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Particle Trajectories in Shallow Water Models

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PARTICLE TRAJECTORIES IN SHALLOW WATER MODELS

A Thesis

by

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ABSTRACT

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In this work, we study particle trajectories under shallow water waves. We examine equations such as the Korteweg-de Vries and systems dealing with Boussinesq and Euler's Equations to find relationships between particles' irrotational velocities. Their solutions and behavior when modeling interacting surface waves will be explored. An attempt to find approximate solutions with different parameters, such as small amplitude and long-crested waves, that will lead to new information and study will be discussed.

DEDICATION

Dedicated to my mother, Carmen, who always makes me feel special and deserving, my grandfather, Virgilio, who never fails to love me, and my best friend, Miriam, who believes in me even when I do not have the energy to do so.

ACKNOWLEDGMENTS

I would like to acknowledge Dr. Vesselin Vatchev for the guidance and mentoring throughout the past two years to perfect my thesis and for encouraging me to pursue great things with my abilities and knowledge in math. I would also like to acknowledge Dr. Zhijun Qiao for his support and help throughout my graduate career. Additional acknowledgements go to the mathematics department staff and professors as well as my committee members Dr. Sergey Grigorian and Dr. Dambaru Bhatta.

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CHAPTER I

INTRODUCTION

The disturbance of the state of equilibrium of a type of wave is known as wave motion. This can apply to most types of waves, whether they be water, light, or sound. In this work, water waves will be our main focus, in particular, surface waves in water. The study of different waves is often a topic of importance in physics and engineering. However, in mathematics, it is also a large contributor of intrigue. The focus of this paper will be on particle trajectories under surface waves and using different methods to study equations and how their possible solutions will lead to new approximations.

1.1 Preface

1.1.1 Solitons and Particle Trajectories

Before introducing the work, we must understand what we will examine. Although waves are our main focus, it is important to note what makes up these waves. The surface of a water wave is a critical component that deserves a lot of attention. It is that barrier that seals any and all particles and matter beneath it. The surface wave travels along the top of a substance. It holds everything together and can be modeled through different mathematical equations that we will go into detail further in this paper. The fluid we will study is going to be referred to as in-compressible and in-viscid. This simply means the fluid flow has a constant density throughout and is considered an ideal fluid with no viscosity.

Additionally, the word soliton will be heard throughout the paper, and it is important to note what it describes. A soliton is a non-dispersive, solitary, traveling wave that can maintain its shape and has constant velocity in its path. Soliton will be used to describe the surface wave and its characteristics. Underneath the surface wave, we have matter. This matter is made up of small particles that travel with the wave in different movements and directions. The word irrotational will be used to describe these particles throughout this paper. Simply put, it means the particles underneath the surface wave do not rotate on their own axis. These particles and their trajectories are going to be a great focus of study in this paper.

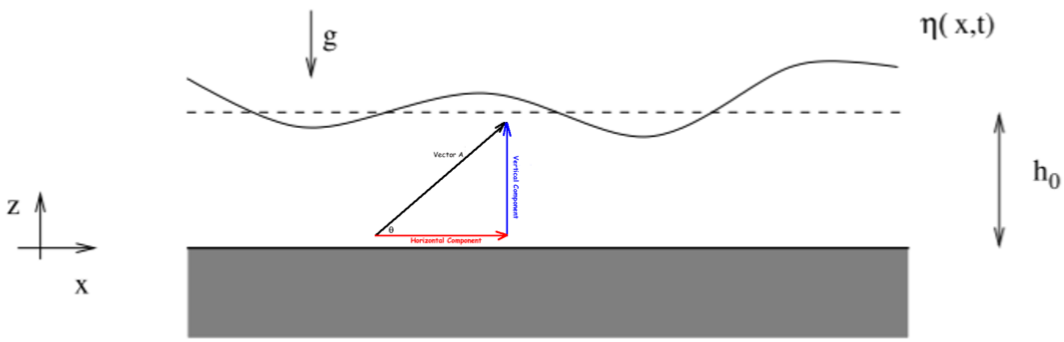

Figure 1.1: Particle Trajectory Model

In Figure 1.1, g represents the gravitational force, $\eta(x,t)$ is representing the surface wave we will study, h_0 is the undisturbed depth, and below the surface wave the particle trajectory model is shown.

1.1.2 Setting

For this thesis our setting will be simple. We will be looking at a cross section of a channel with constant depth and approximate uniformity across the entire fluid. For this scenario, we will have a small amplitude and long wavelength. The situation will be modeled using very small parameters showcasing the circumstances at hand. We will also examine a scenario of a cross section of a channel with multiple waves either traveling in the same or opposite direction. The environment, however, remains the same. The fluid will be in-compressible and in-viscid as well as irrotational. We will also ignore surface tension effects safely with these scenarios.

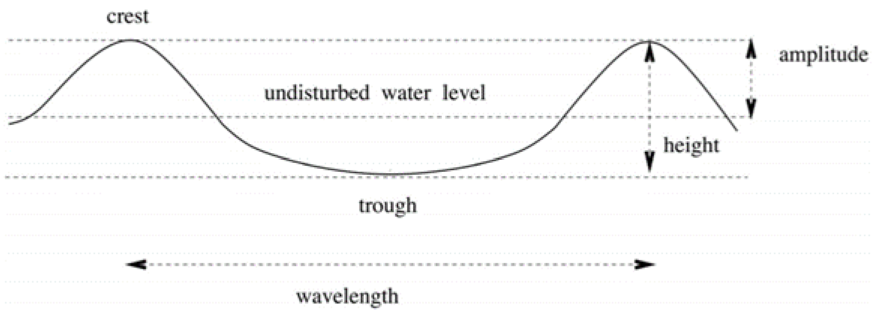

Figure 1.2: Wave Model Characteristics

1.2 Background

To study these water waves and the particles beneath them, we must have a general understanding of how to model the situation. Because of this, we will dive into popular mathematical equations, systems, and approximations that will fit this scenario. More specifically, we will utilize the work of mathematicians such as, Leonhard Euler, Joseph Boussinesq, Diederik Korteweg, Gustav de Vries, and several others.

1.2.1 Euler Model

In the mid 1700s, Leonhard Euler, Swiss Mathematician, studied under the guidance of Daniel Bernoulli, who's great contributions to fluid dynamics and probability and statistics continue to be studied today. Euler's equations, used to relate velocity, pressure, and density of a fluid flow, will be the main topic of study in this work. These equations are a system of low order, non-linear, partial differential equations and are a more general, simplified form of the Navier-Stokes equations, which include temperature and viscosity. Euler's equations can be studied in the form of the following:

$$\frac{Du}{Dt} = -\nabla w + g$$
$$\nabla u = 0$$

where u is the velocity vector, w is work, and g is acceleration. ∇ is the gradient, and $\frac{Du}{Dt}$ is the time derivative. Additionally, for a flow with uniform density, the following holds:

$$w \equiv \nabla\left(\frac{p}{\rho_0}\right) = \frac{1}{\rho_0}\nabla p \tag{1.1}$$

where ρ_0 is the density and p will denote pressure. In this work, Euler's equations will be studied in the two-dimensional form of:

$$u_t + uu_x + vu_z = -p_x \tag{1.2}$$

$$v_t + uv_x + vv_z = -p_z - g \tag{1.3}$$

These equations will be used to study particle trajectories, where u and v are the horizontal and vertical velocities, respectively, p denotes pressure, and g will be the gravitational constant. Unfortunately, this group of equations cannot be solved in a simple manner and the p term creates a new difficulty. Our goal will be to find or approximate equations satisfying u and v in the system. The method we will use to find these solutions will be by using educated assumptions to the solutions for the particle trajectories. By imposing restrictions on u and v and deriving simpler models that preserve the general characteristics of the Euler systems, we can reach a positive outcome.

If we discuss Euler's Equations, it is a must for us to discuss Joseph Boussinesq's, French mathematician's, contributions to this same matter. In fact, Boussinesq introduced the first model dealing with a situation just like this.

1.2.2 Boussinesq

Joseph V. Boussinesq is well known for his paper that responds to theories about solitary waves known as Boussinesq approximation. This approximation takes into account the velocity flow in the horizontal and vertical directions. With this, a system of non-linear partial differential equations (PDEs) are introduced as Boussinesq-type equations. This approximation is set for long-crested wavelengths. Moreover, the Boussinesq approximation can be used to eliminate the vertical coordinate from the equations used, and solely depend on the surface wave and horizontal velocity. This strategy will be discussed later in this paper as well.

The general form of the Boussinesq equation is

$$\eta_{tt} - c^2\eta_{xx} - \frac{3c^2}{h}(\eta_x^2 + \eta\eta_{xx}) - \frac{c^2h^2}{3}\eta_{xxxx} = 0 \tag{1.4}$$

It describes the gravity-induced surface waves as they propagate at a constant linear speed, $c = \sqrt{gh}$, in a canal of uniform depth h. Using the Boussinesq approximation will facilitate things since the surface wave will be the main use.

The most popular way to find the solution for fluid dynamics systems is by asymptotic expansion. The method of asymptotic expansion is used to derive different equations such as the KdV equation or the Camassa-Holm equation by creating a formal series of functions with limiting terms that approximate a given function. It will be studied in further detail as we go on.

Now that we have discussed the importance of using the Boussinesq equation among others, it is imperative to continue with a well known and well defined model of shallow water waves involving Diederik Korteweg and Gustav de Vries, Dutch mathematicians.

1.2.3 Korteweg-de Vries

The origin of this next equation modeling is said to be from Boussinesq and later rediscovered and slightly altered by Dutch mathematicians, Diederik Korteweg and Gustav de Vries in the 1890s. The Korteweg-de Vries equation, is a long studied mathematical equation that models the evolution of long-crested traveling, shallow water waves at the surface of a fluid. The equation is a non-linear PDE of a function, η, that depends on two real variables x, space, and t, time. It models as follows:

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0 \tag{1.5}$$

The subscripts here η_t and η_x are partial derivatives of the function with respect to t and x respectively. With this equation, there are no initial conditions or boundary conditions, and it only represents solitary (alone standing) waves. Additionally, there is one solution that is well known for this equation, generally called the solitary wave solution:

$$\eta(x,t) = \frac{1}{2}c\text{sech}^2\sqrt{c}(x - ct + x_0) \tag{1.6}$$

It is important to note that $c > 0$, leads to the wave always traveling to the right. This solution has several parameters. c is our amplitude and x_0 is the initial placement on the surface wave. We will also see a scenario using the KdV Equation involing 2 solitons traveling in the same direction and examine particle trajectories under the interacting surface waves.

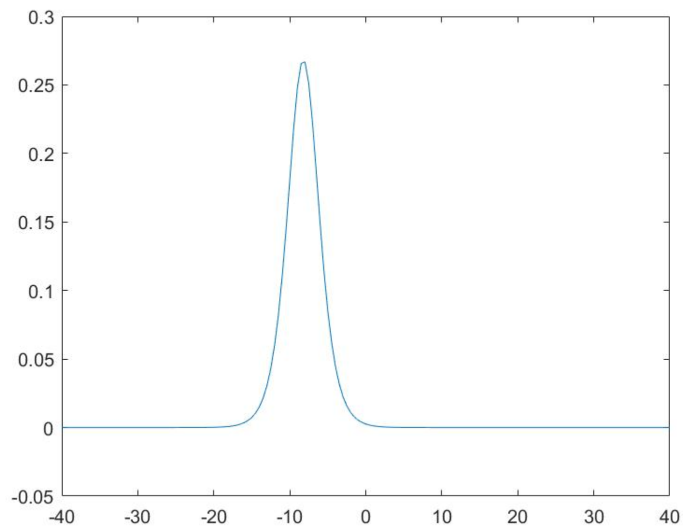

Figure 1.3: Single Traveling Wave Example

CHAPTER II

A KDV MODEL FOR APPROXIMATION

Our topic in this chapter is studying the classical problem of elastic collisions of traveling waves which arise as solutions of nonlinear partial differential equations. Examining these wave collisions with new and intricate methods will allow us to explore properties of multiple solitary waves, which emerge from the subtle balance between non-linearity and dispersion. It is also important to note we are looking closely at the particular solutions of the wave equations during and after collision. After wave collisions, these solutions have the quality of preserving their shapes and velocities during propagation. To study the process during interaction, a simplified explicit solutions or high precision approximations will be sought to facilitate theoretical and numerical approaches.

2.1 Model for Undirectional Waves

The equation under investigation is derived under the assumption of incompressibility and inviscidness of a fluid. Moreover, it is supposed that the fluid flow is irrotational and two-dimensional, and that the free surface can be described by a single-valued function $\eta(x,t)$. Our domain, which extends to infinity in the positive and negative x-direction, is the following:

$$\{(x,z) \in R^2 | 0 < z < h_0 + \eta(x,t)\} \tag{2.1}$$

On this domain, h_0 represents the undisturbed depth of the fluid. Additionally, Euler's equations can be studied and represented in two-dimensional manner previously mentioned as:

$$u_t + uu_x + vu_z = -p_x \tag{2.2}$$

$$v_t + uv_x + vv_z = -p_z - g \tag{2.3}$$

Here (u, v) represents the velocity field while p denotes the pressure and g denotes the gravitational force and is constant. The density is assumed to be unity. These equations are used in fluid dynamics and are used to describe the free surface elevation and the velocity potential on the parameterized free surface. The two dimensional equations are non linear and have dynamical boundary conditions.

The incompressibility of the fluid, and the irrotationality of the flow are expressed, respectively, by $u_x + v_z = 0$ and $u_x - v_z = 0$. The free-surface boundary conditions are given by requiring the pressure to be equal to atmospheric pressure at the surface if surface tension effects are neglected, and the kinematic boundary condition: $p = p_{atm}$ and $\eta_t + u\eta_x = v$ at $z = h_0 + \eta(x,t)$.

We will restrict the scenario by considering only unidirectional waves. By doing this, we assume a relationship of the form $w = \eta + \varepsilon f[\eta]$. This relationship is between w, the horizontal velocity at the mean height, and the elevation η. The horizontal velocity w can be approximated by the surface wave in different rates of approximations by following Borluk's method as:

$$w = \eta + A(\eta) + \delta^2 B(\eta) + \varepsilon^2 C(\eta) + \varepsilon\delta^2 D(\eta) + \delta^4 E(\eta) \tag{2.4}$$

Here ε and δ are arbitrary constants. In this instance $\varepsilon = \delta^2$. This will be broken down later in the paper. However, the Korteweg-de Vries (KdV) equation can be approximated by using some conditions as $A = -\frac{1}{4}\eta^2$ and $B = \frac{1}{6}(2-3\sigma)\eta_{xx}$.

