} \quad)^4}{(\prod_{n=0}^{\infty} 1 - q^{5n+4} \quad)^4 (\prod_{n=0}^{\infty} 1 - q^{5n+2} \quad)^4 (\prod_{n=0}^{\infty} 1 - q^{5n+3} \quad)^4} \tag{4.17}$$

=

$$q * \sum_{n=0}^{\infty} P_{(8,4,4)}(n) q^n \quad . \tag{4.18}$$

So, since we had to multiply $P_{(8,4,4)}$ by q, we can say that the Ramanujan type congruence is of the form:

$$P_{(8,4,4)}(5n+4) \equiv 0 \pmod 5. \tag{4.19}$$

If we iterate U_5 again, the congruence becomes

$$P_{(8,4,4)}(25n+24) \equiv 0 \pmod{25}. \tag{4.20}$$

If we do it once more, the congruence becomes

$$P_{(8,4,4)}(125n+99) \equiv 0 \pmod{125}. \tag{4.21}$$

Thus, the function $\ell(a_0,a_1,a_2)$ defines congruence classes for the congruences.

For any other solution (a_0,a_1,a_2) to the system of inequalities/congruences that we defined in Theorem 13, we can use the same operator U_5 and the same method of finding the basis representation of the product mentioned in 4.2 for each ordered triple (a_0,a_1,a_2). Then, through the use of the matrix A of the linear transformation, we can write the basis representation as a linear combination of the eigenvectors corresponding to to eigenvalues of A that are powers of 5, or divisible by 5. This will allow us to prove that the product in fact satisfies infinitely many Ramanujan type congruences. Not all of the triples (a_0,a_1,a_2) are expected to correspond to products satisfying infinitely Ramanujan type congruences, because many of the eigenvalues for larger even a_0 values are complex.

CHAPTER V

THE ANTICHIMERALITY OF THE PRODUCTS OF MODULAR FORMS OF $\Gamma_1(5)$ AND ADDITIONAL REMARKS

5.1 Antichimerality

In the end of chapter 3, we made a few observations regarding the lattice points (a_0, a_1, a_2) that yielded a modular form that satisfied Ramanujan type congruences. The most important observation was the existence of antichimeral Ramanujan Type congruences, outlined in Definition 17, that only occurred for $a_0 \equiv 6 \pmod{10}$, with $a_0 > 6$. This appeared to be an anomaly. One would expect that if a modular form $f_{(a_0,a_1,a_2)}(\tau)$ does not satisfy a Ramanujan type congruence modulo powers of 5, it will not satisfy any other Ramanujan-type congruences for larger powers of 5. This caused us to examine the matrix of linear transformation a little closer.

We will first examine the simplest case, which is when $a_0 = 16$. Running the code that we had in Figure 3.2 for $a_0 = 16$ and $r = 1$ and $r = 2$, we see that the solutions $(16,-9,1)$, $(16,-8,12)$, $(16,-4,-4)$,$(16,-3,7)$,$(16,1,-9)$,$(16,2,2)$,$(16,7,-3)$,$(16,12,-8)$ all yield modular forms that may satisfy antichimeral Ramanujan type congruences modulo powers of 5^j, where $j \geq 2$ (shown below). In order to prove that these solutions do actually yield antichimeral Ramanujan type

```
In[57]:= Fin[16, 1]
Out[57]= {{a1 == -19 && a2 == 11}, {a1 == -14 && a2 == 6}, {a1 == 3 && a2 == 13}, {a1 == 6 && a2 == -14},
  {a1 == 8 && a2 == 8}, {a1 == 11 && a2 == -19}, {a1 == 13 && a2 == 3}, {a1 == 14 && a2 == 14}}

In[58]:= Fin[16, 2]
Out[58]= {{a1 == -19 && a2 == 11}, {a1 == -14 && a2 == 6}, {a1 == -9 && a2 == 1}, {a1 == -8 && a2 == 12},
  {a1 == -4 && a2 == -4}, {a1 == -3 && a2 == 7}, {a1 == 1 && a2 == -9}, {a1 == 2 && a2 == 2},
  {a1 == 3 && a2 == 13}, {a1 == 6 && a2 == -14}, {a1 == 7 && a2 == -3}, {a1 == 8 && a2 == 8},
  {a1 == 11 && a2 == -19}, {a1 == 12 && a2 == -8}, {a1 == 13 && a2 == 3}, {a1 == 14 && a2 == 14}}
```