We are going to study the KdV equation in the non-dimensional form of:

$$\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} = 0 \tag{2.5}$$

We also focus on the particle paths in the fluid due to the passage of a solitary wave at the surface. The solitary-wave solution of the KdV equation in this case is given by

$$\eta = \eta_0 \text{sech}^2\Big(\frac{\sqrt{3\eta_0}}{2}(x - x_0 - ct)\Big) \tag{2.6}$$

Where η_0 is the amplitude, x_0 is the initial location of the wave crest, and the phase velocity is given by $c = 1 + \frac{\eta_0}{2}$ Additionally, we have the relationship represented by $f[\eta]$ as:

$$f(x,z,t) = \eta - \frac{1}{4}\eta^2 + (\frac{1}{3} - \frac{d^2}{2})\eta_{xx} \tag{2.7}$$

according to Borluk. Here the depth level will be denoted by d. We attempt to find more solutions using the Wronskian method. Let $\theta_{i,j}(x,t) = m_{i,j}x - 4m_{j,i}^3 t$ where $j = 1,2, i = 1,2$ and $\phi_j = \sum_{i=1}^{2} \varepsilon_i e^{\phi_{j,i}}$ and W be the Wronskian determinant defined as

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} \end{vmatrix} \tag{2.8}$$

We obtain solutions to the KdV equation and represent the surface wave as

$$\eta(x,t) = 2(\log(W))_{xx} \tag{2.9}$$

Plotting the surface wave with parameters $m_1 = 0.2$ and $m_2 = 0.4$ at different times will look like the following

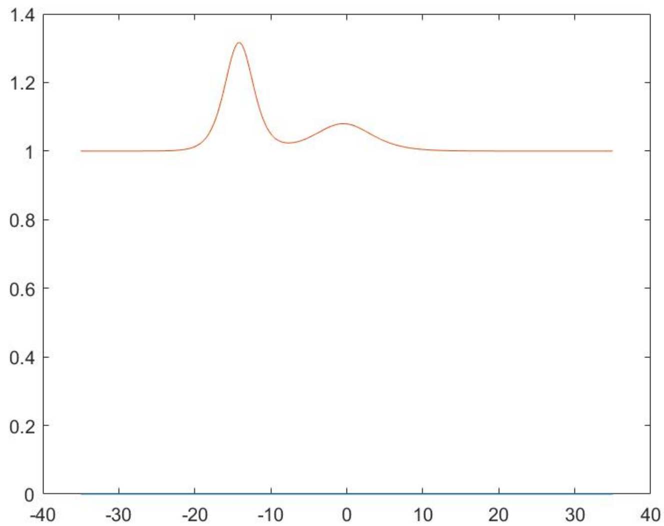

Figure 2.1: KdV Unidirectional Waves t = -20

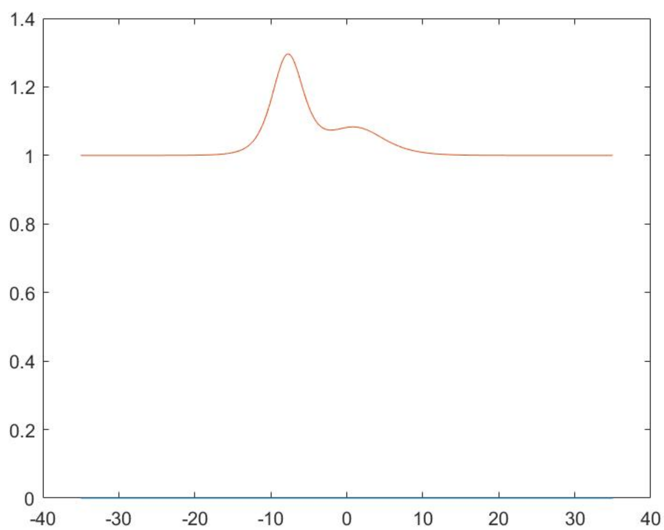

Figure 2.2: KdV Unidirectional Waves t = -10

We see two chasing solitons traveling to the right.

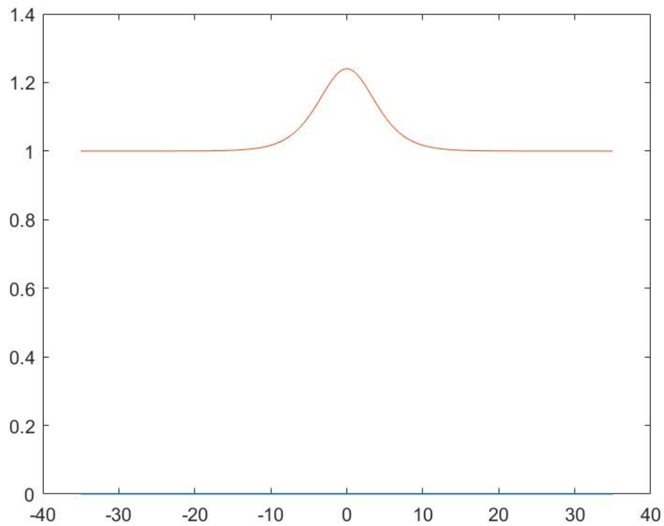

Figure 2.3: KdV Unidirectional Waves t = 0

At $t = 0$ the waves collide and look like a single soliton. However, after the interaction they regain their form and maintain their speed as seen below.

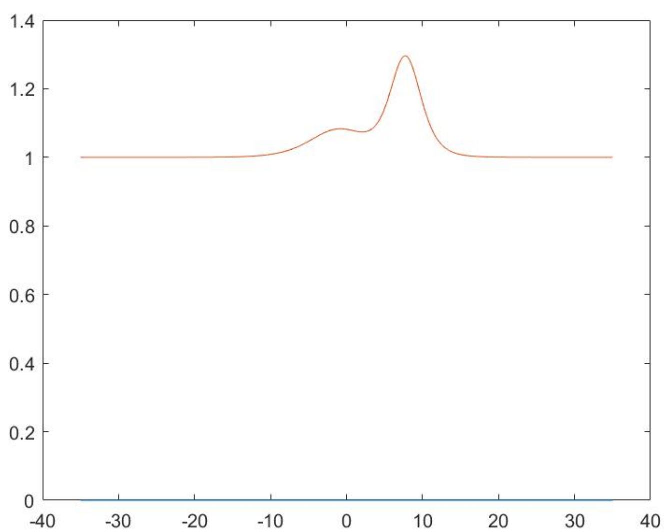

Figure 2.4: KdV Unidirectional Waves t = 10

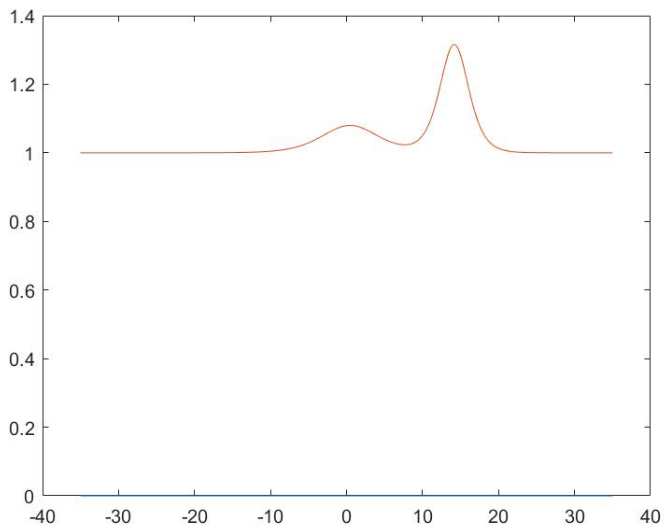

Figure 2.5: KdV Unidirectional Waves t = 20

It will be particularly interesting to examine the particle trajectories below the surface wave before, during, and after soliton interaction. This will be explored in the next section.

2.1.1 Particle Trajectories

We can represent the velocities u, v in Euler's equations approximated by (x,z,t) as the following:

$$u(x,z,t) = f + \frac{1}{2}(d^2 - z^2) f_{xx} \tag{2.10}$$

$$v(x,z,t) = -z f_x \tag{2.11}$$

The velocities, u and v, can now be approximated by η and a new relationship has formed. Taking the functions $\xi(t)$ and $\zeta(t)$ to describe the x-coordinate and z-coordinate, respectively, of a particle originally located at $(x,z) = (\xi_0, \zeta_0)$, the particle motion is described by the nonlinear system of ODE

$$\frac{\partial \xi}{\partial t} = u(\xi(t), \zeta(t), t) \tag{2.12}$$

$$\frac{\partial \zeta}{\partial t} = v(\xi(t), \zeta(t), t) \tag{2.13}$$

In the case of a single soliton or periodic wave trend, the trajectories were studied by Constantin, Borluk and Kalisch.

In order to follow the trajectories and determine a path that they take, we must solve the system with (2.10) (2.11). By doing so using a computer algebra system, we can obtain models such as the following.

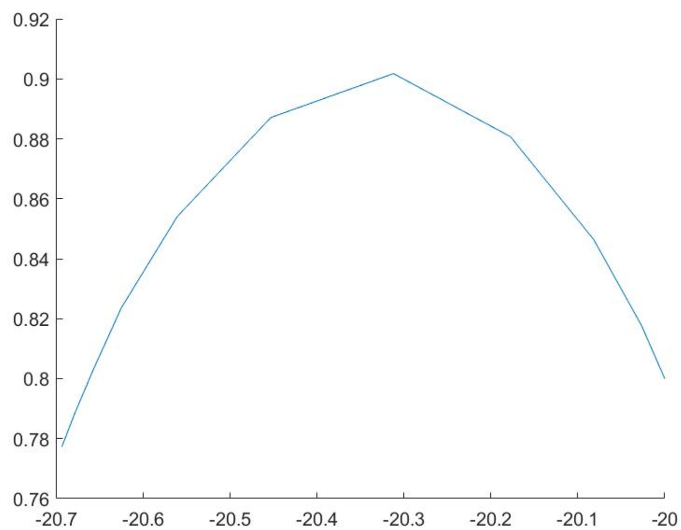

Figure 2.6: KdV Particle Trajectories t = -20 d = 0.8

The figure above shows the path a particle takes under the surface wave at time $t = -20$ and depth $d = 0.8$. It is a short span of time the particle raises and returns to its original depth in a quick motion. This will be more notable when we compare paths at different times.

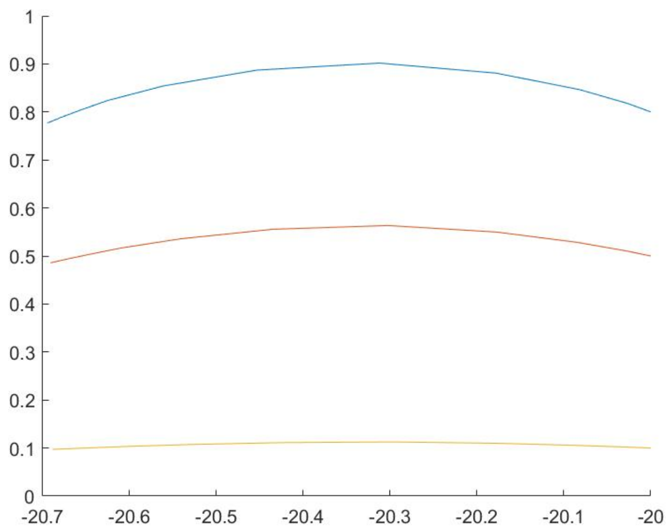

Figure 2.7: KdV Particle Trajectories $t = -20 d = 0.8, 0.5, 0.1$

Figure 2.7 shows particles' path at different depth levels. You can see that the closer to the surface the particle is the more it raises. Overlapping the surface wave and particle trajectories, we can get a clearer picture of how they behave.

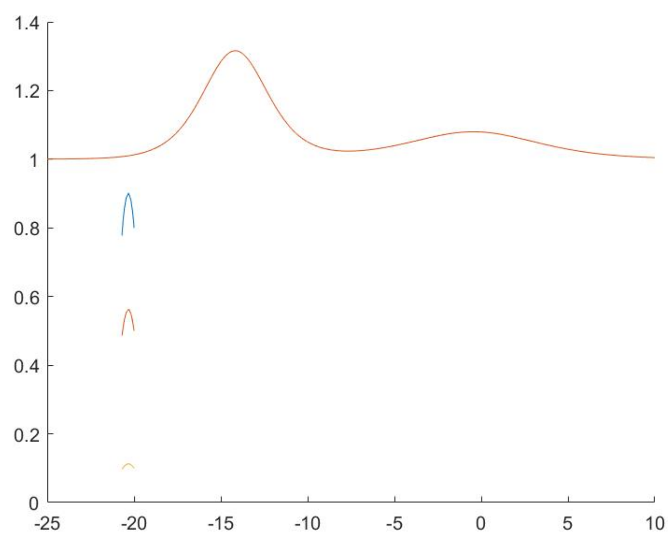

Figure 2.8: Surface Wave and Particle Trajectories Overlapped at $t = -20$

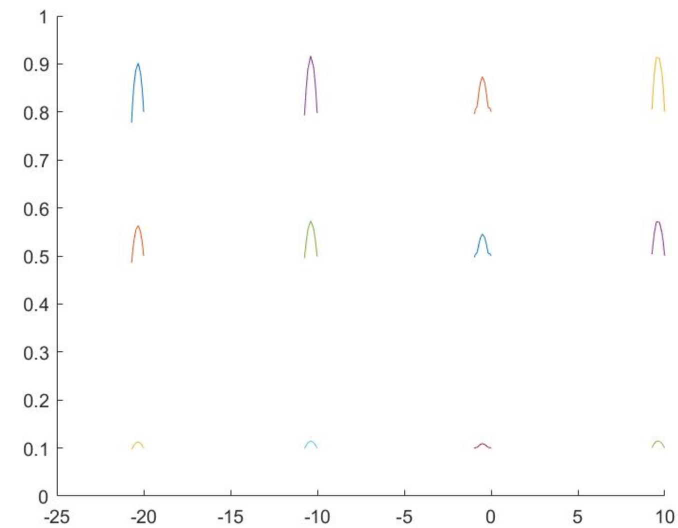

Figure 2.9: Particle Trajectories under Unidirectional Multi-Soliton at $t = -20, -10, 0, 10$ and $d = 0.8, 0.5, 0.1$

The activity when the waves interact is particularly interesting and can be seen up close below.

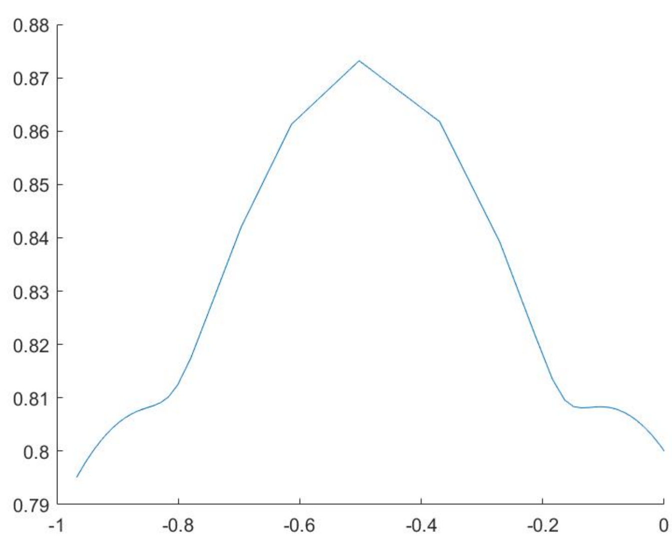

Figure 2.10: Particle Trajectory $t = 0$ and $d = 0.8$

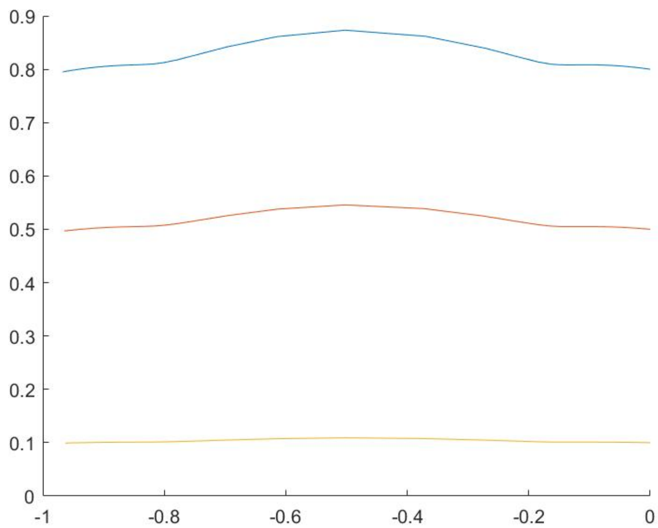

Figure 2.11: Particle Trajectories $t = 0$ and $d = 0.8, 0.5, 0.1$

We can see the particle slowly rises as the first wave comes, then as the waves collide, the particle moves up at a greater rate with the second wave's momentum. Finally, as the waves separate, the particle comes back down slowly.

CHAPTER III

SMALL PARAMETER APPROXIMATION

In this chapter, we will look at approximating systems of different partial differential equations that will relate the surface wave in our scenario to particle trajectory velocities.

3.1 Examining High Order Boussinesq Systems

The systems

$$\begin{aligned} \eta_t + w_x + (w\eta)_x + aw_{xxx} - b\eta_{xxt} &= 0, \\ w_t + \eta_x + ww_x + c\eta_{xxx} - dw_{xxt} &= 0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \eta_t - b\eta_{xxt} + b_1\eta_{xxxxt} =& - w_x - (\eta w)_x - aw_{xxx} \\ & + b(\eta w)_{xxx} - \left(a + b - \frac{1}{3}\right)(\eta w_x x)_x - a_1 w_{xxxxx} \\ w_t - dw_{xxt} + d_1 w_{xxxxt} =& - \eta_x - c\eta_{xxx} - ww_x - c(ww_x)_{xx} - (\eta\eta_{xx})_x \\ & + (c + d - 1)w_x w_{xx} + (c + d)ww_{xxx} - c_1\eta_{xxxxx} \end{aligned} \tag{3.2}$$

of partial differential equations are explained in this section. Deriving these equations take a standard approach, some, however, are derived for the first time. The system (3.1) is considered the Boussinesq systems which are first-order approximations to the Euler equations. In this case, (3.2) are second-order approximations. Higher order approximations are necessary to provide context for further modeling. Let Ω_t, the derivative of Omega with respect to time, be the domain in R^3, the real coordinate space of 3 dimensions, which is occupied by an inviscid, incompressible fluid at time t. The system describing the motion of such fluid, one that is not thick and has a constant density under different pressures, is the classical Euler equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = -g\vec{k}, \quad \text{in } \Omega_t, \tag{3.3}$$

$$\nabla \cdot \vec{v} = 0, \quad \text{in } \Omega_t, \tag{3.4}$$

where g denotes the acceleration of gravity, ∂ is the partial derivative, $\vec{v} = u\vec{i} + v\vec{j} + \vec{k}$ denotes the velocity field, $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along the x—, y—, and z—axis, respectively, in R^3, ∇, the gradient which describes the rate and direction of change in a scalar field, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$ and p denotes the pressure field. Assuming the initial velocity field is irrotational so that $\nabla \times \vec{v} = 0$, the cross product, Helmholtz's vorticity principle that states in the absence of rational external forces, a fluid that is initially irrotational remains irrotational, implies that the velocity field remains irrotational. Counting on a regular solution, we note that

$$\vec{v} = \nabla \phi \tag{3.5}$$

for some potential function $\phi = \phi((x,y,z),t)$. It follows from (3.4) that ϕ satisfies Laplace's equation

$$\Delta \phi = 0 \tag{3.6}$$

in Ω_t, for each t. In this instance, $\Delta = \nabla \cdot \nabla = \nabla^2$ is the Laplace operator, given by the divergence of the gradient of a function.$\nabla \cdot$ is the divergence operator and ∇ is the gradient operator. Here ϕ plays the role of a twice-differentiable real-valued function. The Laplace operator maps a scalar function to another scalar function.

View the boundary of Ω_t as consisting of two parts: the fixed surface located at $z = -h(x,y)$, and the free surface $z = \eta(x,y,t)$. The free surface refers to the changing wave motion style it takes. The domain is taken to be unbounded in the horizontal directions stretching from the negative infinity to positive infinity so that lateral boundaries do not intrude at this stage. Note that $\eta(x,y,t)$ is a fundamental unknown of the problem. On the fixed portion of the boundary, the condition of impermeability (no flow through the solid boundary) $\vec{v}\cdot\vec{n}=0$ is satisfied with $\vec{n}$ being the normal, or perpendicular, direction of the surface, which means

$$\phi_x h_x + \phi_y h_y + \phi_z = 0, \quad \text{on } z = \eta(x,y,t), \tag{3.7}$$

Recall that h is a function of a fixed surface, while η represents the free surface. Since the free surface is a material surface, it satisfies the kinematic condition $\frac{D(\eta-z)}{Dt}=0$, where $\frac{D}{Dt}$ is the usual material derivative $\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$. It is important to note that the material derivative describes the — change of a certain small amount of fluid with time as it flows along its trajectory. Thus, we have

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \quad \text{on } z = \eta(x,y,t). \tag{3.8}$$

Assuming the pressure on the free surface is equal to the ambient air pressure, it follows from (3.3) and (3.5) that the Bernoulli condition, which states that the flow must be steady so that flow parameters at any point cannot change with time, the flow must be incompressible so even though pressure changes, the density must remain constant along a streamline, and the flow's friction by viscous forces must be ignored, given by

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz = 0, \quad \text{on } z = \eta(x,y,t), \tag{3.9}$$

where g is for gravitational acceleration, is satisfied on the free surface as well.

Summarizing the equations for the unknown functions η and ϕ, they consist of (3.6), (3.7), (3.8), and (3.9). This system is challenging to solve either numerically or analytically because Ω_t is changing with time through the evolution of η and the boundary conditions (3.8) and (3.9) on the free surface are nonlinear. Numerical methods are used to solve systems of algebraic equations of linear or non-linear form and are also used to provide approximate solutions to governing equations.

Analytically solving a system means gaining the exact solutions based on mathematical principal which proves to be entirely more difficult in a case such as this. Either route taken can be a challenge. Consider now a simpler case of an open channel in which the fluid motion is irrotational, inviscid, and uniform in the direction of the fluid flow. Suppose the bottom of the channel to be flat and horizontal and let h denote the undisturbed depth of the channel. Then the prior formulation reduces to

$$\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0, \quad \text{in } -h < z < \eta(x,t), \\
\phi_z &= 0, \quad \text{on } z = -h, \\
\eta_t + \phi_x \eta_x - \phi_z &= 0, \quad \text{on } z = \eta(x,t), \\
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz &= 0, \quad \text{on } z = \eta(x,t),
\end{aligned}$$

where the undisturbed free surface is located at $z = 0$, $-\infty < x < +\infty$, for all $t \geq 0$. This system of equations is posed together with suitable boundary conditions as $x \to \pm\infty$ and an initial condition at $t = 0$.

Eliminating the y component, allows restrictions to be imposed. This simplifies the system, which cannot be easily solved, and permits assumptions that will lead to approximate models that preserve the characteristics of the system to be made.

Consider now a practical situation where the free surface has a small amplitude denoted by A, long wavelength denoted by l, a constant channel depth of h, and the classical Stokes number $S = \frac{\alpha}{\beta}$ is of order one. Stokes' law refers to the assumptions for the behavior of a particle in a fluid which are the following: Laminar flow meaning the particles flowing smooth paths in layers, spherical particles, homogeneous or uniform material, smooth surfaces, and particles do not interfere with each other. The Stokes number is a dimensionless number characterising the behavior of particles suspended in the fluid flow often represented as a ratio. We can form the following conditions from the mentioned situation in the following form

$$\alpha = \frac{A}{h} << 1, \qquad \beta = \frac{h^2}{l^2} << 1, \qquad S = \frac{\alpha}{\beta} = \frac{Al^2}{h^3} \approx 1.$$

In this circumstance, the two small parameters α and β may be treated on an equal footing. Choosing β to be the primary parameter, we seek to write solutions of (3.6)-(3.9) in a Taylor series with respect to β, and thereby to obtain approximate equations corresponding to orders of accuracy characterized by β^n for $n = 1, 2, \ldots$

The procedure is most transparent when working with the variables scaled in such a way that the dependent quantities and the initial data appearing in the initial-value problem are all of order one, while the assumptions about small amplitude and long wavelength appear explicitly connected with small parameters in the equation of motion. Simplifying such complex equations can lead us to an approximate solution, and that is the goal. Such consideration leads to the scaled dimensionless variables

$$x = l\tilde{x}, \qquad z = h(\tilde{z} - 1), \qquad \eta = A\tilde{\eta}, \qquad t = l\tilde{t}/c_0, \qquad \phi = gAl\tilde{\phi}/c_0, \tag{3.10}$$

where $c_0 = \sqrt{gh}$. Note that we continue with η being the free surface, h as the undisturbed depth, l is our wavelength, A is the small amplitude, g is the gravitational force, and ϕ is our potential function. In these variables, the last set of equations becomes the system

$$\beta\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0, \quad \text{in } 0 < \tilde{z} < 1 + \alpha\tilde{\eta}(\tilde{x}, \tilde{t}), \tag{3.11}$$

$$\tilde{\phi}_{\tilde{z}} = 0, \quad \text{on } \tilde{z} = 0, \tag{3.12}$$

$$\tilde{\eta}_{\tilde{t}} + \alpha\tilde{\phi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \frac{1}{\beta}\tilde{\phi}_{\tilde{z}} = 0, \quad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}(\tilde{x}, \tilde{t}), \tag{3.13}$$

$$\tilde{\eta} + \tilde{\phi}_{\tilde{t}} + \frac{1}{2}\alpha\tilde{\phi}_{\tilde{x}}^2 = 0, \quad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}(\tilde{x}, \tilde{t}), \tag{3.14}$$

for $-\infty < \tilde{x} < \infty$, $\tilde{t} > 0$. For clarity, we drop the tilde over the new variables in our further machinations.

The next procedure, which is a standard one known as a power series expansion with infinitely many terms, begins by representing ϕ as a formal expansion,

$$\phi(x,z,t)=\sum_{m=0}^{\infty} f_m(x,t)z^m.$$

Demanding that ϕ formally satisfy Laplace's equation, seen in its form of (3.11), leads to the recurrence relation

$$(m+2)(m+1)f_{m+2}(x,t)=-\beta(f_m(x,t))_{xx}, \tag{3.15}$$

for $m=0,1,2,\ldots$. Let $F=f_0$ denote the velocity potential at the bottom $z=0$ and use (3.15) repeatedly to obtain

$$f_{2k}(x,t)=\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}F(x,t)}{\partial x^{2k}}, \qquad k=0,1,2,\ldots$$

Equation (3.12) implies that $f_1(x,t)=0$, so

$$f_{2k+1}(x,t)=0, \qquad k=0,1,2,\ldots,$$

and therefore

$$\phi(x,z,t)=\sum_{k=0}^{\infty}\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}F(x,t)}{\partial x^{2k}}z^{2k}.$$

Substitute the latter representation into (3.13) and (3.14) to obtain a system of equations for $\eta(x,t)$ and $F(x,t)$, as a result we get

$$\eta_t+\alpha\eta_x\sum_{k=0}^{\infty}\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k+1}F}{\partial x^{2k+1}}z^{2k}-\sum_{k=1}^{\infty}\frac{(-1)^k\beta^{k-1}2k}{(2k)!}\frac{\partial^{2k}F}{\partial x^{2k}}z^{2k-1}=0,$$

$$\eta+\sum_{k=0}^{\infty}\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k+1}F}{\partial x^{2k+1}\partial t}z^{2k}+\frac{1}{2}\alpha\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k+1}F}{\partial x^{2k+1}}z^{2k}\right\}^2$$
$$+\frac{1}{2}\frac{\alpha}{\beta}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\beta^k 2k}{(2k)!}\frac{\partial^{2k}F}{\partial x^{2k}}z^{2k-1}\right\}^2=0,$$

on

$$z = 1 + \alpha\eta(x,t).$$

Substituting the value of z into the last two equations leads to

$$\eta_t + \alpha\eta_x \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k+1}} (1+\alpha\eta)^{2k} \right\} \beta^k + \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k+1)!} \frac{\partial^{2k+2} F}{\partial x^{2k+2}} (1+\alpha\eta)^{2k+1} \right\} \beta^k = 0, \tag{3.16}$$

and

$$\eta + \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k} \partial t} (1+\alpha\eta)^{2k} \right\} \beta^k + \frac{1}{2}\alpha \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k+1}} (1+\alpha\eta)^{2k} \beta^k \right\}^2 + \frac{1}{2}\alpha\beta \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\partial^{2k+2} F}{\partial x^{2k+2}} (1+\alpha\eta)^{2k+1} \beta^k \right\}^2 = 0. \tag{3.17}$$

Account is now taken of the formal order of the various terms appearing in (3.16) and (3.17). The parameters α and β have the same small order, while F and η have been scaled so that they and their partial derivatives are of order one. Keeping only the terms in (3.16) and (3.17) which are of lowest order, there obtains the system

$$\eta_t + \frac{\partial^2 F}{\partial x^2} = \text{terms linear in } \alpha,\ \beta,$$
$$\eta + \frac{\partial F}{\partial t} = \text{terms linear in } \alpha,\ \beta.$$

Differentiate the second equation with respect to x and let $\frac{\partial F(x,t)}{\partial x} = u(x,t)$, the scaled horizontal velocity at the bottom of the channel. With this new dependent variable at hand, the last equation becomes

$$\eta_t + u_x = \text{terms linear in } \alpha,\, \beta, \tag{3.18}$$

$$\eta_x + u_t = \text{terms linear in } \alpha,\, \beta, \tag{3.19}$$

which is simply the linear wave equation if the terms of formal order α and β are ignored.

The next order of approximation keeps all the terms in (3.16)-(3.17) which are at most linear in α or β. This leads to the system

$$\eta_t + \frac{\partial^2 F}{\partial x^2} + \alpha \eta_x \frac{\partial F}{\partial x} + \alpha \eta \frac{\partial^2 F}{\partial x^2} - \frac{1}{6}\beta \frac{\partial^4 F}{\partial x^4} = \text{terms quadratic in } \alpha,\, \beta,$$
$$\eta + \frac{\partial F}{\partial t} - \frac{1}{2}\beta \frac{\partial^3 F}{\partial x^2 \partial t} + \frac{1}{2}\alpha \left(\frac{\partial F}{\partial x}\right)^2 = \text{terms quadratic in } \alpha,\, \beta.$$

Differentiate the second equation with respect to x and substitute u for $\frac{dF}{dx}$ as before to recover the first-order Boussinesq system,

$$\eta_t + u_x + \alpha \eta_x u + \alpha \eta u_x - \frac{1}{6}\beta u_{xxx} = \text{terms quadratic in } \alpha,\, \beta, \tag{3.20}$$

$$\eta_x + u_t + \alpha u u_x - \frac{1}{2}\beta u_{xxt} = \text{terms quadratic in } \alpha,\, \beta. \tag{3.21}$$

CHAPTER IV

A BOUSSINESQ MODEL FOR 2 SOLITONS

In this chapter, we will use the Boussinesq equation and model system for traveling surface waves in unidirectional and opposite directions.

4.1 Unidirectional Waves

The focus will be the 2-soliton solutions, thus we will look at the dimensionless form of the Boussinesq equation as such:

$$u_{tt} + a_1 u_{xx} + a_2 (u^2)_{xx} + a_3 u_{xxxx} = 0 \tag{4.1}$$

where $a_i, i = 1,2,3$ are real numbers and $a_2 a_3 \neq 0$. In the special case where $a_3 > 0$ it is known as the good Boussinesq equation and is the following:

$$v_{tt} + (v^2)_{xx} + v_{xxxx} = 0 \tag{4.2}$$

by using the transformation

$$u(x,t) = -\frac{a_1}{2a_2} + \frac{a_3}{a_2} v(x, \sqrt{a_3} t). \tag{4.3}$$

The 'good' Boussinesq equation can be defined as:

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3} u_{xxxx} = 0 \tag{4.4}$$

This will create a relationship dealing with a uni-directional velocity and eliminate the need of the vertical component. The equation is also invariant when changing the variables $x = -x, t = -t$.

One of our goals is to provide constructive analysis of the process of asymptotic expansion using this particular equation. We are going to consider the general 'good' Boussinesq equation in the form

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3}u_{xxxx} = 0 \tag{4.5}$$

Then, solutions can be obtained by using the Hirota 'log' substitution, $v(x,t) = 2log(\phi)_{xx}$ for some potential function ϕ, which are solutions to

$$\phi_{xxx} - \frac{3}{4}\phi_x = \lambda\phi \tag{4.6}$$

for some real λ. The equation above (4.6) is the space equation from the Lax pair. A Lax pair is a pair of operators that correspond to a differential equation. From here, We will continue to look for solutions and approximations using various Boussinesq systems.

We can use the seed solution of

$$\phi(x,t) = A_1 e^{m_1(x-m_1t)} + A_2 e^{m_2(x-m_2t)} + A_3 e^{m_3(x-m_3t)} \tag{4.7}$$

for equation (4.6). Here A_1, A_2, A_3 are real constants and m_1, m_2, m_3 are the roots of the cubic equation

$$\mu^3 - \frac{3}{4}\mu = \lambda \tag{4.8}$$

In an attempt to obtain more solutions, we can use the Wronskian method and the Hirota 'log' substitution. Let $\theta_{j,i}(x,t) = m_{j,i}x - m_{j,i}^2 t + \alpha_{j,i}$, $j = 1,\ldots,n, i = 1,2,3$ and $\phi_j = \sum_{i=1}^{3} \varepsilon_i e^{\theta_{j,i}}$ and W be the Wronskian determinant defined as

$$W(\phi_1,\ldots,\phi_N) = \begin{vmatrix} \phi_1 & \ldots & \phi_N \\ \vdots & \vdots & \vdots \\ \frac{\partial^j \phi_1}{\partial x^j} & \ldots & \frac{\partial^j \phi_N}{\partial x^j} \\ \vdots & \vdots & \vdots \\ \frac{\partial^{N-1} \phi_1}{\partial x^{N-1}} & \ldots & \frac{\partial^{N-1} \phi_N}{\partial x^{N-1}} \end{vmatrix} \tag{4.9}$$

We can obtain a solution to the 'good' Boussinesq equation as the following

$$v = 2(\log W)_{xx} = 2\frac{WW_{xx} - W_x^2}{W^2} \tag{4.10}$$

We will study this to gain a better understanding of the dynamics between multiple solutions. From here we will use the notations of J. L. Bona, M. Chen, and J.-C Saut. Consider the potential for a function $f = logW$, which will be defined later, and $z = 0$, recall h is the undisturbed depth of a channel,

$$\phi(x,z,t) = -f_t(x,t) + \frac{z^2}{2} f_{txx}(x,t) \tag{4.11}$$

, the surface wave $\eta(x,t) = f_{xx}$, the auxillary function $u(x,t) = f_{tx}(x,t)$ satisfy the following relation

$$\eta_t + u_x + \alpha\eta_x u + \alpha\eta u_x - \frac{1}{6}\beta u_{xxx} = \text{terms quadratic in } \alpha,\ \beta, \tag{3.20}$$

$$\eta_x + u_t + \alpha u u_x - \frac{1}{2}\beta u_{xxt} = \text{terms quadratic in } \alpha,\ \beta. \tag{3.21}$$

Here we set the potential function (4.11) in a special form. From here we will be approximating the solution.