Figure 5.1: Do loop for $a_0 = 16$ and $r = 1$ and $r = 2$

congruences, we first must look at the Jordan decomposition of the matrix of linear transformation A of U_5. We can use the Jordan decomposition to relatively easily (computation-wise) compute many applications of the U_5 operator on a basis representation of a solution. For $a_0 = 16$, it can be shown, using the same method in chapter 4, that a basis of $M_8(\Gamma_1(5))$ is the set

$$\left[A^{40}, A^{35}B^{5}, A^{30}B^{10}, A^{25}B^{15}, A^{20}B^{20}, A^{15}B^{25}A^{10}B^{30}, A^{5}B^{35}, B^{40}\right],$$

and the matrix A of linear transformation of the operator U_5 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -55552 & 5090 & -470 & 44 & -4 & 1 & 0 & 0 & 0 \\ 6463260 & -577750 & 51775 & -4633 & 418 & -2 & 240 & 355 & 144 \\ -27630160 & 2506025 & -228150 & 20765 & -1840 & 885 & 7000 & 52900 & 199992 \\ 8545390 & -764225 & 68900 & -6215 & 1015 & 6215 & 68900 & 764225 & 8545390 \\ -199992 & 52900 & -7000 & 885 & 1840 & 20765 & 228150 & 2506025 & 27630160 \\ 144 & -355 & 240 & 2 & 418 & 4633 & 51775 & 577750 & 6463260 \\ 0 & 0 & 0 & 1 & 4 & 44 & 470 & 5090 & 55552 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we consider the Jordan decomposition of the matrix A of linear transformation of U_5 for $a_0 = 16$, we see that $A = VJV^{-1}$, where V is composed of eigenvectors of A, and J is diagonal (see Figure 5.2). We should note that J need not be diagonal in order for VJV^{-1} to be in Jordan canonical form, but for this case, J is diagonal. If we then take our basis representation of a solution that may correspond to an antichimeral Ramanujan type congruence, like $(a_0, a_1, a_2) = (16, -9, 1)$, we see that this triple has the basis representation

$$(0,0,0,0,0,1,0,0,0),$$

which is the elementary vector e_6 (see Figure 5.3). Applying U_5 a total of n iterations is the equivalent of raising the matrix of linear transformation A to the nth power. Therefore, $A^n =$

```
V = MatrixForm[FullSimplify[JordanDecomposition[U][[1]]]]
```

Out[46]//MatrixForm=

$$
\begin{pmatrix}
0 & 0 & -338456957 & -14101382 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -8123537664 & -338456957 & -1 & 109 & -1 & 1 & 1 \\
0 & 1 & -58031790860 & -2417820978 & -33 & -11880 & 109 & 19-2i\sqrt{29} & 19+2i\sqrt{29} \\
241 & 22 & -106236629568 & -4426217715 & -235 & 52551 & -482 & 87-22i\sqrt{29} & 87+22i\sqrt{29} \\
1287 & 119 & 15531286290 & 647211630 & 110 & -15768 & 146 & 0 & 0 \\
-241 & -22 & 317438784 & 14047629 & 235 & 1899 & -13 & 87-22i\sqrt{29} & 87+22i\sqrt{29} \\
0 & 1 & 56707828 & 1913674 & -33 & 0 & 1 & -19+2i\sqrt{29} & -19-2i\sqrt{29} \\
1 & 0 & 0 & 62851 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 62851 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

```
In[45]:= J = MatrixForm[FullSimplify[JordanDecomposition[U][[2]]]]
```