Different parameters will yield different results. In this case, we will be using the parameters where $m_1 = \cos(a)$ and $m_2 = \cos(b\frac{4\pi}{3})$ and a and b will be set to $\frac{\pi}{4}$ and $\frac{\pi}{12}$, respectively. Setting this conditions, we are able to plot the surface wave and see the following results.

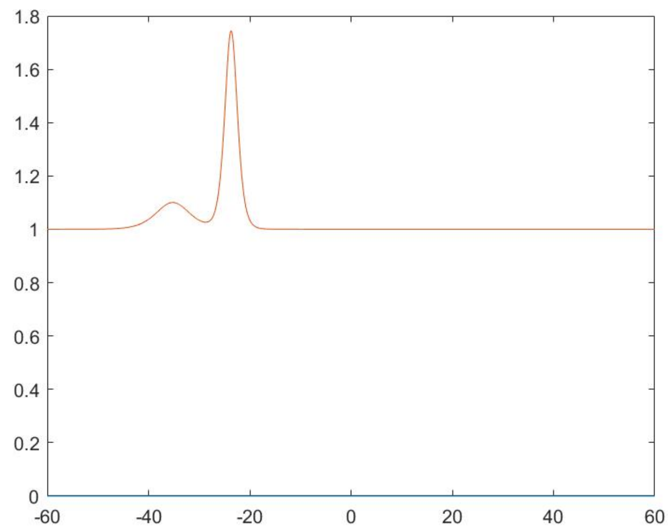

Figure 4.1: Boussinesq Unidirectional Waves t = -35

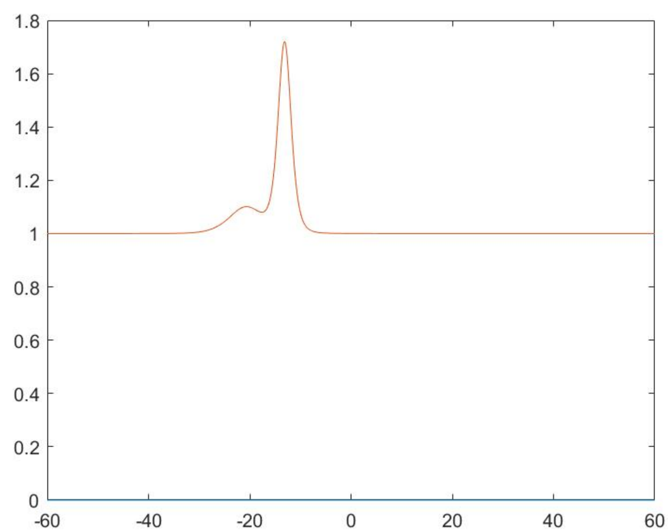

Figure 4.2: Boussinesq Unidirectional Waves t = -20

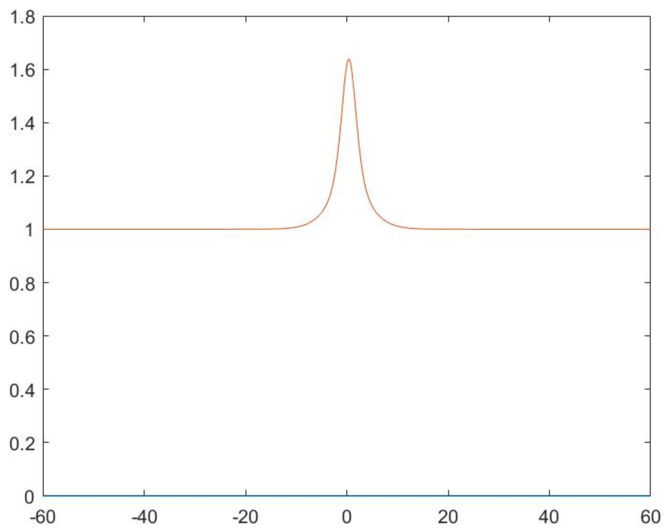

Figure 4.3: Boussinesq Unidirectional Waves at t = 0

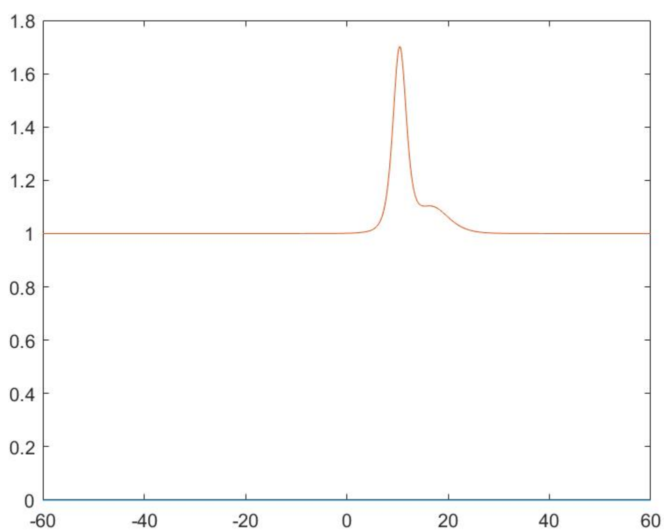

Figure 4.4: Boussinesq Unidirectional Waves at t = 15

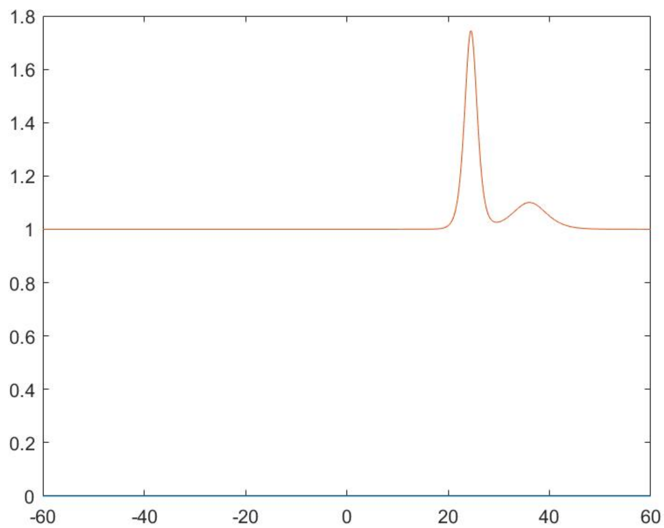

Figure 4.5: Boussinesq Unidirectional Waves at t = 35

In this case, the smaller wave is chasing the larger wave. It is peculiar compared to the previous model from chapters before using the KdV approximation where the larger wave chased the smaller wave. Obviously, these are different conditions when plotting the surface wave, but it is still important to note. It could mean an error was made at some point. However, the parameters that were used led to this.

4.1.1 Particle Trajectories

Substituting ϕ and $\eta = f_{xx}$ directly into Euler's Equations we get for $0 < z < 1+\eta$

$$\phi_{xx} + \phi_{zz} = -ftxx + \frac{z^2}{2} f_{txxx} + f_{txx} = \frac{z^2}{2} f_{txxx} = O(\lambda^5) \tag{4.12}$$

for $z = 0$, $\phi_z = z f_{txx} = 0$ and on the surface level $z = 1+\eta = 1+f_{xx}$

$$\begin{aligned} \eta_t + \phi_x \eta_x - \phi_z &= f_{xxt} + (-f_{tx} + \frac{z^2}{2} f_{txxx}) f_{xxx} - z f_{txx}|_{z=1+\eta} \\ &= \frac{(1+f_{xx})^2}{2} f_{txxx} f_{xxx} - f_{xx} f_{txx} - f_{tx} f_{xxx} = O(\lambda^5) \end{aligned} \tag{4.13}$$

$$\eta + \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) = f_{xx} - f_{tt} + \frac{(1+f_{xx})^2}{2} f_{ttxx} \\ + \frac{1}{2}\Big((-f_{xt} + \frac{z^2}{2} f_{txxx})^2 + (1+f_{xx})^2 f_{txx}^2\Big) \\ = f_{xx} - f_{tt} + \frac{1}{2} f_{tx}^2 + \frac{1}{2} f_{ttxx} + O(\lambda^5). \tag{4.14}$$

Solving this system in a similar manner as before, will give us the particle trajectory paths underneath the surface wave at different depth levels. The results are shown below.

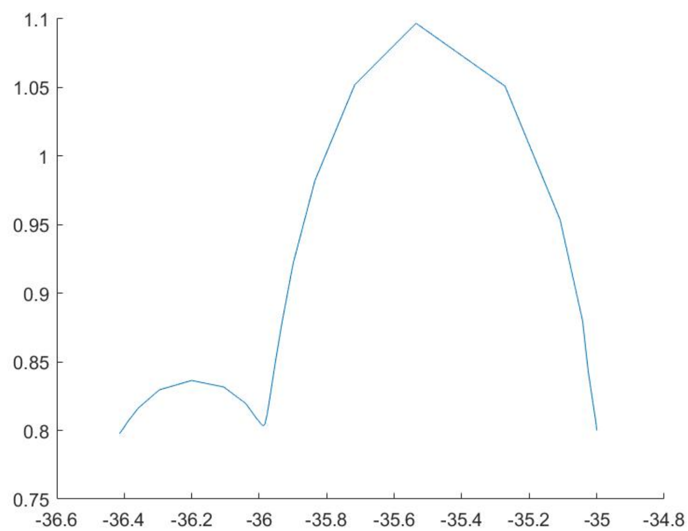

Figure 4.6: Particle Trajectory Path at t = -35 z = 0.8

Figure 4.6 shows a particle path beneath the surface wave at time $t = -35$ and depth level of $z = 0.8$. The behavior of this particle raises some intrigue. The particle seems to move up and come back down to the original depth level before moving up again at a much greater height then returning. Below we can see the behavior at the same time for different levels of depth.

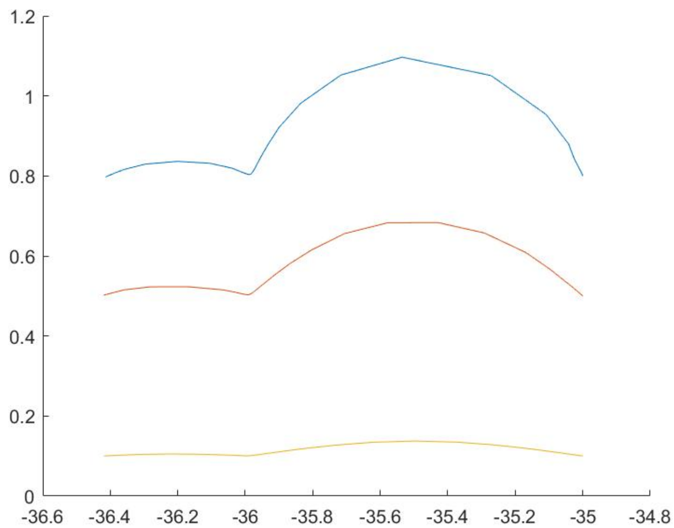

Figure 4.7: Particle Trajectory Path at t = -35 and z = 0.8, 0.5, 0.1

There are similar behaviors at these various depth levels with the largest variant closer to the surface. This can be examined closely in the figure below where the surface wave and particle trajectories are overlapped.

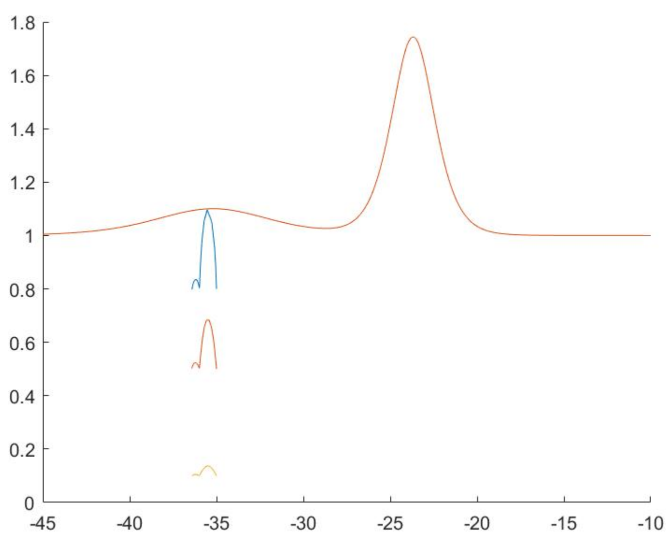

Figure 4.8: Surface Wave and Particle Trajectories Overlapped at t = -35

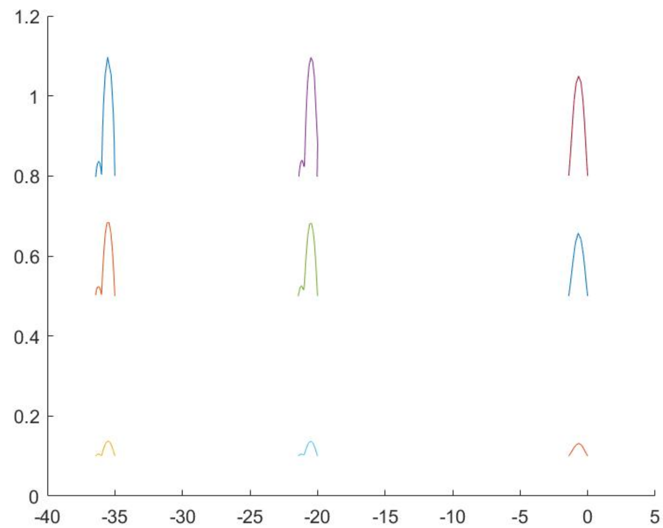

Figure 4.9: Particle Tarjectory Paths at t = -35, -20, 0 and z = 08, .5, 0.1

At different times and depth levels the particles behave accordingly. Figure 4.9 shows the difference in paths taken by each particle depending on time and depth. We can also try to examine the situation at $t = 0$ a little more closely since the path changes from its previous forms. This is the moment of wave interaction.

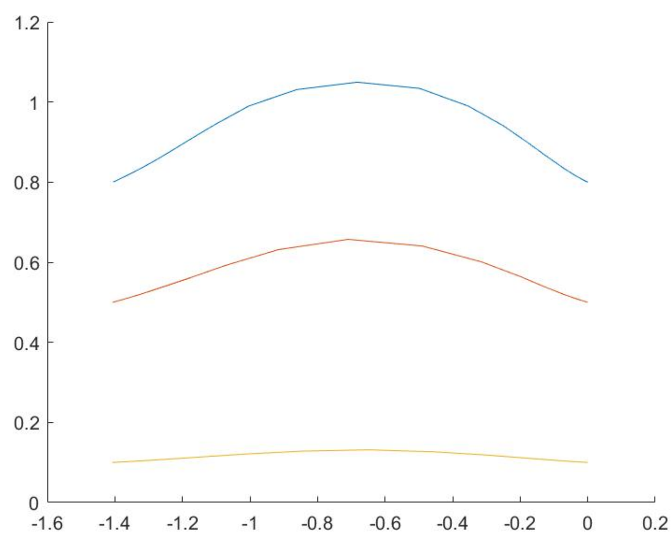

Figure 4.10: Particle Trajectory Path at t = 0 and z = 0.8, 0.5, 0.1

These path have a similar flow as the ones in chapter 2 where we modeled using the KdV equation. There is no deviation from the path once it returns to its original depth level.

4.2 Particular Function for Opposite Traveling Waves

The goal in this section is to study the Wronskian solution for the following equation

$$f_{xx} - f_{tt} + \frac{1}{2}(f_{tx})^2 + \frac{1}{2}f_{ttxx} = 0 \tag{4.15}$$

If we take the mixed derivative f_{tx} and replace it with f_{xx} we can attain the regularized Boussinesq equation. For a single traveling wave $f_t = cf_x$ and the two equations are equivalent to that of Boussinesq's. We will consider a case with higher order approximation of λ^{4+} to the original Euler system in the setting of asymptotic approximation for the parameter λ compared to previously used method of $\lambda \approx (\alpha)^{\frac{1}{2}}(\beta)^{\frac{1}{4}}$.

The one soliton or single soliton solution considered will be

$$f = 6\log(1 + e^{2kx \pm \frac{2k}{\sqrt{1-2k^2}}t}) \tag{4.16}$$

Two solitons traveling in opposite directions can be obtained by setting

$$m_2 = -m_1, n_2 = m_1\left(1 - \frac{m_1^2}{2}\right)^{-\frac{1}{2}}$$

and thus,

$$W(x,t) = 1 + e^{m_1x - n_1t} + e^{-m_1x - n_1t} + e^{-n_1t} = \cosh m_1x + \cosh n_1t \geq 1. \tag{4.17}$$

Additionally, the surface wave will be defined as

$$f_{xx} = 6m_1^2 \frac{\cosh m_1x \cosh n_1t + 1}{(\cosh m_1x + \cosh n_1t)^2} \tag{4.18}$$

for $f = 6\log W$.

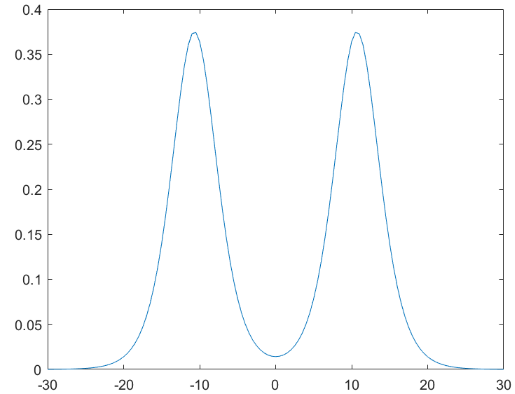

Figure 4.11: Opposite Traveling Waves, m1 = 0.1, t = -10

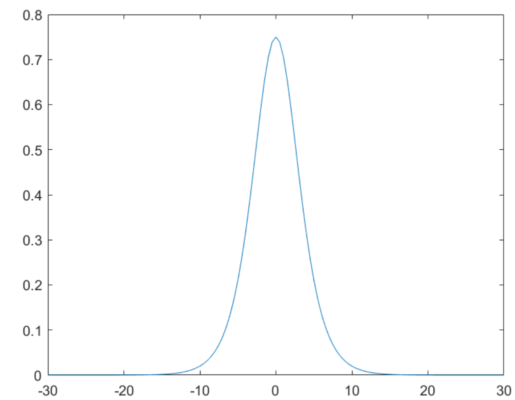

Figure 4.12: Opposite Traveling Waves, m1 = 0.1, t = 0

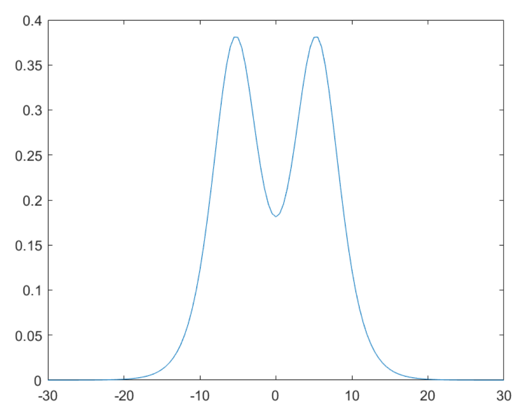

Figure 4.13: Opposite Traveling Waves, m1 = 0.1, t = 5

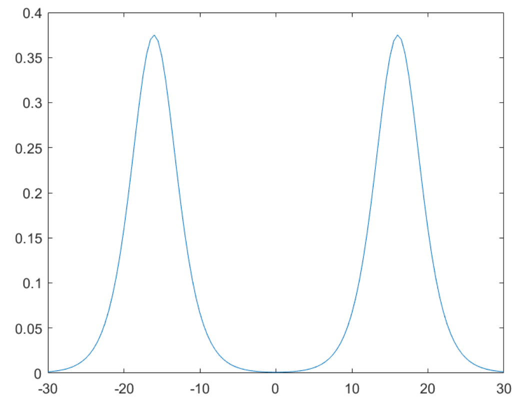

Figure 4.14: Opposite Traveling Waves, m1 = 0.1, t = 15

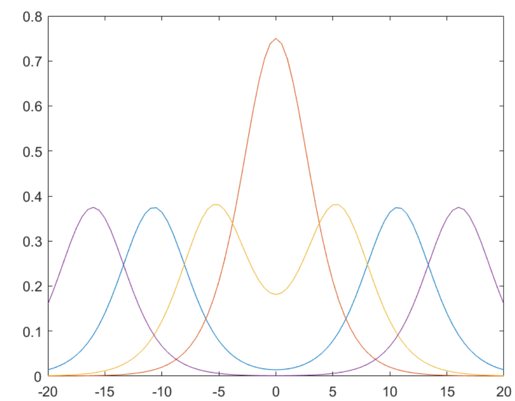

Figure 4.15: Opposite Traveling Waves Overlapped t = -10,0,5,15

The equation (4.18) can be considered as approximating f_{xx} and we define the 'error' function

$$\begin{aligned} E(x,t) &= f_{xx} - \left(f_{tt} + 3(f_{tx})^2 + \frac{1}{2} f_{ttxx} \right) \\ &= \frac{m_1^2 - n_1^2 - m_1^2 n_1^2 + (m_1^2 - n_1^2 + \frac{1}{2} m_1^2 n_1^2) \cosh m_1 x \cosh n_1 t}{(\cosh m_1 x + \cosh n_1 t)^2}. \end{aligned}$$

Setting the relation

$$m_1^2 - n_1^2 + \frac{1}{2} m_1^2 n_1^2 = 0$$

it follows that $n_1 = \pm\frac{m_1}{\sqrt{z-\frac{m_1^2}{2}}}$ and

$$|E(x,t)| = |-\frac{3}{2}\frac{m_1^2 n_1^2}{W^2}| \leq \frac{3}{2}\frac{m_1^4}{\sqrt{1-\frac{m_1^2}{2}}} = O(\lambda^4)$$

since $\lambda = |m_1|$. The magnitude of f_{xx} is as expected $O(m_1^2)$ and it approaches the exact single soliton solution when $|t| \to \infty$.

4.2.1 Particle Trajectories

In this section we will examine the particles underneath the surface wave. Here we focus on the path the particles take. The system for particle trajectories with potential $\phi = -f_{tx} + \frac{z^2}{2} f_{txxxx}$ is

$$\xi'(t) = -f_{xt}(\xi(t),t) + \frac{\xi^2(t)}{2} f_{txxx}(\xi(t),t) \tag{4.19}$$

$$\zeta'(t) = \zeta(t) f_{txx}(\xi(t),t) \tag{4.20}$$

with initial conditions $\xi(t_0) = x_0, \zeta(t_0) = z_0$ describing the particle position at time t_0. We can use if ofr general f, including for the Boussinesq equation, but for the particular case described in the section above f simplifies.

In the special case $m_2 = -m_1, n_2 = n_1 = m_1\left(1-\frac{m_1^2}{2}\right)^{-\frac{1}{2}}$,

$$f(x,t) = 6\log(\cosh m_1 x + \cosh n_1 t) > 0.$$

the derivatives then are

$$f_{tx} = -m_1^2 n_1 \frac{\sinh m_1 x \sinh n_1 t}{(\cosh m_1 x + \cosh n_1 t)^2}$$

$$f_{txx} = -m_1 n_1 \frac{\sinh n_1 t(\cosh m_1 x \cosh n_1 t - \cosh^2 m_1 x) + 2}{(\cosh m_1 x + \cosh n_1 t)^3}$$

$$f_{txxx} = -m_1^3 n_1 \frac{\sinh m_1 x \sinh n_1 t(\cosh^2 m_1 x + \cosh^2 n_1 t - 4\cosh m_1 x \cosh n_1 t) - 6}{(\cosh m_1 x + \cosh n_1 t)^4}$$

Using MatLab, the system above can be solved. The code can be found in Appendix A. For $m_1 = 0.5$ the trajectories at different times will be shown below.

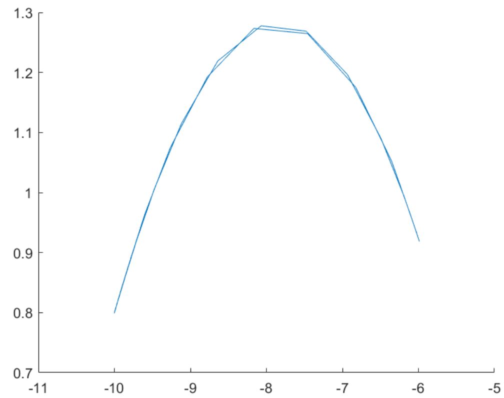

Figure 4.16: Particle Trajectory Path at t = -10, z = 0.8

Figure 2.6 shows a particle trajectory at time -10 and depth 0.8. The close up shows how the left wave distributes the particle upwards and to the left, then the right wave arrives and moves the particle backwards.

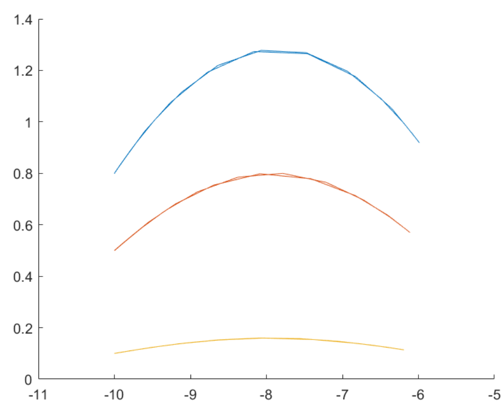

Figure 4.17: Particle Trajectory Path at t = -10, z = 0.8, 0.5, 0.1

At different depths the particles behave accordingly. Their amplitude changes the closer it is to the surface of the wave. From top to bottom we have the particle trajectories at depths 0.8, 0.5, and 0.1 respectively, all at the same time $t = -10$.

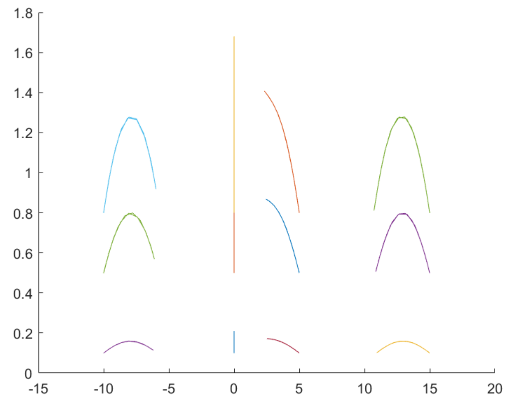

Figure 4.18: Particle Trajectory Path at t = -10, 0, 5, 15, z = 0.8, 0.5, 0.1

From left to right the particle trajectories shown are for the times $t = -10, 0, 5, 15$. From top to bottom, the particles follow the depth from the previous figure 2.7. It is particularly interesting the activity at $t = 0$. We can examine closer by using the figure below.

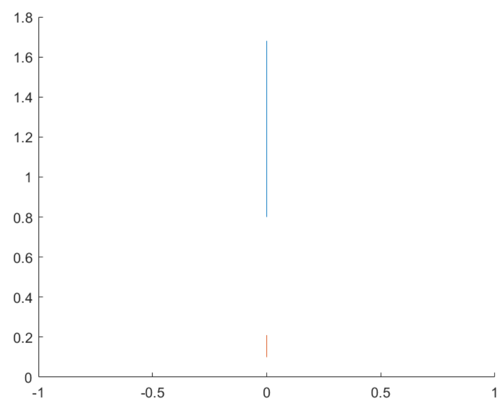

Figure 4.19: Particle Trajectory Path at t = 0, z = 0.8, 0.1

The particle follows an almost asymptotic behavior and goes vertically up and back down. This occurs when the waves collide and pass through each other.

CHAPTER V

CONCLUSION

It has been evident that the procedures that were taken in this paper have been done prior by Borluk for the KdV model of approximation. However, using that work along with the higher order Boussinesq system and with the help of the MatLab programs, new results were attained. The steps for examination were consistent throughout the work for multiple mathematical models beginning with plotting the soliton solutions, creating PDE or ODE systems using Euler's Equations, then solving said system to approximate the particle trajectory paths. The particle trajectories were able to be studied in an efficient and effective manner for unidirectional and opposite traveling surface waves.

This topic is extensive and can be continually studied to gain more information and methods of approximation. Additionally, more mathematical models can be used for these approximations using the same methods described in the paper. There is no doubt that the research and work for a topic as broad as this one will be sought out for many years to come.

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APPENDIX A

APPENDIX A

MATLAB CODES

1.1 KdV Unidirectional Multi Soliton Waves

1.1.1 Surface Wave

syms m1 r1 n1 q1 m2 r2 n2 q2 x z t; r1 = 4*m1^3;$q1 = 4*n1^3; h1 = exp(m1*x - r1*t) + exp(2*(m1*x - r1*t)); h2 = exp(n1*x - q1*t) - exp(2*(n1*x - q1*t)); h1x = diff(h1,x); h2x = diff(h2,x); f = h1*h2x - h1x*h2; etaxx = 2*diff(log(f),x,2);$

function surfwavekdv=surfwavekdv(x,t)

m1=.2;n1=.4;

etaxx =(2.*(n1.$^3.*cosh(-4.*t.*m1.^3+x.*m1).*cosh(-4.*t.*n1.^3+x.*n1)-m1.^3.*sinh(-4.*t.*m1.^3+x.*m1).*sinh(-4.*t.*n1.^3+x.*n1)+m1.*n1.^2.*sinh(-4.*t.*m1.^3+x.*m1).*sinh(-4.*t.*n1.^3+x.*n1)-m1.^2.*n1.*cosh(-4.*t.*m1.^3+x.*m1).*cosh(-4.*t.*n1.^3+x.*n1))) ./(n1.*cosh(-4.*t.*m1.^3+x.*m1).*cosh(-4.*t.*n1.^3+x.*n1)-m1.*sinh(-4.*t.*m1.^3+x.*m1).*sinh(-4.*t.*n1.^3+x.*n1))-(2.*(m1.^2.*cosh(-4.*t.*m1.^3+x.*m1).*sinh(-4.*t.*n1.^3+x.*n1)-n1.^2.*cosh(-4.*t.*m1.^3+x.*m1).*sinh(-4.*t.*n1.^3+x.*n1)).^2)./(n1.*cosh(m1.*x-4.*m1.^3.*t).*cosh(n1.*x-4.*n1.^3.*t)-m1.*sinh(m1.*x-4.*m1.^3.*t).*sinh(n1.*x-4.*n1.^3.*t)).^2;$

figure(10);plot(x,x.*0,x,1+etaxx); end

Note, in this code, the surface wave is reffered to as 'etaxx' where as in the paper it is η

input example surfwaveKDV([-35:.1:35],20); where [-35:.1:35] is the range for x and t = 20

to see a continuous flow from one time to another input the following:

```
for t=-35:0.5:35 surfwaveKDV([-50:.1:50],t);end;
```