Out[45]//MatrixForm=

$$
\begin{pmatrix}
-125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 125 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 78125 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 78125 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 75-50i\sqrt{29} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 75+50i\sqrt{29}
\end{pmatrix}
$$

Figure 5.2: Jordan decomposition for the matrix A of linear transformation for the operator U_5 and $a_0 = 16$

```
In[30]:= b[a0_, a1_, a2_] :=
  Solve[Table[SeriesCoefficient[Sum[a[i] × A[q]^(5 (8 - i)) × B[q]^(5 i), {i, 0, 8}], {q, 0, l}] == SeriesCoefficient[f[a0, a1, a2] , {q, 0, l}],
    {l, 0, 10}], Table[a[i], {i, 0, 8}]];

In[31]:=
  b[16, -9, 1]

Out[31]= {{a[0] → 0, a[1] → 0, a[2] → 0, a[3] → 0, a[4] → 0, a[5] → 1, a[6] → 0, a[7] → 0, a[8] → 0}}
```

Figure 5.3: Basis representation of the solution $(16, -9, 1)$

$(VJV^{-1})^n = VJV^{-1}VJV^{-1}VJV^{-1}\ldots VJV^{-1} = VJIJI\ldots V^{-1} = VJ^nV^{-1}$. This allows us to see relatively easily what the operator U_5 does to the basis representation of the solution $(16,-9,1)$ after n applications. So, evaluating $VJ^nV^{-1}b$, where $b = e_6$ is the basis representation for the solution $(16,-9,1)$, will show us whether the modular form associated with this solution will satisfy a Ramanujan type congruence modulo powers of 5^j, for some specific $j \in \mathbb{Z}^+$. This will occur if and only if each entry in the result is divisible by 5^j.

With this, we are finally ready to prove our final theorem.

Theorem 17. *The triple $(a_0, a_1, a_2) = (16,-9,1)$ corresponds to the modular form $f_{(16,-9,1)}(\tau)$ yields an antichimeral Ramanujan type congruence.*

Proof. We will need to prove two things, which are that for some integer j, $0 < j < m$, the modular form will not yield a Ramanujan type congruence, and that for every other $m > j$, the modular form will yield a Ramanujan type congruence. We first compute $VJ^nV^{-1}e_6$, where VJV^{-1} is the Jordan canonical form of the matrix A of linear transformation for U_5 and $a_0 = 16$. Also, e_6 is the basis representation of the solution $(16,-9,1)$, as stated earlier. This is computed in Figure 5.4.

```
In[32]:= FullSimplify[V.MatrixPower[J, n].Inverse[V].{0, 0, 0, 0, 0, 1, 0, 0, 0}, n ∈ Integers]
```

Out[32]=

$$\Big\{0,\ \frac{(75+50i\sqrt{29})^n(-58-127i\sqrt{29})}{418760}+\frac{(75-50i\sqrt{29})^n(-58+127i\sqrt{29})}{418760}+\frac{125^{-1+n}(196\times 5^{2+4n}+361(-2817+3367(-1)^n))}{5875636},$$

$$\frac{9(75+50i\sqrt{29})^n(696-281i\sqrt{29})}{418760}+\frac{9(75-50i\sqrt{29})^n(696+281i\sqrt{29})}{418760}+\frac{125^{-1+n}(-5284\times 5^{2+4n}+3971(-8451+2951(-1)^n))}{5875636},$$

$$\frac{-14050842(-125)^n+607024\times 5^{1+7n}-95592078\times 125^n+101725(75+50i\sqrt{29})^n(524-85i\sqrt{29})+101725(75-50i\sqrt{29})^n(524+85i\sqrt{29})}{293781800},$$

$$\frac{11\times 5^{-2+3n}(2817-4277(-1)^n+292\times 5^{1+4n})}{8138},$$

$$\frac{5^{-2+3n}(47796039+7025421(-1)^n+7753108\times 5^{1+4n})}{5875636}+\frac{(75+50i\sqrt{29})^n(524-85i\sqrt{29})}{2888}+\frac{(75-50i\sqrt{29})^n(524+85i\sqrt{29})}{2888},$$

$$-\frac{9i(75-50i\sqrt{29})^n(-696i+281\sqrt{29})}{418760}+\frac{9i(75+50i\sqrt{29})^n(696i+281\sqrt{29})}{418760}+\frac{125^{-1+n}(1752524\times 5^{2+4n}+3971(-8451+2951(-1)^n))}{5875636},$$

$$\frac{804\times 5^{7n}}{1468909}+\frac{125^{-1+n}(2817-3367(-1)^n)}{16276}+\frac{(75+50i\sqrt{29})^n(-58-127i\sqrt{29})}{418760}+\frac{(75-50i\sqrt{29})^n(-58+127i\sqrt{29})}{418760},\ 0\Big\}$$

Figure 5.4: Evaluation of $A^n e_6$ through Jordan Decomposition

Looking at this result carefully, we see that for $n = 1$, after reducing each fraction to lowest terms, there are 4 terms in different entries that are not divisible by 5. These terms are the last term in the first, second, and second to last lines, as well as the second term in the last line. Every other fraction is divisible by 5, so it follows then that in the main computation, the second, third, seventh, and eighth entries in the resultant vector will not be divisible by 5, thus proving the first condition for antichimerality. To prove the second condition requires a little more analysis.

We can clearly see that if $n \geq 2$, every term in Figure 5.4 is clearly divisible by 5^n, so the only thing we must do is prove that every entry in this resultant vector is actually an integer. To do this, we must first consider our ordered basis for $M_k(\Gamma_1(5))$, which is

$$\left[A^{5k}, A^{5k-5}B^5, \ldots B^{5k}\right].$$

We'll denote the basis representation of a product in $M_k(\Gamma_1(5))$ to be

$$(c_1, c_2, \ldots, c_{k+1}),$$

and define the transformation of this basis representation under the operator U_5^n as

$$(c_{1,n}, c_{2,n}, \ldots, c_{k+1,n}).$$

We also note that the two bases $A^5(q)$ and $B^5(q)$ that make up the ordered basis for $M_k(\Gamma_1(5))$ have series of the form

$$A^5 = q + O(q),$$
$$B^5 = 1 + O(q).$$

This means that the smallest nonzero term of the series expansion for the basis element $A^{5m}B^{5n}$ is q^m, since $A^{5m} = (q+O(q))^m = q^m + O(q^{m+1})$. We also note that, since A^5 and B^5 have integer coefficients,

$$U_5^n(f_{(16,-9,1)}) = \sum_{n=0}^{\infty} k_n q^n,$$

where $k_n \in \mathbb{Z}$. We also define the following function

$$\left[\sum_{n=0}^{\infty} r_n q^n\right]_k = r_k.$$

From what we deduced above and the fact that a Hecke operator is a linear transformation from the vector space $M_k(\Gamma_1(5))$ to itself,

$$U_5^n(f) = c_{1,n}A^{5k} + c_{2,n}A^{5k-5}B^5 + \cdots + c_{k+1,n}B^{5k},$$

which can be written, via the elements generating the bases $A^5 = q + O(q), B^5 = 1 + O(q)$ as

$$c_{k+1,n} + (c_{k,n} + c_{k+1,n}[B^{5k}]_1)q + (c_{k-1,n} + c_{k,n}[A^5B^{5k-5}]_2 + c_{k+1,n}[B^{5k}]_2)q^2 + \ldots.$$

Since A^5 and B^5 are power series with integer coefficients, we have that $c_{k+1,n} \in \mathbb{Z}$, and that $(c_{k,n} + c_{k+1,n}[B^{5k}]_1) \in \mathbb{Z}$, which means that $c_{k,n} \in \mathbb{Z}$. We can use an inductive argument to show that $c_{1,n}, c_{2,n}, \ldots, c_{k+1,n} \in \mathbb{Z}$.