1.1.2 Particle Trajectories

Same input from previous section along with the system

```
u = diff(diff(f,x),t); ux = diff(u,x); uxx = diff(u,x,2);

function dy = trajKdV(t,y)

dy=zeros(2,1);

m1=.2; n1=.4;

u =((m1.^2.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-n1.^2.*
cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)).*(4.*m1.^4.*cosh(-4.*t.*
m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*n1.^4.*cosh(-4.*t.*m1.^3+y(1).*m1).*
sinh(-4.*t.*n1.^3+y(1).*n1)+4.*m1.*n1.^3.*cosh(-4.*t.*n1.^3+y(1).*n1).*sinh(-4.*t.*
m1.^3+y(1).*m1)-4.*m1.^3.*n1.*cosh(-4.*t.*n1.^3+y(1).*n1).*sinh(-4.*t.*m1.^3+y(1).*
m1)))./(n1.*cosh(m1.*y(1)-4.*m1.^3.*t).*cosh(n1.*y(1)-4.*n1.^3.*t)-m1.*sinh(m1.*
y(1)-4.*m1.^3.*t).*sinh(n1.*y(1)-4.*n1.^3.*t)).^2-(4.*n1.^5.*cosh(-4.*t.*m1.^3+y(1).*
m1).*cosh(-4.*t.*n1.^3+y(1).*n1)-4.*m1.^5.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*
t.*n1.^3+y(1).*n1)-4.*m1.^2.*n1.^3.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+
y(1).*n1)+4.*m1.^3.*n1.^2.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*
n1))./(n1.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+y(1).*n1)-m1.*sinh(-4.*
t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1));

ux=(4.*m1.^6.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*
n1.^6.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)+4.*m1.^2.*n1.^4.*
cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*m1.^4.*n1.^2.*cosh(-4.*
t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*m1.*n1.^5.*cosh(-4.*t.*n1.^3+
y(1).*n1).*sinh(-4.*t.*m1.^3+y(1).*m1)+4.*m1.^5.*n1.*cosh(-4.*t.*n1.^3+y(1).*n1).*
sinh(-4.*t.*m1.^3+y(1).*m1))./(n1.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+
y(1).*n1)-m1.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1))-(2.*
(m1.^2.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-n1.^2.*cosh(-4.*t.*
```

$m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)).*(4.*n1.^{5}.*cosh(-4.*t.*m1.^{3}+y(1).*$
$m1).*cosh(-4.*t.*n1.^{3}+y(1).*n1)-4.*m1.^{5}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*$
$t.*n1.^{3}+y(1).*n1)-4.*m1.^{2}.*n1.^{3}.*cosh(-4.*t.*m1.^{3}+y(1).*m1).*cosh(-4.*t.*n1.^{3}+$
$y(1).*n1)+4.*m1.^{3}.*n1.^{2}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*$
$n1))))./(n1.*cosh(m1.*y(1)-4.*m1.^{3}.*t).*cosh(n1.*y(1)-4.*n1.^{3}.*t)-m1.*sinh(m1.*$
$y(1)-4.*m1.^{3}.*t).*sinh(n1.*y(1)-4.*n1.^{3}.*t)).^{2}-((n1.^{3}.*cosh(-4.*t.*m1.^{3}+y(1).*$
$m1).*cosh(-4.*t.*n1.^{3}+y(1).*n1)-m1.^{3}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*$
$n1.^{3}+y(1).*n1)+m1.*n1.^{2}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*$
$n1)-m1.^{2}.*n1.*cosh(-4.*t.*m1.^{3}+y(1).*m1).*cosh(-4.*t.*n1.^{3}+y(1).*n1)).*(4.*$
$m1.^{4}.*cosh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)-4.*n1.^{4}.*cosh(-4.*$
$t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)+4.*m1.*n1.^{3}.*cosh(-4.*t.*n1.^{3}+$
$y(1).*n1).*sinh(-4.*t.*m1.^{3}+y(1).*m1)-4.*m1.^{3}.*n1.*cosh(-4.*t.*n1.^{3}+y(1).*n1).*$
$sinh(-4.*t.*m1.^{3}+y(1).*m1))))./(n1.*cosh(m1.*y(1)-4.*m1.^{3}.*t).*cosh(n1.*y(1)-4.*$
$n1.^{3}.*t)-m1.*sinh(m1.*y(1)-4.*m1.^{3}.*t).*sinh(n1.*y(1)-4.*n1.^{3}.*t)).^{2}+(2.*(m1.^{2}.*$
$cosh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)-n1.^{2}.*cosh(-4.*t.*m1.^{3}+$
$y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)).^{2}.*(4.*m1.^{4}.*cosh(-4.*t.*m1.^{3}+y(1).*m1).*$
$sinh(-4.*t.*n1.^{3}+y(1).*n1)-4.*n1.^{4}.*cosh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*$
$n1.^{3}+y(1).*n1)+4.*m1.*n1.^{3}.*cosh(-4.*t.*n1.^{3}+y(1).*n1).*sinh(-4.*t.*m1.^{3}+y(1).*$
$m1)-4.*m1.^{3}.*n1.*cosh(-4.*t.*n1.^{3}+y(1).*n1).*sinh(-4.*t.*m1.^{3}+y(1).*m1))))./(n1.*$
$cosh(m1.*y(1)-4.*m1.^{3}.*t).*cosh(n1.*y(1)-4.*n1.^{3}.*t)-m1.*sinh(m1.*y(1)-4.*m1.^{3}.*$
$t).*sinh(n1.*y(1)-4.*n1.^{3}.*t)).^{3};$

uxx =(3.*(4.*n1.$^{5}.*cosh(-4.*t.*m1.^{3}+y(1).*m1).*cosh(-4.*t.*n1.^{3}+y(1).*n1)-$
$4.*m1.^{5}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)-4.*m1.^{2}.*n1.^{3}.*$
$cosh(-4.*t.*m1.^{3}+y(1).*m1).*cosh(-4.*t.*n1.^{3}+y(1).*n1)+4.*m1.^{3}.*n1.^{2}.*sinh(-4.*$
$t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*n1)).*(n1.^{3}.*cosh(-4.*t.*m1.^{3}+y(1).*$
$m1).*cosh(-4.*t.*n1.^{3}+y(1).*n1)-m1.^{3}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*$
$n1.^{3}+y(1).*n1)+m1.*n1.^{2}.*sinh(-4.*t.*m1.^{3}+y(1).*m1).*sinh(-4.*t.*n1.^{3}+y(1).*$

$n1) - m1.^{2}. \ast n1. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1))). /(n1. \ast$

$cosh(m1. \ast y(1) - 4. \ast m1.^{3}. \ast t). \ast cosh(n1. \ast y(1) - 4. \ast n1.^{3}. \ast t) - m1. \ast sinh(m1. \ast y(1) - 4. \ast m1.^{3}. \ast$

$t). \ast sinh(n1. \ast y(1) - 4. \ast n1.^{3}. \ast t)).^{2} - (4. \ast n1.^{7}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast cosh(-4. \ast t. \ast$

$n1.^{3} + y(1). \ast n1) - 4. \ast m1.^{7}. \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) +$

$8. \ast m1. \ast n1.^{6}. \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) + 4. \ast m1.^{4}. \ast$

$n1.^{3}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - 4. \ast m1.^{3}. \ast n1.^{4}. \ast$

$sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - 8. \ast m1.^{6}. \ast n1. \ast cosh(-4. \ast$

$t. \ast m1.^{3} + y(1). \ast m1). \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1))./(n1. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast$

$m1). \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - m1. \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast$

$n1.^{3} + y(1). \ast n1)) + (3. \ast (m1.^{2}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast$

$n1) - n1.^{2}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1)). \ast (4. \ast m1.^{6}. \ast$

$cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - 4. \ast n1.^{6}. \ast cosh(-4. \ast t. \ast$

$m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) + 4. \ast m1.^{2}. \ast n1.^{4}. \ast cosh(-4. \ast t. \ast m1.^{3} +$

$y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - 4. \ast m1.^{4}. \ast n1.^{2}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast$

$sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - 4. \ast m1. \ast n1.^{5}. \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1). \ast sinh(-4. \ast t. \ast$

$m1.^{3} + y(1). \ast m1) + 4. \ast m1.^{5}. \ast n1. \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1). \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast$

$m1)))./(n1. \ast cosh(m1. \ast y(1) - 4. \ast m1.^{3}. \ast t). \ast cosh(n1. \ast y(1) - 4. \ast n1.^{3}. \ast t) - m1. \ast sinh(m1. \ast$

$y(1) - 4. \ast m1.^{3}. \ast t). \ast sinh(n1. \ast y(1) - 4. \ast n1.^{3}. \ast t)).^{2} + ((m1.^{4}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast$

$m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - n1.^{4}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast$

$n1.^{3} + y(1). \ast n1) - 2. \ast m1. \ast n1.^{3}. \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1). \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast$

$m1) + 2. \ast m1.^{3}. \ast n1. \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1). \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1)). \ast (4. \ast$

$m1.^{4}. \ast cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - 4. \ast n1.^{4}. \ast cosh(-4. \ast$

$t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) + 4. \ast m1. \ast n1.^{3}. \ast cosh(-4. \ast t. \ast n1.^{3} +$

$y(1). \ast n1). \ast sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1) - 4. \ast m1.^{3}. \ast n1. \ast cosh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1). \ast$

$sinh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1)))./(n1. \ast cosh(m1. \ast y(1) - 4. \ast m1.^{3}. \ast t). \ast cosh(n1. \ast y(1) - 4. \ast$

$n1.^{3}. \ast t) - m1. \ast sinh(m1. \ast y(1) - 4. \ast m1.^{3}. \ast t). \ast sinh(n1. \ast y(1) - 4. \ast n1.^{3}. \ast t)).^{2} - (6. \ast (m1.^{2}. \ast$

$cosh(-4. \ast t. \ast m1.^{3} + y(1). \ast m1). \ast sinh(-4. \ast t. \ast n1.^{3} + y(1). \ast n1) - n1.^{2}. \ast cosh(-4. \ast t. \ast m1.^{3} +$

$y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)).^2.*(4.*n1.^5.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+y(1).*n1)-4.*m1.^5.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*m1.^2.*n1.^3.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+y(1).*n1)+4.*m1.^3.*n1.^2.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1))).../(n1.*cosh(m1.*y(1)-4.*m1.^3.*t).*cosh(n1.*y(1)-4.*n1.^3.*t)-m1.*sinh(m1.*y(1)-4.*m1.^3.*t).*sinh(n1.*y(1)-4.*n1.^3.*t)).^3+(6.*(m1.^2.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-n1.^2.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)).^3.*(4.*m1.^4.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*n1.^4.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)+4.*m1.*n1.^3.*cosh(-4.*t.*n1.^3+y(1).*n1).*sinh(-4.*t.*m1.^3+y(1).*m1)-4.*m1.^3.*n1.*cosh(-4.*t.*n1.^3+y(1).*n1).*sinh(-4.*t.*m1.^3+y(1).*m1))).../(n1.*cosh(m1.*y(1)-4.*m1.^3.*t).*cosh(n1.*y(1)-4.*n1.^3.*t)-m1.*sinh(m1.*y(1)-4.*m1.^3.*t).*sinh(n1.*y(1)-4.*n1.^3.*t)).^4-(6.*(m1.^2.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-n1.^2.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)).*(n1.^3.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+y(1).*n1)-m1.^3.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)+m1.*n1.^2.*sinh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-m1.^2.*n1.*cosh(-4.*t.*m1.^3+y(1).*m1).*cosh(-4.*t.*n1.^3+y(1).*n1)).*(4.*m1.^4.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)-4.*n1.^4.*cosh(-4.*t.*m1.^3+y(1).*m1).*sinh(-4.*t.*n1.^3+y(1).*n1)+4.*m1.*n1.^3.*cosh(-4.*t.*n1.^3+y(1).*n1).*sinh(-4.*t.*m1.^3+y(1).*m1)-4.*m1.^3.*n1.*cosh(-4.*t.*n1.^3+y(1).*n1).*sinh(-4.*t.*m1.^3+y(1).*m1))).../(n1.*cosh(m1.*y(1)-4.*m1.^3.*t).*cosh(n1.*y(1)-4.*n1.^3.*t)-m1.*sinh(m1.*y(1)-4.*m1.^3.*t).*sinh(n1.*y(1)-4.*n1.^3.*t)).^3;$

dy(1)=u-y(2)$^2/2*uxx$;

dy(2)=ux*y(2);

input example [ti,po]=ode45(@trajKdV,[-35 35],[-10; 0.8]); figure(7); hold on; plot(po(:,1),po(:,2)); hold off;

where t = -10 (time) and z = 0.8 (depth level)

1.2 Boussinesq Unidirectional Waves

1.2.1 Surface Wave

syms m1 r1 n1 q1 m2 r2 n2 q2 x z t; r1 = 4*m1^3;$q1 = 4*n1^3; h1 = exp(m1*x - r1*t) + exp(2*(m1*x - r1*t)); h2 = exp(n1*x - q1*t) - exp(2*(n1*x - q1*t)); h1x = diff(h1,x); h2x = diff(h2,x); f = h1*h2x - h1x*h2; etaxx = 6*diff(log(f),x,2);$

function surfwavebuni=surfwavebuni(x,t)

a=pi/4;b=pi/12;

m1=cos(a);m2=cos(a+4*pi/3);n1=cos(b);n2=cos(b+4*pi/3);

etaxx=(2.*((n1.^{3}$.*exp(-t.*n1.^2+x.*n1)-n2.^3.*exp(-t.*n2.^2+x.*n2)).*(exp(-t.*m1.^2+x.*m1)+exp(-t.*m2.^2+x.*m2))-(m1.^3.*exp(-t.*m1.^2+x.*m1)+m2.^3.*exp(-t.*m2.^2+x.*m2)).*(exp(-t.*n1.^2+x.*n1)-exp(-t.*n2.^2+x.*n2))-(m1.^2.*exp(-t.*m1.^2+x.*m1)+m2.^2.*exp(-t.*m2.^2+x.*m2)).*(n1.*exp(-t.*n1.^2+x.*n1)-n2.*exp(-t.*n2.^2+x.*n2))+(n1.^2.*exp(-t.*n1.^2+x.*n1)-n2.^2.*exp(-t.*n2.^2+x.*n2)).*(m1.*exp(-t.*m1.^2+x.*m1)+m2.*exp(-t.*m2.^2+x.*m2)))) ./((n1.*exp(-t.*n1.^2+x.*n1)-n2.*exp(-t.*n2.^2+x.*n2)).*(exp(-t.*m1.^2+x.*m1)+exp(-t.*m2.^2+x.*m2))-(exp(-t.*n1.^2+x.*n1)-exp(-t.*n2.^2+x.*n2)).*(m1.*exp(-t.*m1.^2+x.*m1)+m2.*exp(-t.*m2.^2+x.*m2)))-(2.*((n1.^2.*exp(-t.*n1.^2+x.*n1)-n2.^2.*exp(-t.*n2.^2+x.*n2)).*(exp(-t.*m1.^2+x.*m1)+exp(-t.*m2.^2+x.*m2))-(m1.^2.*exp(-t.*m1.^2+x.*m1)+m2.^2.*exp(-t.*m2.^2+x.*m2)).*(exp(-t.*n1.^2+x.*n1)-exp(-t.*n2.^2+x.*n2))).^2)./((n1.*exp(n1.*x-n1.^2.*t)-n2.*exp(n2.*x-n2.^2.*t)).*(exp(m1.*x-m1.^2.*t)+exp(m2.*x-m2.^2.*t))-(exp(n1.*x-n1.^2.*t)-exp(n2.*x-n2.^2.*t)).*(m1.*exp(m1.*x-m1.^2.*t)+m2.*exp(m2.*x-m2.^2.*t))).^2;$

figure(12);plot(x,x.*0,x,1+etaxx);

input example surfwavebuni([-45:.1:-10],-35);

1.2.2 Particle Trajectories

function dy = trajGBuni(t,y)

```
dy=zeros(2,1);
a=pi/4;b=pi/12;
m1=cos(a);m2=cos(a+4*pi/3);n1=cos(b);n2=cos(b+4*pi/3);
u =(((n1.^2.*exp(-t.*n1.^2+y(1).*n1)-n2.^2.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^2.*exp(-t.*m1.^2+y(1).*m1)+m2.^2.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2))).*((n1.^3.*exp(-t.*n1.^2+y(1).*n1)-n2.^3.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^3.*exp(-t.*m1.^2+y(1).*m1)+m2.^3.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2))+(m1.^2.*exp(-t.*m1.^2+y(1).*m1)+m2.^2.*exp(-t.*m2.^2+y(1).*m2)).*(n1.*exp(-t.*n1.^2+y(1).*n1)-n2.*exp(-t.*n2.^2+y(1).*n2))-(n1.^2.*exp(-t.*n1.^2+y(1).*n1)-n2.^2.*exp(-t.*n2.^2+y(1).*n2)).*(m1.*exp(-t.*m1.^2+y(1).*m1)+m2.*exp(-t.*m2.^2+y(1).*m2)))))./((n1.*exp(n1.*y(1)-n1.^2.*t)-n2.*exp(n2.*y(1)-n2.^2.*t)).*(exp(m1.*y(1)-m1.^2.*t)+exp(m2.*y(1)-m2.^2.*t))-(exp(n1.*y(1)-n1.^2.*t)-exp(n2.*y(1)-n2.^2.*t)).*(m1.*exp(m1.*y(1)-m1.^2.*t)+m2.*exp(m2.*y(1)-m2.^2.*t))).^2-((n1.^4.*exp(-t.*n1.^2+y(1).*n1)-n2.^4.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^4.*exp(-t.*m1.^2+y(1).*m1)+m2.^4.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2)))./((n1.*exp(-t.*n1.^2+y(1).*n1)-n2.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2)).*(m1.*exp(-t.*m1.^2+y(1).*m1)+m2.*exp(-t.*m2.^2+y(1).*m2)));
ux=(((n1.^3.*exp(-t.*n1.^2+y(1).*n1)-n2.^3.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^3.*exp(-t.*m1.^2+y(1).*m1)+m2.^3.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2))+(m1.^2.*exp(-t.*m1.^2+y(1).*m1)+m2.^2.*exp(-t.*m2.^2+y(1).*m2)).*(n1.*exp(-t.*n1.^2+y(1).*n1)-n2.*exp(-t.*n2.^2+y(1).*n2))-(n1.^2.*exp(-t.*n1.^2+y(1).*n1)-n2.^2.*exp(-t.*n2.^2+y(1).*n2)).*(m1.*exp(-t.*m1.^2+y(1).*m1)+m2.*exp(-t.*m2.^2+y(1).*m2))).*
```

$((n1.^{3}.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.^{3}.\ast exp(-t.\ast n2.^{2}+y(1).\ast n2)).\ast (exp(-t.\ast m1.^{2}+y(1).\ast$
$m1)+exp(-t.\ast m2.^{2}+y(1).\ast m2))-(m1.^{3}.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.^{3}.\ast exp(-t.\ast$
$m2.^{2}+y(1).\ast m2)).\ast (exp(-t.\ast n1.^{2}+y(1).\ast n1)-exp(-t.\ast n2.^{2}+y(1).\ast n2))-(m1.^{2}.\ast exp(-t.\ast$
$m1.^{2}+y(1).\ast m1)+m2.^{2}.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)).\ast (n1.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.\ast$
$exp(-t.\ast n2.^{2}+y(1).\ast n2))+(n1.^{2}.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.^{2}.\ast exp(-t.\ast n2.^{2}+y(1).\ast$
$n2)).\ast (m1.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)))))./((n1.\ast exp(n1.\ast$
$y(1)-n1.^{2}.\ast t)-n2.\ast exp(n2.\ast y(1)-n2.^{2}.\ast t)).\ast (exp(m1.\ast y(1)-m1.^{2}.\ast t)+exp(m2.\ast y(1)-$
$m2.^{2}.\ast t))-(exp(n1.\ast y(1)-n1.^{2}.\ast t)-exp(n2.\ast y(1)-n2.^{2}.\ast t)).\ast (m1.\ast exp(m1.\ast y(1)-$
$m1.^{2}.\ast t)+m2.\ast exp(m2.\ast y(1)-m2.^{2}.\ast t))).^{2}-(2.\ast ((n1.^{2}.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.^{2}.\ast$
$exp(-t.\ast n2.^{2}+y(1).\ast n2)).\ast (exp(-t.\ast m1.^{2}+y(1).\ast m1)+exp(-t.\ast m2.^{2}+y(1).\ast m2))-$
$(m1.^{2}.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.^{2}.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)).\ast (exp(-t.\ast n1.^{2}+$
$y(1).\ast n1)-exp(-t.\ast n2.^{2}+y(1).\ast n2))).^{2}.\ast ((n1.^{3}.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.^{3}.\ast exp(-t.\ast$
$n2.^{2}+y(1).\ast n2)).\ast (exp(-t.\ast m1.^{2}+y(1).\ast m1)+exp(-t.\ast m2.^{2}+y(1).\ast m2))-(m1.^{3}.\ast exp(-t.\ast$
$m1.^{2}+y(1).\ast m1)+m2.^{3}.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)).\ast (exp(-t.\ast n1.^{2}+y(1).\ast n1)-exp(-t.\ast$
$n2.^{2}+y(1).\ast n2))+(m1.^{2}.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.^{2}.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)).\ast$
$(n1.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.\ast exp(-t.\ast n2.^{2}+y(1).\ast n2))-(n1.^{2}.\ast exp(-t.\ast n1.^{2}+$
$y(1).\ast n1)-n2.^{2}.\ast exp(-t.\ast n2.^{2}+y(1).\ast n2)).\ast (m1.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.\ast$
$exp(-t.\ast m2.^{2}+y(1).\ast m2))))./((n1.\ast exp(n1.\ast y(1)-n1.^{2}.\ast t)-n2.\ast exp(n2.\ast y(1)-n2.^{2}.\ast$
$t)).\ast (exp(m1.\ast y(1)-m1.^{2}.\ast t)+exp(m2.\ast y(1)-m2.^{2}.\ast t))-(exp(n1.\ast y(1)-n1.^{2}.\ast t)-$
$exp(n2.\ast y(1)-n2.^{2}.\ast t)).\ast (m1.\ast exp(m1.\ast y(1)-m1.^{2}.\ast t)+m2.\ast exp(m2.\ast y(1)-m2.^{2}.\ast$
$t))).^{3}-((n1.^{5}.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.^{5}.\ast exp(-t.\ast n2.^{2}+y(1).\ast n2)).\ast (exp(-t.\ast$
$m1.^{2}+y(1).\ast m1)+exp(-t.\ast m2.^{2}+y(1).\ast m2))-(m1.^{5}.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.^{5}.\ast$
$exp(-t.\ast m2.^{2}+y(1).\ast m2)).\ast (exp(-t.\ast n1.^{2}+y(1).\ast n1)-exp(-t.\ast n2.^{2}+y(1).\ast n2))-(m1.^{4}.\ast$
$exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.^{4}.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)).\ast (n1.\ast exp(-t.\ast n1.^{2}+y(1).\ast$
$n1)-n2.\ast exp(-t.\ast n2.^{2}+y(1).\ast n2))+(n1.^{4}.\ast exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.^{4}.\ast exp(-t.\ast$
$n2.^{2}+y(1).\ast n2)).\ast (m1.\ast exp(-t.\ast m1.^{2}+y(1).\ast m1)+m2.\ast exp(-t.\ast m2.^{2}+y(1).\ast m2)))./((n1.\ast$
$exp(-t.\ast n1.^{2}+y(1).\ast n1)-n2.\ast exp(-t.\ast n2.^{2}+y(1).\ast n2)).\ast (exp(-t.\ast m1.^{2}+y(1).\ast m1)+$

$$exp(-t.*m2.^{2}+y(1).*m2))-(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+m2.*exp(-t.*m2.^{2}+y(1).*m2)))+(2.*((n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))).*((n1.^{4}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{4}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{4}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{4}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))))./((n1.*exp(n1.*y(1)-n1.^{2}.*t)-n2.*exp(n2.*y(1)-n2.^{2}.*t)).*(exp(m1.*y(1)-m1.^{2}.*t)+exp(m2.*y(1)-m2.^{2}.*t))-(exp(n1.*y(1)-n1.^{2}.*t)-exp(n2.*y(1)-n2.^{2}.*t)).*(m1.*exp(m1.*y(1)-m1.^{2}.*t)+m2.*exp(m2.*y(1)-m2.^{2}.*t))).^{2};$$

$$uxx=(6.*((n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))).^{3}.*((n1.^{3}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{3}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{3}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{3}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))+(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*exp(-t.*m2.^{2}+y(1).*m2)).*(n1.*exp(-t.*n1.^{2}+y(1).*n1)-n2.*exp(-t.*n2.^{2}+y(1).*n2))-(n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+m2.*exp(-t.*m2.^{2}+y(1).*m2)))))./((n1.*exp(n1.*y(1)-n1.^{2}.*t)-n2.*exp(n2.*y(1)-n2.^{2}.*t)).*(exp(m1.*y(1)-m1.^{2}.*t)+exp(m2.*y(1)-m2.^{2}.*t))-(exp(n1.*y(1)-n1.^{2}.*t)-exp(n2.*y(1)-n2.^{2}.*t)).*(m1.*exp(m1.*y(1)-m1.^{2}.*t)+m2.*exp(m2.*y(1)-m2.^{2}.*t))).^{4}-((n1.^{6}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{6}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))+(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*exp(-t.*m2.^{2}+y(1).*m2)).*(n1.^{4}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{4}.*exp(-t.*n2.^{2}+y(1).*n2))-(m1.^{4}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{4}.*exp(-t.*m2.^{2}+y(1).*m2)).*(n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+$$

$y(1).*n2))-(m1.^{6}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{6}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*$
$n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))-2.*(m1.^{5}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{5}.*$
$exp(-t.*m2.^{2}+y(1).*m2)).*(n1.*exp(-t.*n1.^{2}+y(1).*n1)-n2.*exp(-t.*n2.^{2}+y(1).*$
$n2))+2.*(n1.^{5}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{5}.*exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*$
$m1.^{2}+y(1).*m1)+m2.*exp(-t.*m2.^{2}+y(1).*m2)))./((n1.*exp(-t.*n1.^{2}+y(1).*n1)-$
$n2.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-$
$(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+$
$m2.*exp(-t.*m2.^{2}+y(1).*m2)))+(((n1.^{3}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{3}.*exp(-t.*n2.^{2}+$
$y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{3}.*exp(-t.*$
$m1.^{2}+y(1).*m1)+m2.^{3}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*$
$n2.^{2}+y(1).*n2))+(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*exp(-t.*m2.^{2}+y(1).*m2)).*$
$(n1.*exp(-t.*n1.^{2}+y(1).*n1)-n2.*exp(-t.*n2.^{2}+y(1).*n2))-(n1.^{2}.*exp(-t.*n1.^{2}+$
$y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+m2.*$
$exp(-t.*m2.^{2}+y(1).*m2))).*((n1.^{4}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{4}.*exp(-t.*n2.^{2}+$
$y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{4}.*exp(-t.*$
$m1.^{2}+y(1).*m1)+m2.^{4}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*$
$n2.^{2}+y(1).*n2))-2.*(m1.^{3}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{3}.*exp(-t.*m2.^{2}+y(1).*$
$m2)).*(n1.*exp(-t.*n1.^{2}+y(1).*n1)-n2.*exp(-t.*n2.^{2}+y(1).*n2))+2.*(n1.^{3}.*exp(-t.*$
$n1.^{2}+y(1).*n1)-n2.^{3}.*exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+$
$m2.*exp(-t.*m2.^{2}+y(1).*m2))))./((n1.*exp(n1.*y(1)-n1.^{2}.*t)-n2.*exp(n2.*y(1)-$
$n2.^{2}.*t)).*(exp(m1.*y(1)-m1.^{2}.*t)+exp(m2.*y(1)-m2.^{2}.*t))-(exp(n1.*y(1)-n1.^{2}.*$
$t)-exp(n2.*y(1)-n2.^{2}.*t)).*(m1.*exp(m1.*y(1)-m1.^{2}.*t)+m2.*exp(m2.*y(1)-m2.^{2}.*$
$t))).^{2}-(6.*((n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*$
$m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*$
$exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))).^{2}.*$
$((n1.^{4}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{4}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*$
$m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{4}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{4}.*exp(-t.*$

$m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))))./((n1.*exp(n1.*$

$y(1)-n1.^{2}.*t)-n2.*exp(n2.*y(1)-n2.^{2}.*t)).*(exp(m1.*y(1)-m1.^{2}.*t)+exp(m2.*y(1)-$

$m2.^{2}.*t))-(exp(n1.*y(1)-n1.^{2}.*t)-exp(n2.*y(1)-n2.^{2}.*t)).*(m1.*exp(m1.*y(1)-$

$m1.^{2}.*t)+m2.*exp(m2.*y(1)-m2.^{2}.*t))).^{3}+(3.*((n1.^{4}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{4}.*$

$exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-$

$(m1.^{4}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{4}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+$

$y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))).*((n1.^{3}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{3}.*exp(-t.*$

$n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{3}.*exp(-t.*$

$m1.^{2}+y(1).*m1)+m2.^{3}.*exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*$

$n2.^{2}+y(1).*n2))-(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*exp(-t.*m2.^{2}+y(1).*m2)).*$

$(n1.*exp(-t.*n1.^{2}+y(1).*n1)-n2.*exp(-t.*n2.^{2}+y(1).*n2))+(n1.^{2}.*exp(-t.*n1.^{2}+$

$y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+m2.*$

$exp(-t.*m2.^{2}+y(1).*m2))))./((n1.*exp(n1.*y(1)-n1.^{2}.*t)-n2.*exp(n2.*y(1)-n2.^{2}.*$

$t)).*(exp(m1.*y(1)-m1.^{2}.*t)+exp(m2.*y(1)-m2.^{2}.*t))-(exp(n1.*y(1)-n1.^{2}.*t)-$

$exp(n2.*y(1)-n2.^{2}.*t)).*(m1.*exp(m1.*y(1)-m1.^{2}.*t)+m2.*exp(m2.*y(1)-m2.^{2}.*$

$t))).^{2}+(3.*((n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*$

$m1.^{2}+y(1).*m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{2}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{2}.*$

$exp(-t.*m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))).*$

$((n1.^{5}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{5}.*exp(-t.*n2.^{2}+y(1).*n2)).*(exp(-t.*m1.^{2}+y(1).*$

$m1)+exp(-t.*m2.^{2}+y(1).*m2))-(m1.^{5}.*exp(-t.*m1.^{2}+y(1).*m1)+m2.^{5}.*exp(-t.*$

$m2.^{2}+y(1).*m2)).*(exp(-t.*n1.^{2}+y(1).*n1)-exp(-t.*n2.^{2}+y(1).*n2))-(m1.^{4}.*exp(-t.*$

$m1.^{2}+y(1).*m1)+m2.^{4}.*exp(-t.*m2.^{2}+y(1).*m2)).*(n1.*exp(-t.*n1.^{2}+y(1).*n1)-n2.*$

$exp(-t.*n2.^{2}+y(1).*n2))+(n1.^{4}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{4}.*exp(-t.*n2.^{2}+y(1).*$

$n2)).*(m1.*exp(-t.*m1.^{2}+y(1).*m1)+m2.*exp(-t.*m2.^{2}+y(1).*m2))))./((n1.*exp(n1.*$

$y(1)-n1.^{2}.*t)-n2.*exp(n2.*y(1)-n2.^{2}.*t)).*(exp(m1.*y(1)-m1.^{2}.*t)+exp(m2.*y(1)-$

$m2.^{2}.*t))-(exp(n1.*y(1)-n1.^{2}.*t)-exp(n2.*y(1)-n2.^{2}.*t)).*(m1.*exp(m1.*y(1)-$

$m1.^{2}.*t)+m2.*exp(m2.*y(1)-m2.^{2}.*t))).^{2}-(6.*((n1.^{2}.*exp(-t.*n1.^{2}+y(1).*n1)-n2.^{2}.*$

$exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^2.*exp(-t.*m1.^2+y(1).*m1)+m2.^2.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2))).*((n1.^3.*exp(-t.*n1.^2+y(1).*n1)-n2.^3.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^3.*exp(-t.*m1.^2+y(1).*m1)+m2.^3.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2))+(m1.^2.*exp(-t.*m1.^2+y(1).*m1)+m2.^2.*exp(-t.*m2.^2+y(1).*m2)).*(n1.*exp(-t.*n1.^2+y(1).*n1)-n2.*exp(-t.*n2.^2+y(1).*n2))-(n1.^2.*exp(-t.*n1.^2+y(1).*n1)-n2.^2.*exp(-t.*n2.^2+y(1).*n2)).*(m1.*exp(-t.*m1.^2+y(1).*m1)+m2.*exp(-t.*m2.^2+y(1).*m2))).*((n1.^3.*exp(-t.*n1.^2+y(1).*n1)-n2.^3.*exp(-t.*n2.^2+y(1).*n2)).*(exp(-t.*m1.^2+y(1).*m1)+exp(-t.*m2.^2+y(1).*m2))-(m1.^3.*exp(-t.*m1.^2+y(1).*m1)+m2.^3.*exp(-t.*m2.^2+y(1).*m2)).*(exp(-t.*n1.^2+y(1).*n1)-exp(-t.*n2.^2+y(1).*n2))-(m1.^2.*exp(-t.*m1.^2+y(1).*m1)+m2.^2.*exp(-t.*m2.^2+y(1).*m2)).*(n1.*exp(-t.*n1.^2+y(1).*n1)-n2.*exp(-t.*n2.^2+y(1).*n2))+(n1.^2.*exp(-t.*n1.^2+y(1).*n1)-n2.^2.*exp(-t.*n2.^2+y(1).*n2)).*(m1.*exp(-t.*m1.^2+y(1).*m1)+m2.*exp(-t.*m2.^2+y(1).*m2)))))./((n1.*exp(n1.*y(1)-n1.^2.*t)-n2.*exp(n2.*y(1)-n2.^2.*t)).*(exp(m1.*y(1)-m1.^2.*t)+exp(m2.*y(1)-m2.^2.*t))-(exp(n1.*y(1)-n1.^2.*t)-exp(n2.*y(1)-n2.^2.*t)).*(m1.*exp(m1.*y(1)-m1.^2.*t)+m2.*exp(m2.*y(1)-m2.^2.*t))).^3$;

dy(1)=u-y(2)$^2/2*uxx$;

dy(2)=ux*y(2);

1.3 Boussinesq Opposite Traveling Waves

1.3.1 Surface Wave

function twosoliton = twosoliton(x, t);

range for x and fixed time t

m1=.5;n1=m1/sqrt(1-m1$^2/2$);

define the surface wave

fxx=6*m1$^2*(cosh(m1*x).*cosh(n1*t)+1)./(cosh(m1*x)+cosh(n1*t)).^2$;

figure(20);plot(x,fxx);

twosoliton=fxx;

input example twosoliton([-20:0.5:20], 10);

1.3.2 Particle Trajectories

function dy = TrajNew(t,y)

dy=zeros(2,1);

m1=.5;n1=m1/sqrt(1-m1$^2/2)$;

dy(1) =(6*m1*n1*sinh(n1*t)*sinh(m1*y(1)))/(cosh(n1*t) + cosh(m1*y(1)))$^2 - (3*m1^3*n1*y(2)^2*sinh(n1*t)*sinh(m1*y(1))*(cosh(n1*t)^2+cosh(m1*y(1))^2-4*cosh(n1*t)*cosh(m1*y(1))-6))/(cosh(n1*t)+cosh(m1*y(1)))^4$;

dy(2)=-(6*m1$^2*n1*y(2)*sinh(n1*t)*(cosh(n1*t)*cosh(m1*y(1))-cosh(m1*y(1))^2+2))/(cosh(n1*t)+cosh(m1*y(1)))^3$;

input example [T,Y] = ode45(@TrajNew, [-50 50],[-6,.7]); figure(71); hold on;plot(Y(:,1),Y(:,2));hold off;

APPENDIX B

APPENDIX B

EXTENDED RESEARCH

2.1 Further Examination of Euler's Equations

To expand our research we revisit Euler's Equations and approximate a solution.

$$u_t + uu_x + vu_z = -p_x \tag{2.2}$$

$$v_t + uv_x + vv_z = -p_z - g \tag{2.3}$$

The most popular way to find the solution is by Asymptotic expansion of equation (2.4). This method was mentioned earlier. Using the KdV equation, whcih can be solved, we can approximate the solutions to the Euler's Equations.

If we know the solutions to the asymptotic expansion, we can go back to the Euler's equations. Using u and v from equations (2.8) and (2.9) we want to see how they relate to Euler's Equations. With this equations, we are going to avoid pressure. Therefore, if u and v are solutions, then there is a necessary condition to consider when looking at the right hand side of the equations. That is

$$p_{xz} = p_{zx} \tag{2.1}$$

Once the partial derivatives are applied to the entire equations we get

$$(u_t + uu_x + vu_z = -p_x)_z = u_{tz} + u_z u_x + uu_{xz} + v_z u_z + vu_{zz} = -p_{xz} \tag{2.2}$$

$$(v_t + uv_x + vv_z = -p_z - g)_x = v_{tx} + u_x v_x + uv_{xx} + v_x v_z + vv_{zx} = -p_{zx} \tag{2.3}$$

This should yield identical or to the very least similar graphs when plotted. We take the partial derivatives for each equation that results to equations (2.14) and (2.15) and plot the surface.

2.1.1 Single Soliton Plots

For a single soliton, we have the following results. First, we have the surface wave below.

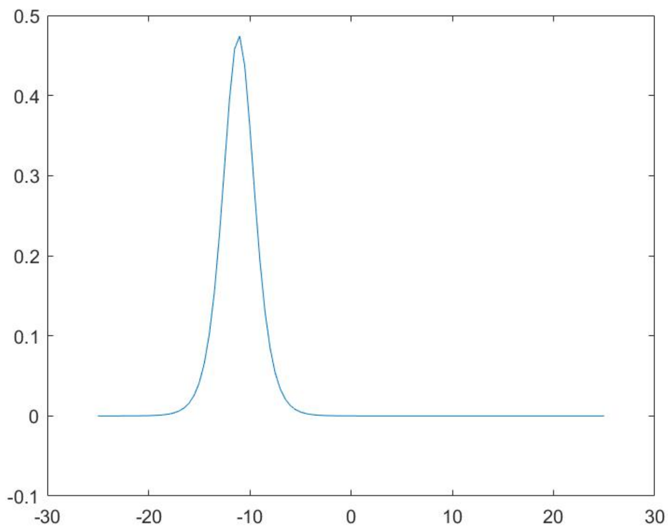

Figure B.1: Single Soliton $t = -10, z = 0.1$

Now, we plot the surface of equations (2.14) and (2.15) using the same parameters as the surface wave. Note that these equations are completely dependant on η. When we do this plotting we get the following for $t = -10$ and $z = 0.1$.

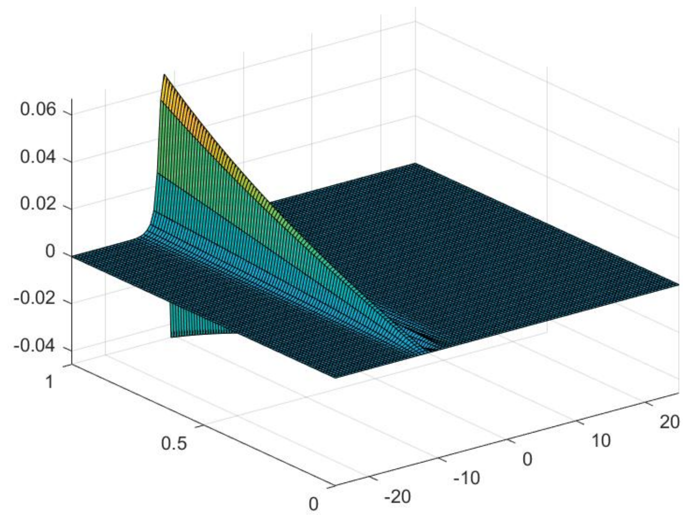

Figure B.2: $u_{tz} + u_z u_x + u u_{xz} + v_z u_z + v u_{zz} = -p_{xz}$ (2.14)

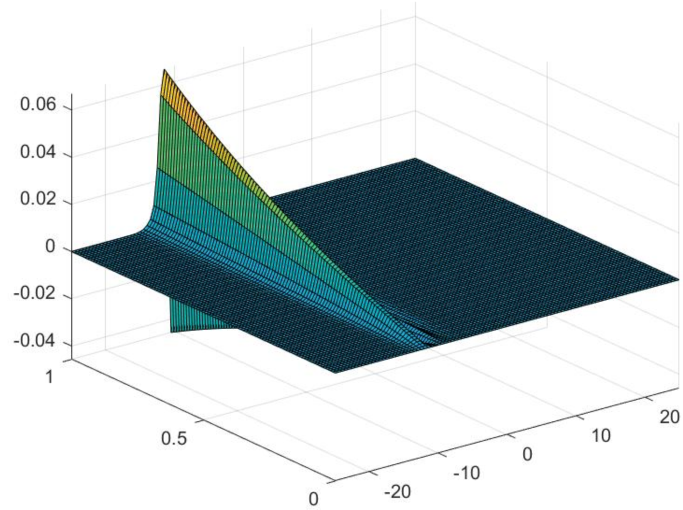

Figure B.3: $v_{tx} + u_x v_x + u v_{xx} + v_x v_z + v v_{zx} = -p_{zx}$ (2.15)

To truly see how close these surface plots are, we can plot the difference error and examine

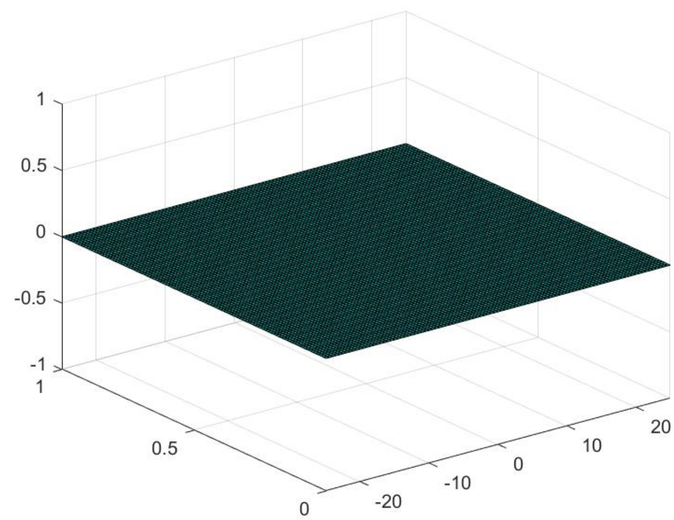

Figure B.4: Difference Error in Single Soliton Plot

Visually, we can see a completely flat surface at 0. Additionally, the calculations bring the result of the difference to 0. This means that our approximations of u and v are solutions to the two dimensional Euler Equations from before.

2.1.2 Multi Soliton or 2-Soliton Plots

The procedure here will be similar to the one before. We begin by plotting the surface wave. Notice here two unidirection waves.

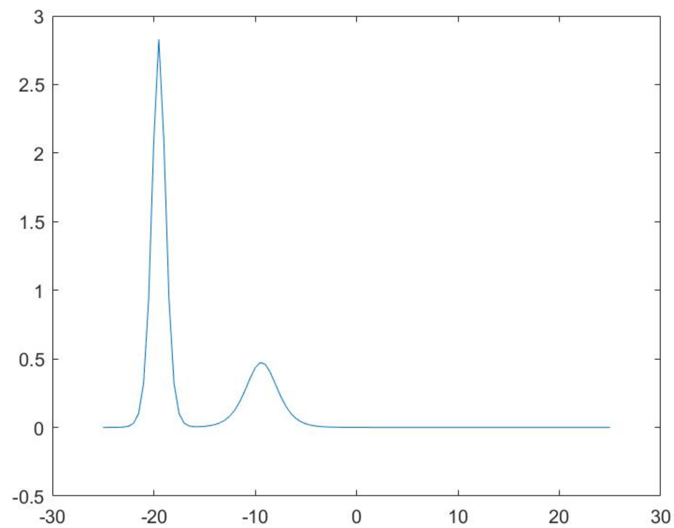

Figure B.5: Unidirectional Waves $t = -10, z = 0.1$

Now, we plot the same equations from before (2.14) (2.15) with these new parameters.

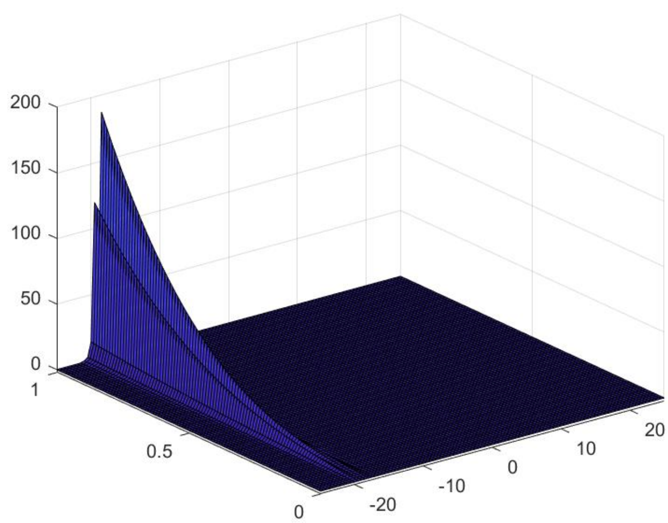

Figure B.6: $u_{tz} + u_z u_x + u u_{xz} + v_z u_z + v u_{zz} = -p_{xz}$ (2.14)

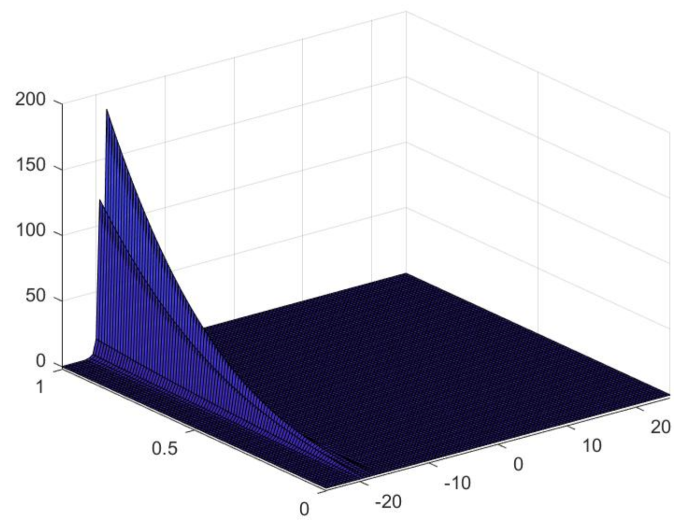

Figure B.7: $v_{tx} + u_x v_x + u v_{xx} + v_x v_z + v v_{zx} = -p_{zx}$ (2.15)

From here, we can now plot the difference error to see how close these surface plots are.

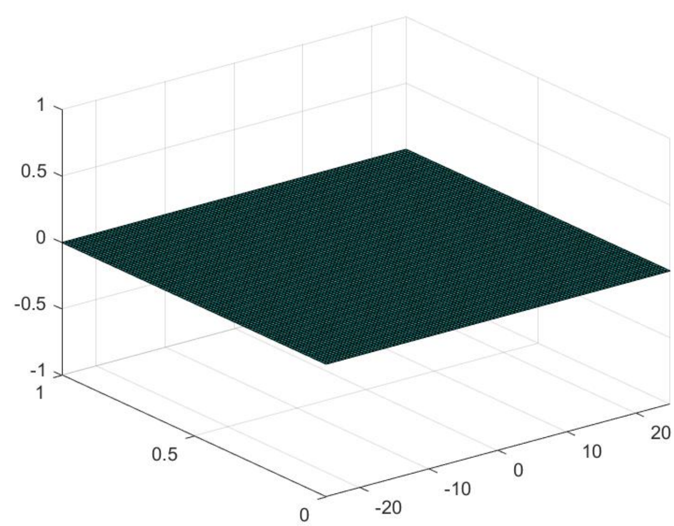

Figure B.8: Difference Error for 2-Soliton Plot

The results are similar as before.From our trials and different parameters we see a nice result and can actually plot the difference of the two surfaces and see that it is 0. Here we notice the solutions for the KdV can be used to approximate the Euler's Equations.

BIOGRAPHICAL SKETCH

Diana Torres was born to immigrant parents in the border town of Brownsville, Texas. Her childhood was spent in a typical Mexican household filled with love and the smell of tortillas. Hardships occurred, as they tend to do, but were almost always overcome. Fast forward a bit over the personal and emotional bits, and she graduated with honors from Brownsville Early College High School in 2016 and was able to attain over 60 college credit hours there. Her love of math and teaching grew and she decided to venture into that field. She continued her education in her home town and completed and earned a Bachelor's Degree in Mathematics with a teaching certification in May of 2019. From there, she decided to take on the task of completing her Master's Degree at the same university in Applied Mathematics while continuing to help her family and attempting to enjoy life. Two years later, on August 12, 2021, and in the middle of a global pandemic, she defended her thesis and received her Master's Degree in Applied Math. She is now a math teacher at a high school and enjoys her job dearly, but hopes that one day she can explore other careers that involve math, too. She can be contacted through her personal email address torres.diana2016@gmail.com or through twitter "@diana_twrs".