We now consider the expressions in Figure 5.4 again. For the solution $(a_0, a_1, a_2) = (16, -9, 1)$, we notice that we can get all of the fractions in lowest terms, or in other words,

$$c_{r,n} = \frac{5^{n\alpha_r} b_{r,n}}{d},$$

where $\gcd(5,d) = 1$, and $\alpha_r \in \mathbb{Z}^+$. We'd like to show that $c_{r,n}$ is an integer multiple of $5^{n\alpha_r}$. We first need to show that $b_{r,n} \in \mathbb{Z}$, which we can deduce by noting that $b_{r,n} \in \mathbb{R}$ and that $b_{r,n} \in \mathbb{Z}[i\sqrt{29}] = \{a + i\sqrt{29}b | a, b \in \mathbb{Z}\}$, since every imaginary part in Figure 5.4 always appears with its complex conjugate. This implies that $b_{r,n} \in \mathbb{R}$, and therefore,

$$b_{r,n} \in \mathbb{R} \cap \mathbb{Z}[i\sqrt{29}],$$

which means that $b_{r,n} \in \mathbb{Z}$, as desired. The second thing that we have to do is show that $d | b_{r,n}$, which we can easily deduce because for every r, $c_{r,n} \in \mathbb{Z}$, and d is relatively prime to 5. Thus, we have that $d|b_{r,n}$, which shows us that the computation VJ^nV^{-1} yields a vector with entries that are integral multiples of $5^{n\alpha_r}$, where $\alpha_r \geq 1$. Thus, the modular form $f_{(16,-9,1)}(\tau)$ associated with the solution $(16,-9,1)$ yields an antichimeral congruence modulo powers of 5^j. That is, for $j \geq 2$, the Fourier coefficients $\mathscr{P}_{(16,-9,1)}(5^j n-3)$ of $f_{(16,-9,1)}$ satisfy a Ramanujan type congruence. Indeed,

$$f_{(16,-9,1)} = q^3 \sum_{n=1}^{\infty} \mathscr{P}_{(16,-9,1)}(n)q^n = q^3 + 9q^4 + 44q^5 + 155q^6 + 450q^7 + 1143q^8 + \cdots,$$

and since $44 \not\equiv 0 \pmod 5$, we have established that the Ramanujan type congruence is not satisfied modulo 5. □

A similar method of proof should suffice to prove each antichimeral congruence with $a_0 = 16$, since the matrix U_5 is fixed for each specific a_0. We then compute the Jordan decomposition of the matrix of linear transformation A for the operator U_5 and $a_0 = 16$, and compute the basis representation of the solution in question.

5.2 Additional Remarks

As we've discovered, there were many topics that we could have expanded on, if we had the time to devote to them. A couple of the directions that deserve exploration are these:

- It would be beneficial to find a way to characterize conditions to determine whether or not a solution (a_0, a_1, a_2) yields a modular form that satisfies an antichimeral Ramanujan type congruence.
- It would also be beneficial to find ways to reduce computation times, especially for high values of a_0.
- While antichimeral Ramanujan type congruences were only observed for the prime $p = 5$, it is conjectured that for other primes p, there are sets of exponents $(a_0, a_1, a_2, \ldots, a_{(p-1)/2}$ that will give a modular form that satisfies an antichimeral Ramanujan type congruence.

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BIOGRAPHICAL SKETCH

Ryan Mowers earned a Master's of Science degree from the University of Texas Rio Grande Valley in May 2023. When he was young, he realized that he had a love for all things math-related thanks to the UIL Academics program that was run throughout the state of Texas. He was even recognized as one of the best people in the program during his high school years by being a state champion. Throughout his college days, he learned many things about the world and how it works, and is currently on the path to publish his findings regarding these modular forms in a joint research paper. He can be contacted by email at ryan.mowers.southhawks@gmail.com.